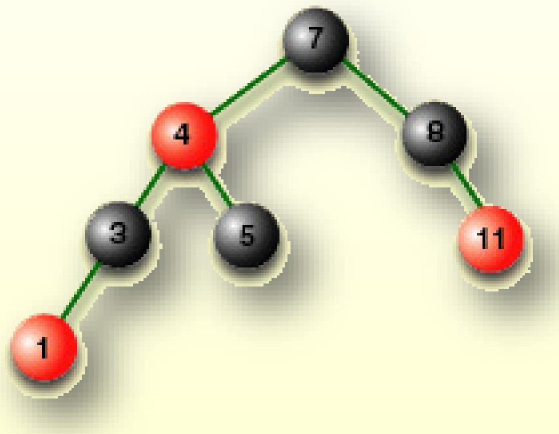
CS161:
Design and Analysis of Algorithms



Lecture 3
Leonidas Guibas

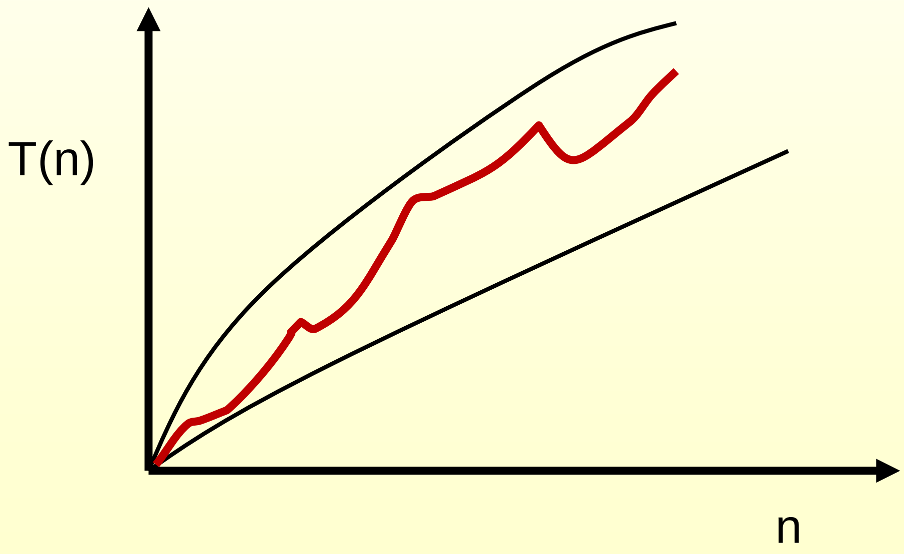
Outline

- Review of last lecture (asymptotic notations, recurrence relations)
- Key Topic: Solving Recurrences
  - using recursion trees (or iteration)
  - the master method
  - the substitution method

Slides modified from
- http://www.cs.virginia.edu/~luebke/cs332/

Asymptotic Bounds on Algorithm Performance

- Worst-case and average-case are difficult to analyze precisely -- the details can be very complicated



It may be easier to talk about upper and lower bounds on the function T(n).

Review: Asymptotic Notations

- O: Big-Oh
- Ω: Big-Omega
- Θ: Theta
- o: Small-oh
- ω: Small-omega

Big O

- Informally, O(g(n)) is the set of all functions with a smaller or same order of growth as g(n), within a constant multiple

Intuitively, O is like ≤

an upper bound notation

- If we say f(n) is in O(g(n)), this means that g(n) is an asymptotic upper bound on f(n)
  - Formally. $\exists$ C (>0) & n_0, f(n) ≤ Cg(n) for $\forall$ n >= n_0

g(n) should be a “simple” function

Big Ω

- Informally, Ω(g(n)) is the set of all functions with a larger or same order of growth as g(n), within a constant multiple

- f(n) ∈ Ω(g(n)) means g(n) is an asymptotic lower bound of f(n)
  - Intuitively, it is like f(n) ≥ g(n)

