Sampling and Fourier Theory

CS 148, Summer 2012
Introduction to Computer Graphics and Imaging

Justin Solomon
Important Issues We’ve Ignored

Jaggies
Important Issues We’ve Ignored

Sampling issues
Important Issues We’ve Ignored

Small image

Large texture

Texture lookup spacing
Important Issues We’ve Ignored

Moiré patterns
More Curiosities

Wagon wheel effect

http://www.youtube.com/watch?v=jHSgJGkEOmA
More Curiosities

Original

Photoshop “Median”

Kass and Solomon, “Smoothed Local Histogram Filters,” SIGGRAPH 2010

Broken image filtering
What’s Broken?

Continuous world

Discrete sensors and displays
What’s Broken?

What goes in each pixel?
Our Strategy

Employ mathematical and perceptual tricks to compensate for our inexact sampling.
Employ mathematical and perceptual tricks to compensate for our inexact sampling.
WARNING

THE OPPOSITE OF RIGOROUS
Our Model

$y = f(x)$
Our Model

1D signals
Sampling

Multiply by “impulse train”
Multiply by “impulse train”
Sampling

Multiply by "impulse train"
Sampling

Reconstruct
Sampling

Original signal

Reconstruct
Sampling

High frequencies lost!

Reconstruct
Four samples won’t suffice!
More Obvious Example

Four samples won’t suffice!
Worst Possible Example

http://www.svi.nl/wikiimg/Aliasing-plot.png
The Issue at Hand

Too many wiggles between consecutive samples.
The Issue at Hand

Too many wiggles between consecutive samples.

Need a way to formalize
THEOREM(-ISH).
Any function can be written as a combination of simple wiggles.
Fourier Transform
Fourier Transform

\[ f(x) \]

\[ \mathcal{F}[f](\xi) \]

\[ \mathcal{F}[\cdot] \]

\[ \mathcal{F}^{-1}[\cdot] \]
Fourier Transform

\[ f(x) \]

\[ \mathcal{F}[\cdot] \]

\[ \mathcal{F}[f](\xi) \]

\[ \mathcal{F}^{-1}[\cdot] \]

Spatial

Frequency
Frequency Domain Filtering

2D Fourier transform
Frequency Domain Filtering

High pass filter
Frequency Domain Filtering

Low pass filter
Frequency Domain Filtering

Band pass filter
Evaluating Fourier Transform

$f(x)$

Simplest possible thing
Evaluating Fourier Transform

\( \sin(2\pi \xi x) \)

Simplest possible thing
Evaluating Fourier Transform

Multiply and integrate.
The value of this integral for all $\xi$. 
The value of this integral for all $\xi$.

Invertible if you do this for cosine and sine!

The value of this integral for all $\xi$. 
An Unfortunate Phenomenon

$f(x)$

$\mathcal{F}[f](\xi)$
An Unfortunate Phenomenon

Sampled function

Frequencies repeat
An Unfortunate Phenomenon

Sampled function

More samples...

Frequencies repeat

Fewer repeats!
What’s Going On?

One interval fits multiple frequencies!
String Harmonics
Why is this a problem?

Can “confuse” frequencies
Can “confuse” frequencies

Irrecoverable mixing!
When Isn’t This a Problem?

\[ \mathcal{F}[f](\xi) \]

Everywhere else zero
When Isn’t This a Problem?

\[ \mathcal{F}[f](\xi) \]

Bandwidth
When Isn’t This a Problem?

No interference
Nyquist rate
[nahy-kwist reyt]:
The lowest alias-free sample rate; two times the bandwidth of a band-limited signal.
Perfect Reconstruction

Box function

Multiply in frequency space
Perfect Reconstruction

What happened in the spatial domain?

Box function

Multiply in frequency space
Convolution

Replace function with weighted average
Convolutions

More when we discuss image processing.

Replace function with weighted average
Convolution Theorem

Multiplication in frequency domain is convolution in spatial domain.
Sinc

Fourier transform of box

https://upload.wikimedia.org/wikipedia/commons/thumb/2/2b/Sinc_Filter.svg/700px-Sinc_Filter.svg.png
A Perfect Story

Spatial domain

Frequency domain

Input function
1. Sample the function

**Spatial domain**

**Frequency domain**
A Perfect Story

Spatial domain

Frequency domain

2. Reconstruct using sinc
2. Reconstruct using sinc
Practical signals cannot have finite bandwidth.
Practical signals cannot have finite bandwidth.

“Heisenberg Uncertainty Principle”
Back to Reality

Negative lobe!
Back to Reality

Use positive approximation
Frequency paradigm isn’t perfect

Sharp edges need special treatment!

http://vcg.isti.cnr.it/~corsini/publications/thumbs/tn_fast-bilateral-filter.png
Sample above the Nyquist rate.
Sampling Below the Nyquist Rate

\[ y = \text{wavrecord}(750 \times 20, 750); \]
\[ \text{wavplay}(y \times 300, 750) \]
Practical Conclusions

Reconstruct using filters that look like sinc.
Sampling and Fourier Theory

CS 148, Summer 2012
Introduction to Computer Graphics and Imaging
Justin Solomon