Basic Signal Processing:
Sampling, Aliasing, Antialiasing

Key Concepts

Frequency space
Filters and convolution
Sampling and the Nyquist frequency
Aliasing and Antialiasing
Frequency Space

Sines and Cosines

\[ \sin 2\pi x \]

\[ \cos 2\pi x \]
**Frequencies** \( \cos 2\pi f x \)

\[
f = \frac{1}{T} \quad f = 1
\]

\[
\cos 2\pi x
\]

\[
\cos 4\pi x
\]

Recall Complex Exponentials

**Euler’s Formula**

\[
e^{jx} = \cos x + j \sin x
\]

**Odd (-x)**

\[
e^{-jx} = \cos -x + j \sin -x = \cos x - j \sin x
\]

Therefore

\[
\cos x = \frac{e^{jx} + e^{-jx}}{2} \quad \sin x = \frac{e^{jx} - e^{-jx}}{2j}
\]

Hence, use complex exponentials for sines/cosines
Constant

\[ \sin\left(\frac{2\pi}{32}\right) x \]

Spatial Domain

Frequency Domain

Frequency = 1/32; 32 pixels per cycle
\[ \sin \left( \frac{2\pi}{16} \right) x \]

Spatial Domain  Frequency Domain

\[ \sin \left( \frac{2\pi}{16} \right) y \]

Spatial Domain  Frequency Domain
\[ \sin\left(\frac{2\pi}{32}\right)x \times \sin\left(\frac{2\pi}{16}\right)y \]

\[ e^{-r^2/16^2} \]
$$e^{-r^2/32^2}$$

Spatial Domain  
Frequency Domain

$$e^{-x^2/32^2} \times e^{-y^2/16^2}$$

Spatial Domain  
Frequency Domain
Rotating 45°

\[ e^{-\frac{x^2}{32^2}} \times e^{-\frac{y^2}{16^2}} \]

Spatial Domain | Frequency Domain

Filtering
My Humble Frequencies

Spatial Domain

Frequency Domain

Remove Low Frequencies (Edges)

Spatial Domain

Frequency Domain
Remove High Frequencies (Blur)

Spatial Domain  Frequency Domain

Remove Low and High Frequencies

Spatial Domain  Frequency Domain
Remove Low and High Frequencies

Spatial Domain          Frequency Domain

Filters = Convolution
Convolution

\[
\begin{array}{cccccc}
1 & 3 & 0 & 4 & 2 & 1 \\
1 & 2 & & & & \\
\end{array}
\]

\[1 \times 1 + 3 \times 2 = 7\]

\[
\begin{array}{cccccc}
7 & & & & & \\
\end{array}
\]
Convolution

\[
\begin{array}{cccccc}
1 & 3 & 0 & 4 & 2 & 1 \\
\hline
1 & 2 \\
\end{array}
\]

\[
3 \times 1 + 0 \times 2 = 3
\]

\[
\begin{array}{ccc}
7 & 3 & \\
\end{array}
\]

Convolution

\[
\begin{array}{cccccc}
1 & 3 & 0 & 4 & 2 & 1 \\
\hline
1 & 2 \\
\end{array}
\]

\[
0 \times 1 + 4 \times 2 = 8
\]

\[
\begin{array}{cc}
7 & 3 \\
\end{array}
\]
Convolution Theorem

A filter can be implemented in the spatial domain using convolution

A filter can also be implemented in the frequency domain

- Convert image to frequency domain
- Convert filter to frequency domain
- Multiply filter times image in frequency domain
- Convert result to the spatial domain

Box Filter

\[
\begin{array}{ccc}
 1 & 1 \\
 1 & & 1 \\
 1 & 1 \\
\end{array}
\]
Box Filter = Low-Pass Filter

Wider Filters, Lower Frequencies
Size of Filter

As a filter is localized in space, it spreads out in frequency. Conversely, as a filter is localized in frequency, it spreads out in space.

A box filter is very localized in space; it has infinite extent in frequency space.

Efficiency?

When would it be faster to apply the filter in the spatial domain?

When would it be faster to apply the filter in the frequency domain?
Sampling

Image Generation = Sampling

Evaluating a function at a point is sampling

\[
\begin{align*}
\text{for( int } x = 0; x < \text{xmax; } x++ ) \\
& \quad \text{for( int } y = 0; y < \text{ymax; } y++ ) \\
& \quad \text{Image}[x][y] = f(x,y);
\end{align*}
\]

Rasterization is equivalent to evaluating the function inside(triangle,x,y)
Sampling Causes Jaggies

Retort, by Don Mitchell

Sampling in Computer Graphics

Artifacts due to sampling - Aliasing

- Jaggies – sampling in space
- Wagon wheel effect – sampling in time
- Temporal strobing – sampling in space-time
- Moire – sampling texture coordinates
- Sparkling highlights – sampling normals

Preventing these artifacts - Antialiasing
Aliasing

Wagon Wheel Effect

http://www.michaelbach.de/ot/mot_wagonWheel/
"Aliases"

These two sine waves are indistinguishable. Indistinguishable frequencies are called "aliases".

Nyquist Frequency

Definition: The Nyquist frequency is $\frac{1}{2}$ the sampling frequency ($1/T_s$).

Frequencies above the Nyquist frequency appear as aliases.

No aliases appear if the function being sampled has no frequencies above the Nyquist frequency.
Antialiasing

Simple idea:

Remove frequencies above the Nyquist frequency before sampling

How? Filtering before sampling
Prefiltering by Computing Coverage

A 1 pixel box filter removes frequencies whose period is less than or equal to 1 pixel

Original

Filtered

Point- vs. Area-Sampled

Checkerboard sequence by Tom Duff
Antialiasing

Jaggies

Prefilter

Antialiasing vs. Blurred Aliases

Blurred Jaggies

Prefilter
Things to Remember

Signal processing
- Frequency domain vs. spatial domain
- Filters in the frequency domain
- Filters in the spatial domain = convolution

Sampling and aliasing
- Image generation involves sampling
  - May also sample geometry, motion, ...
- Nyquist frequency is ½ the sampling rate
- Frequencies above the Nyquist frequency appear as other frequencies – aliases
- Antialiasing – Filter before sampling

Extra Slides

Supersampling
Supersampling

Approximate a box filter by taking more samples and averaging them together

4 x 4 supersampling

Point-sampling vs. Super-sampling

Point 4x4 Super-sampled

Checkerboard sequence by Tom Duff
Area-Sampling vs. Super-sampling

Exact Area 4x4 Super-sampled