Intuitively, Ω is like ≥

a lower bound notation

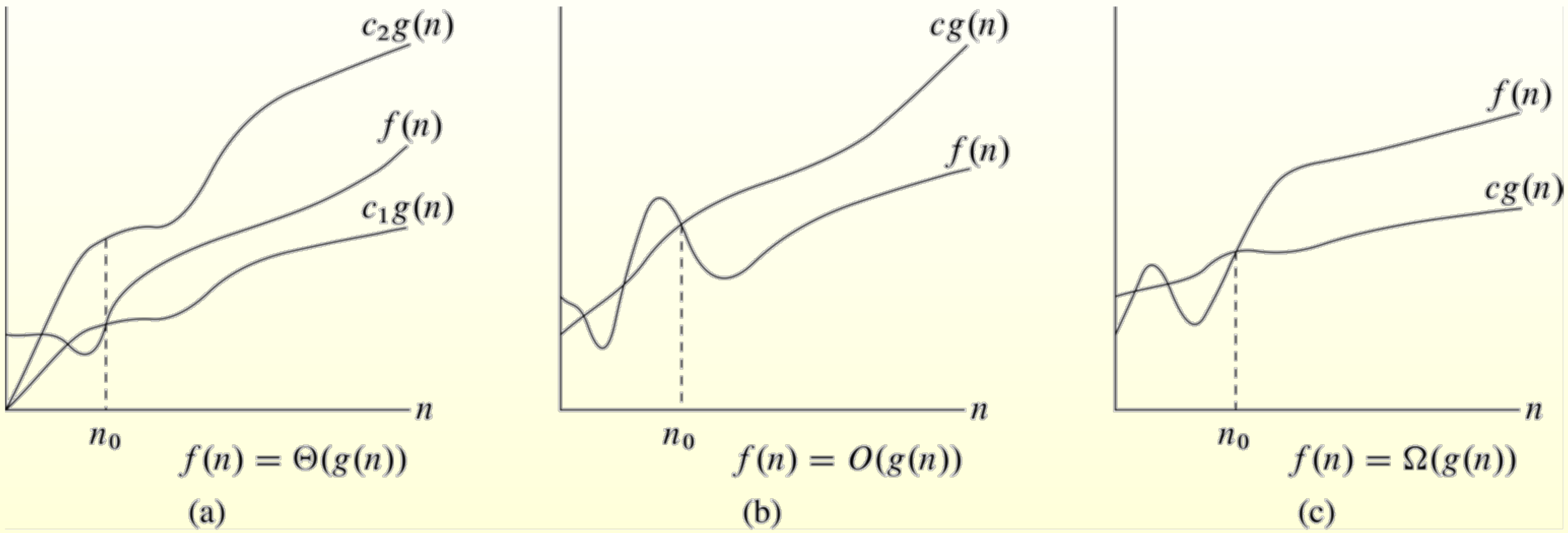
Theta (Θ): Θ = O and Ω

- Informally, Θ(g(n)) is the set of all functions with the same order of growth as g(n), within a constant multiple

Θ is like =

- f(n) $\in$ Θ(g(n)) means g(n) is an asymptotically tight bound on f(n)
  - Intuitively, it is like f(n) = g(n)

O, Ω, and Θ



The definitions imply a constant n_0 *beyond which* they are satisfied. We do not care about small values of n.

Algorithm Efficiency via Recurrences

$$T(n) = T(n-1) + 1$$

$$T(n) = T(n-1) + n$$

$$T(n) = T(n/2) + 1$$

$$T(n) = 2T(n/2) + 1$$

Challenge: how to solve the recurrence to get a tight bound, e.g. $T(n) = \Theta(n^2)$ or $T(n) = \Theta(n \lg n)$, or at least an upper bound such as $T(n) = O(n^2)$?

Solving Recurrences

- The running time of many algorithms can be expressed in one of the following two recursive forms

$$T(n) = aT(n-b) + f(n)$$

or

$$T(n) = aT(n/b) + f(n)$$

Both can be hard to solve. We focus on relatively easy ones, which you will encounter frequently in many real algorithms (and exams…)

Solving Recurrences

1. Recursion tree / iteration method
2. Master method
3. Substitution method

The Recursion Tree Method

Review: Back to MergeSort

$T(n)$	**MERGE-SORT** $A[1 \,..\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
$\Theta(n)$	***3. "Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Recurrence for MergeSort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We saw that the cost of the Merge step is $\Theta(n)$.
- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion Tree

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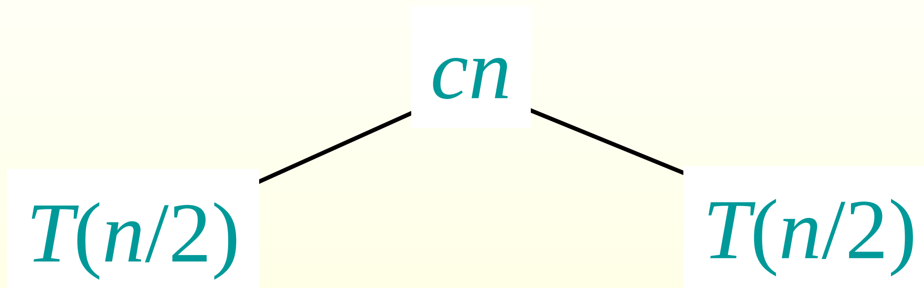

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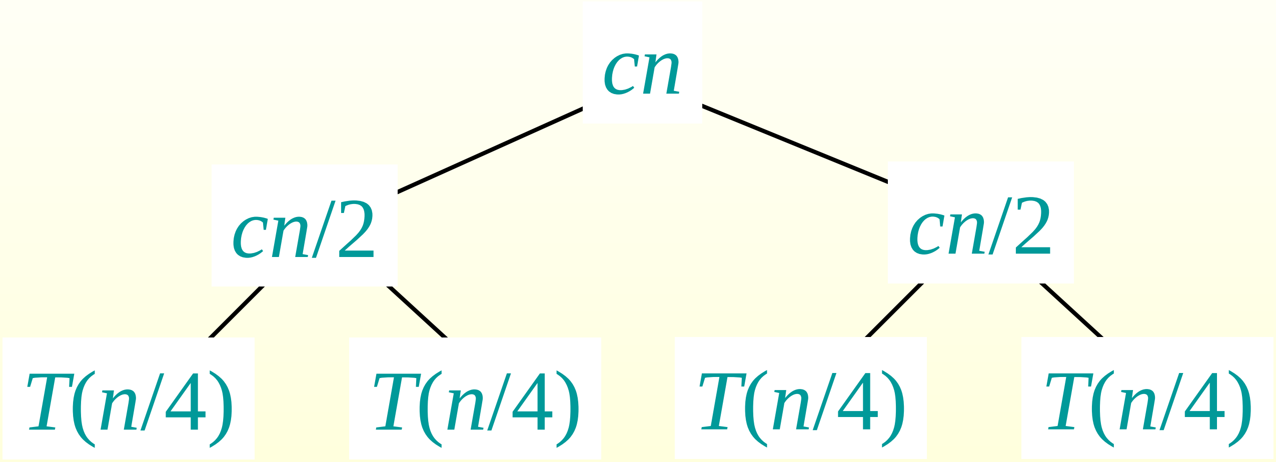

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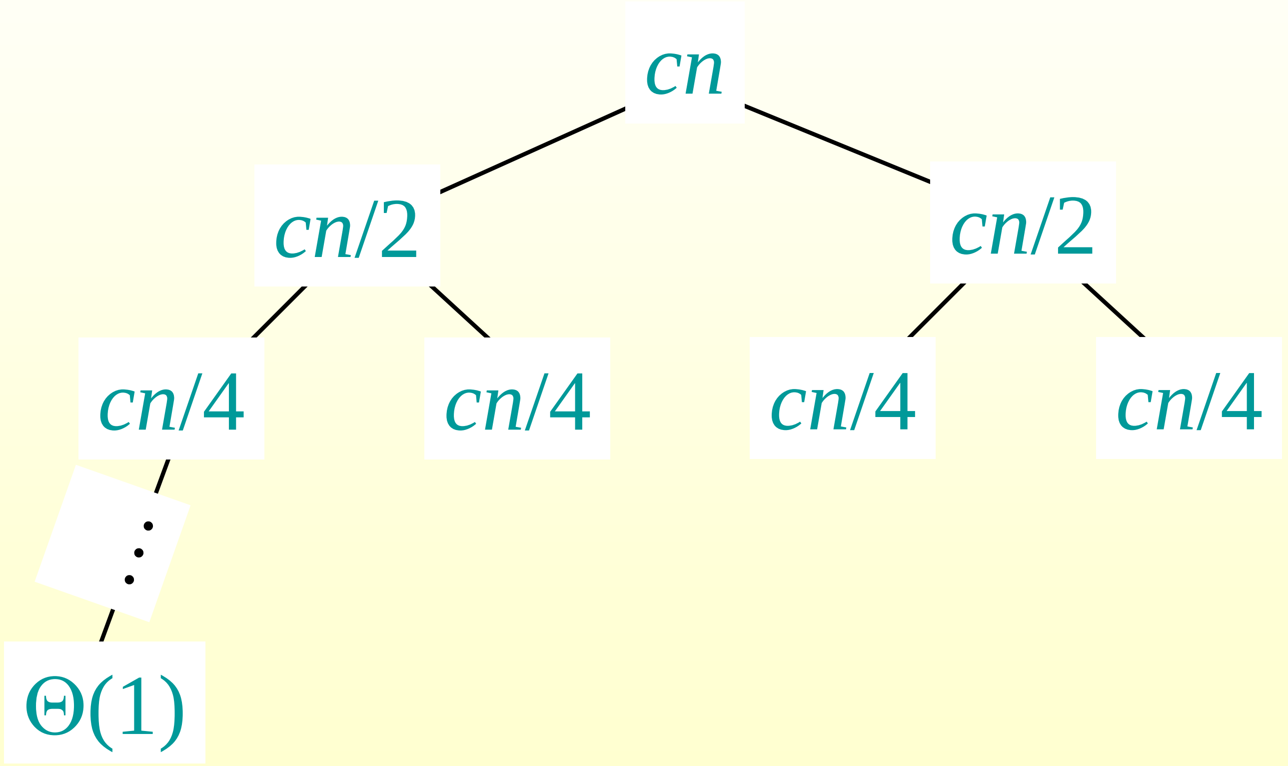

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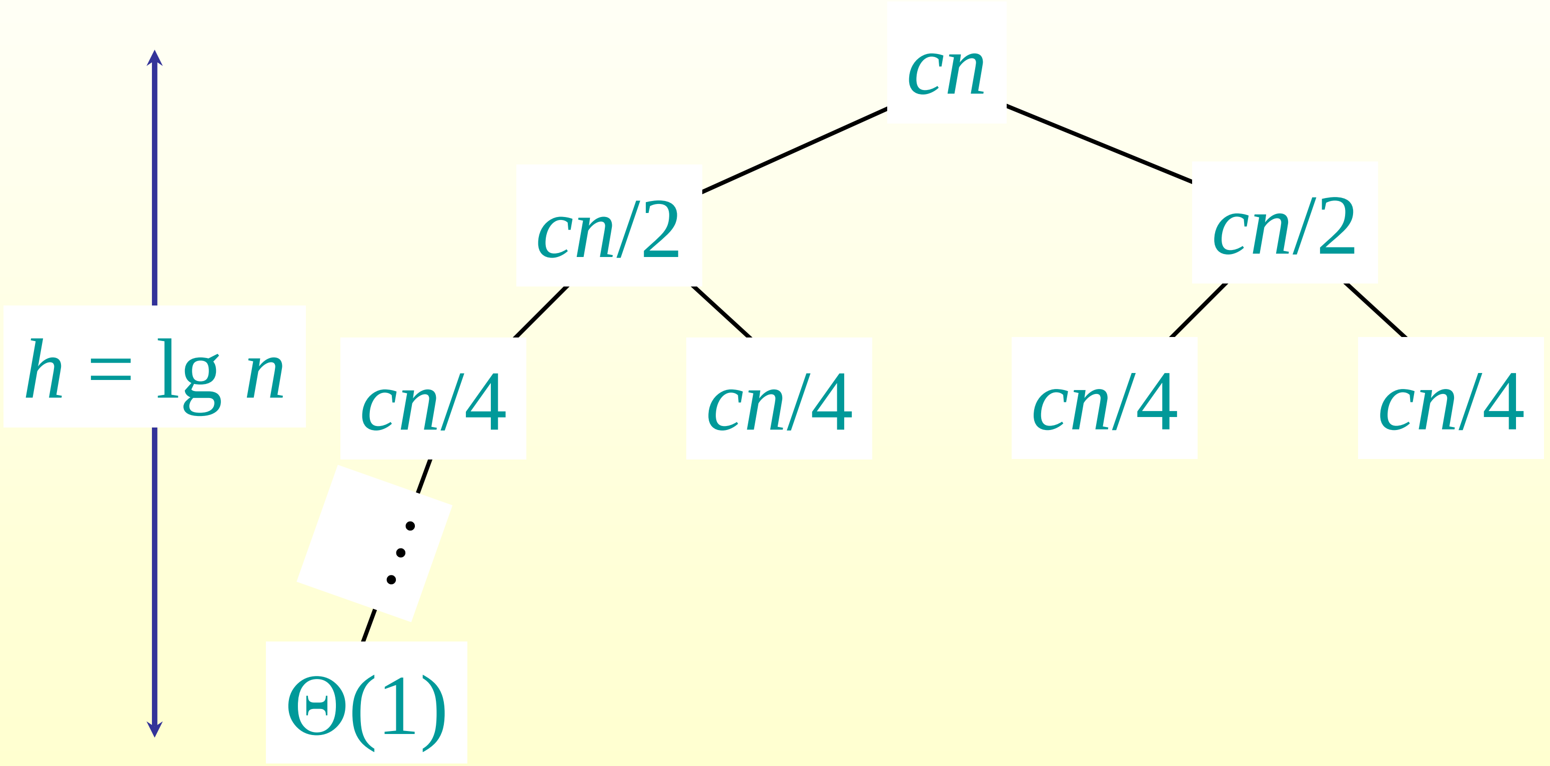

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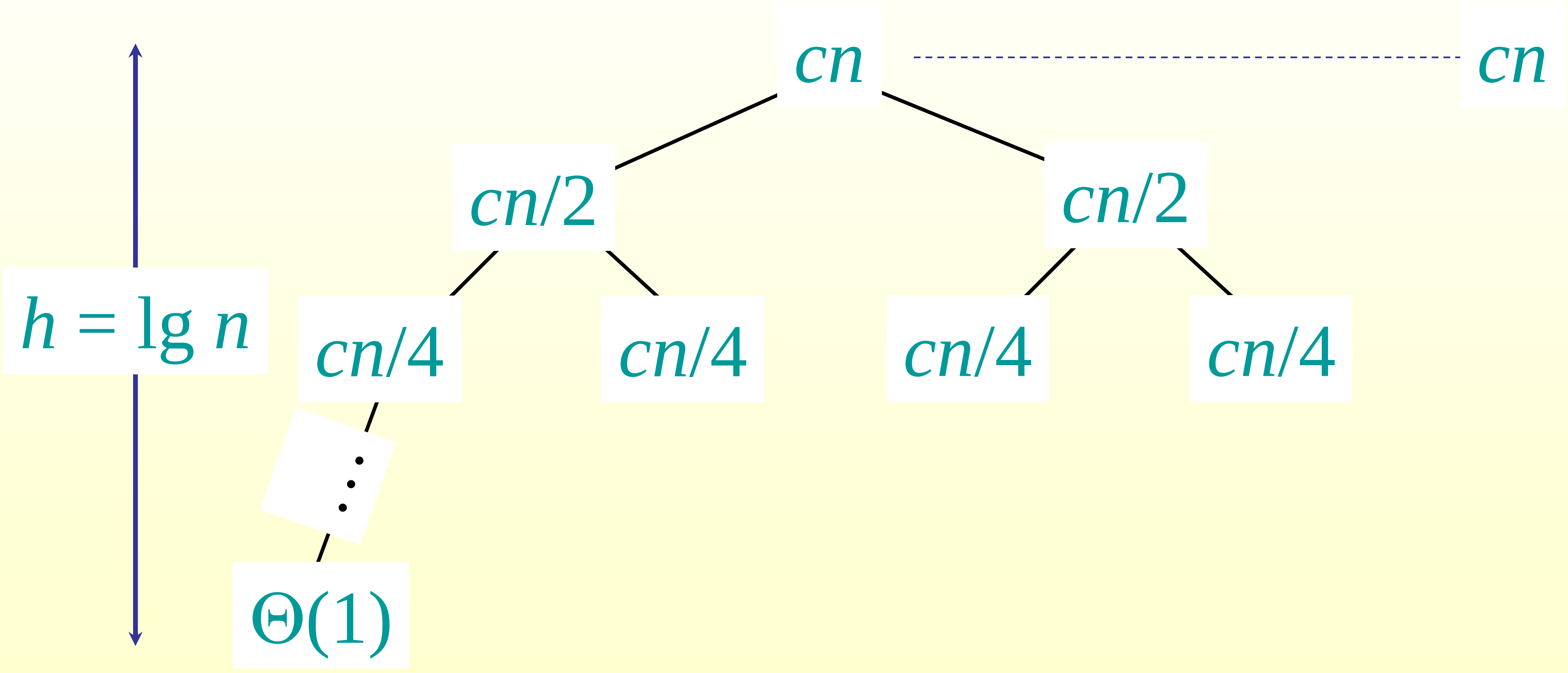

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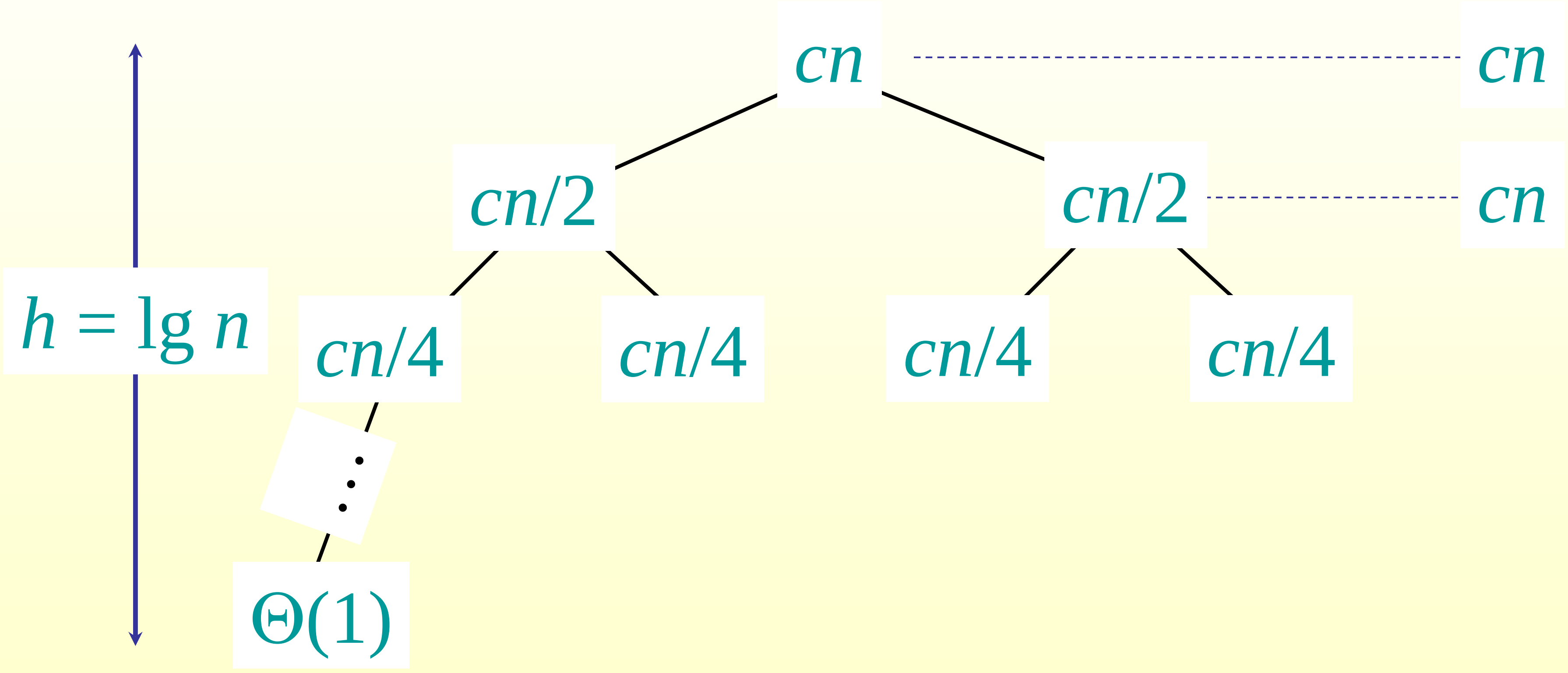

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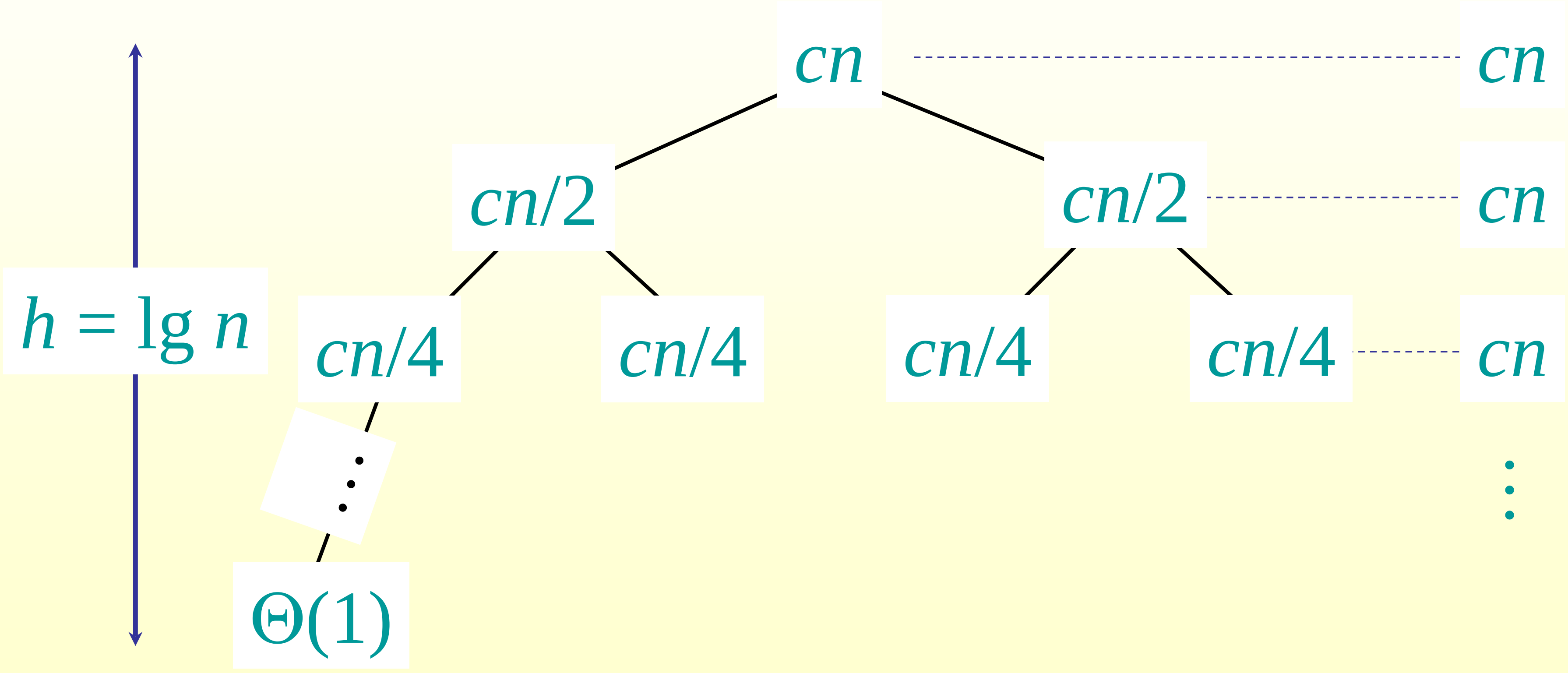

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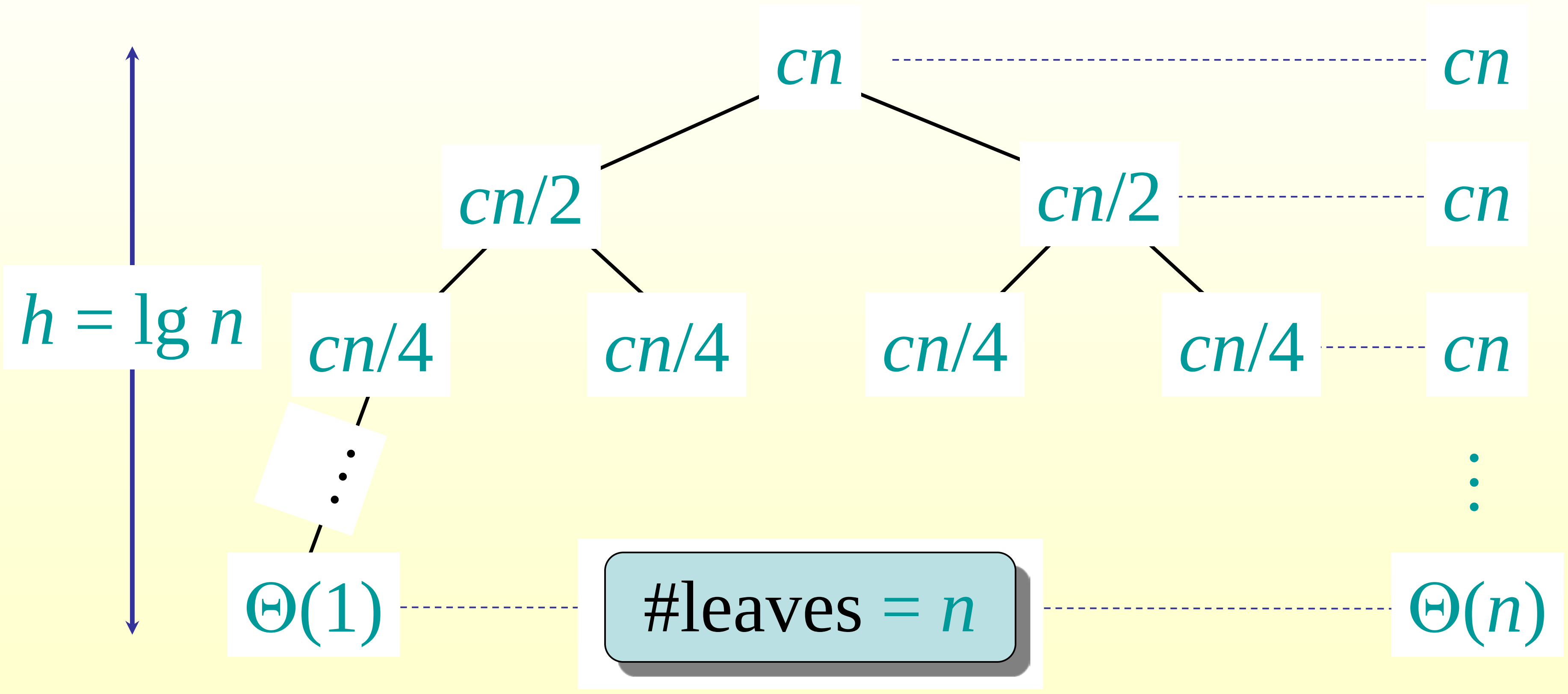

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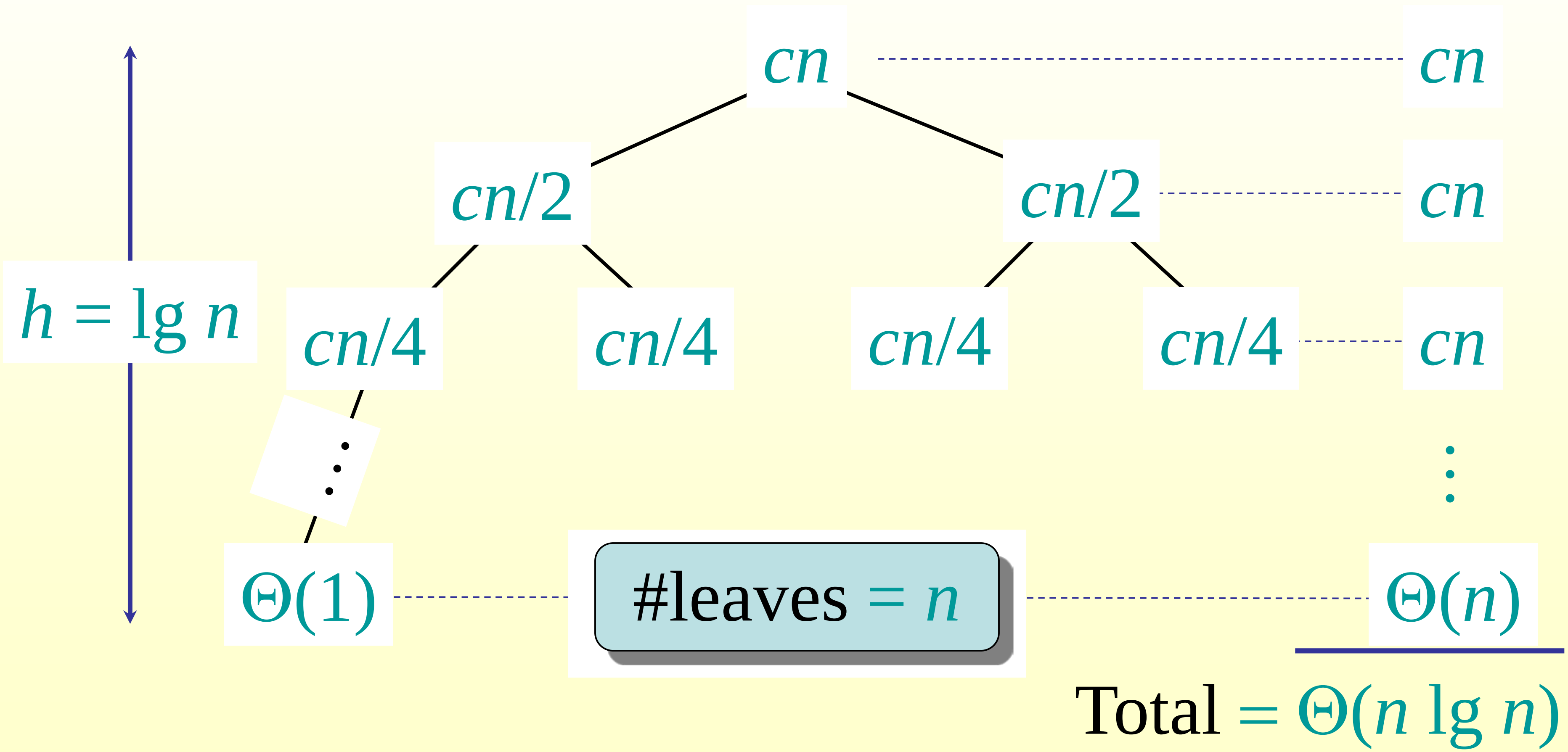


Another Example

- How many multiplications do we need to compute 3^{16}?

$3^{16} = 3 \times 3 \times 3 \; .... \times 3$

Answer: 15

$3^{16} = 3^8 \times 3^8$

$3^8 = 3^4 \times 3^4$

$3^4 = 3^2 \times 3^2$

$3^2 = 3 \times 3$

Answer: 4

Pseudocode for Recursion

```
int pow (b, n)    // compute b^n
   m = n >> 1;    // divide by 2
   p = pow (b, m);
   p = p * p;
   if (n % 2)
       return p * b;
   else
       return p;
```

Pseudocode Variations

```
int pow (b, n)
    m = n >> 1;
    p = pow (b, m);
    p = p * p;
    if (n % 2)
        return p * b;
    else
        return p;
```

```
int pow (b, n)
    m = n >> 1;
    p = pow(b,m) * pow(b,m);
    if (n % 2)
        return p * b;
    else
        return p;
```

Recurrence for Computing Power

Alg1

```
int pow (b, n)
   m = n >> 1;
   p = pow (b, m);
   p = p * p;
   if (n % 2)
      return p * b;
   else
      return p;
```

$T(n) = T(n/2)+\Theta(1)$

Alg2

```
int pow (b, n)
   m = n >> 1;
   p=pow(b,m)*pow(b,m);
   if (n % 2)
      return p * b;
   else
      return p;
```

$T(n) = 2T(n/2)+\Theta(1)$

Which algorithm is more efficient asymptotically?

Time Complexity for Alg1

Solve $T(n) = T(n/2) + 1$

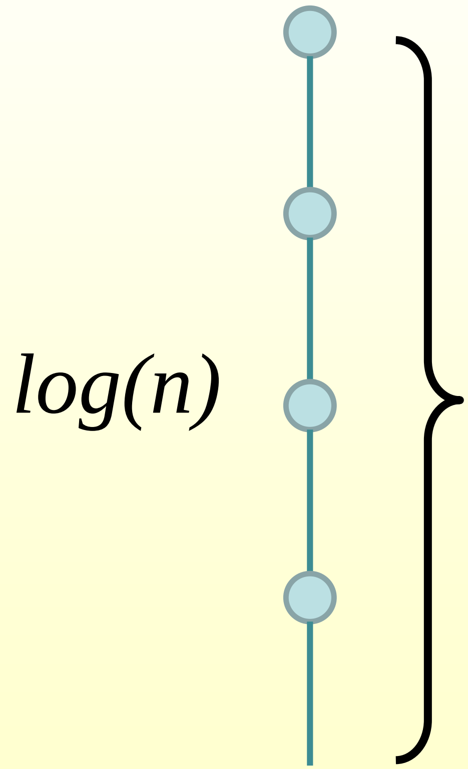
- $T(n) = T(n/2) + 1$

$$= T(n/4) + 1 + 1$$

$$= T(n/8) + 1 + 1 + 1$$

$$= T(1) + \underbrace{1 + 1 + \ldots + 1}_{log(n)}$$

$$= \Theta\,(log(n))$$



Iteration method

Time Complexity for Alg2

Solve $T(n) = 2T(n/2) + 1$.

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$$T(n)$$

Time Complexity for Alg2

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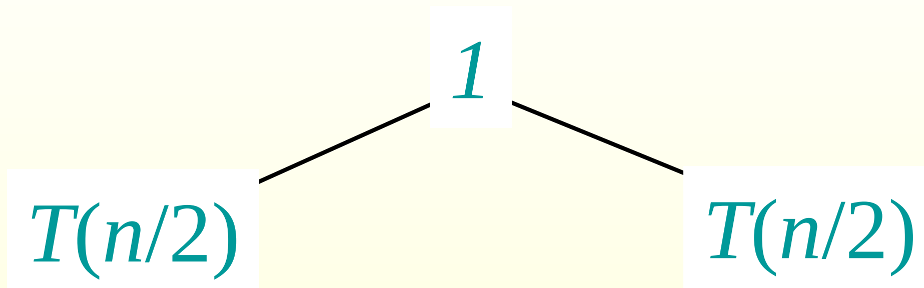
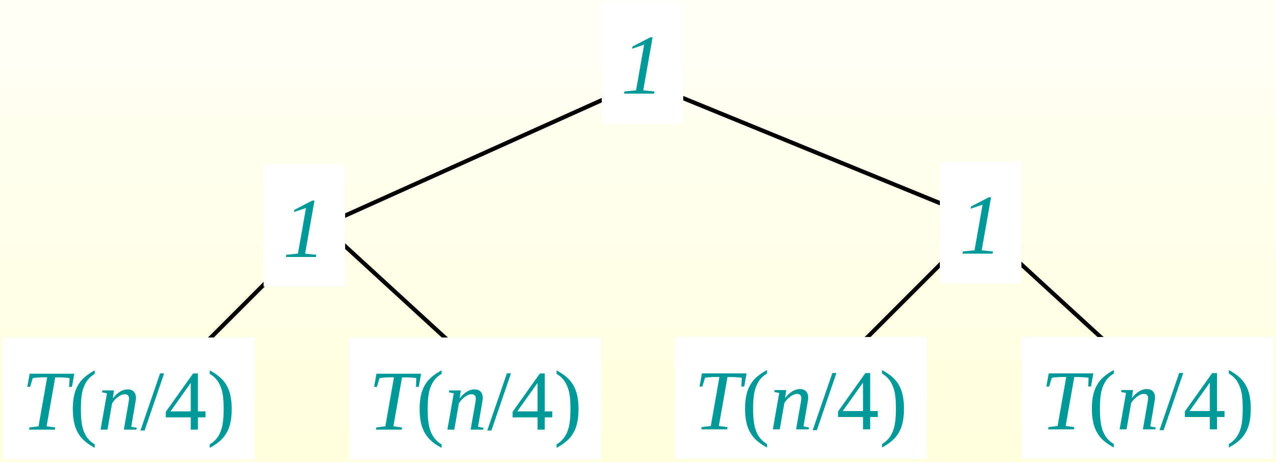


Time Complexity for Alg2

Solve $T(n) = 2T(n/2) + 1$.



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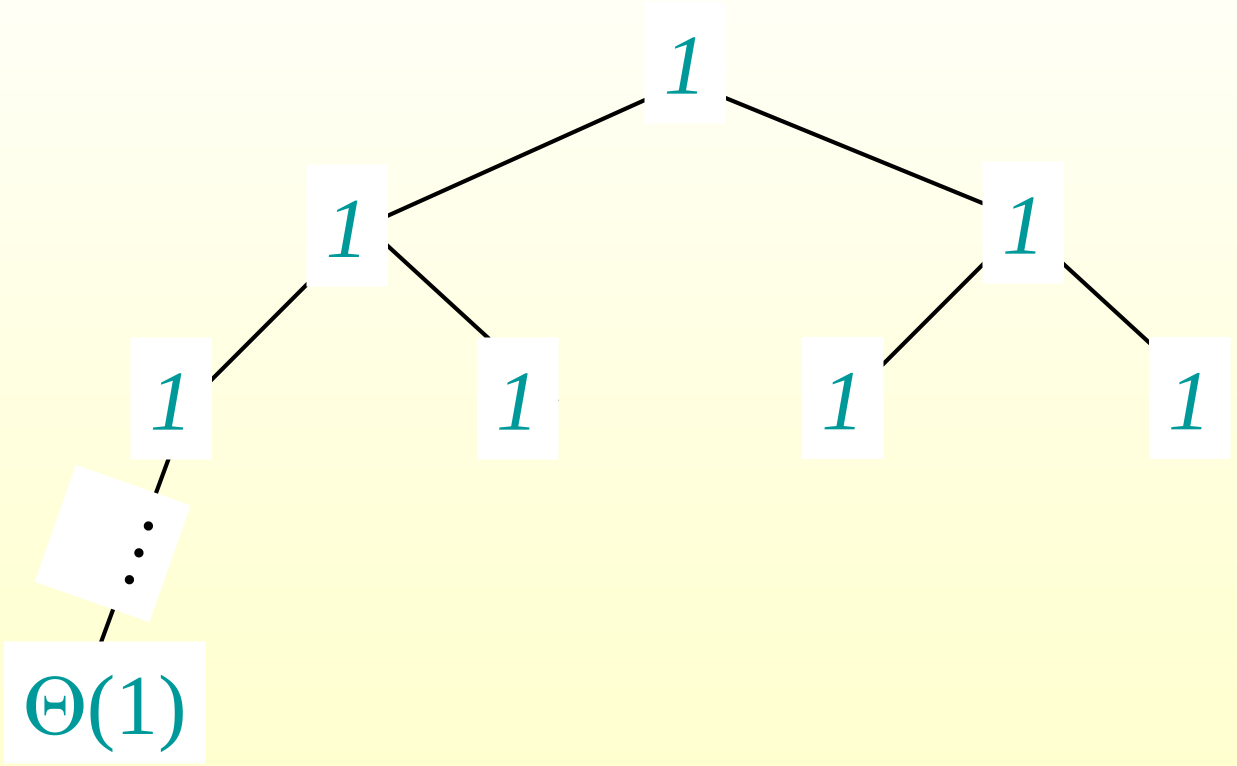

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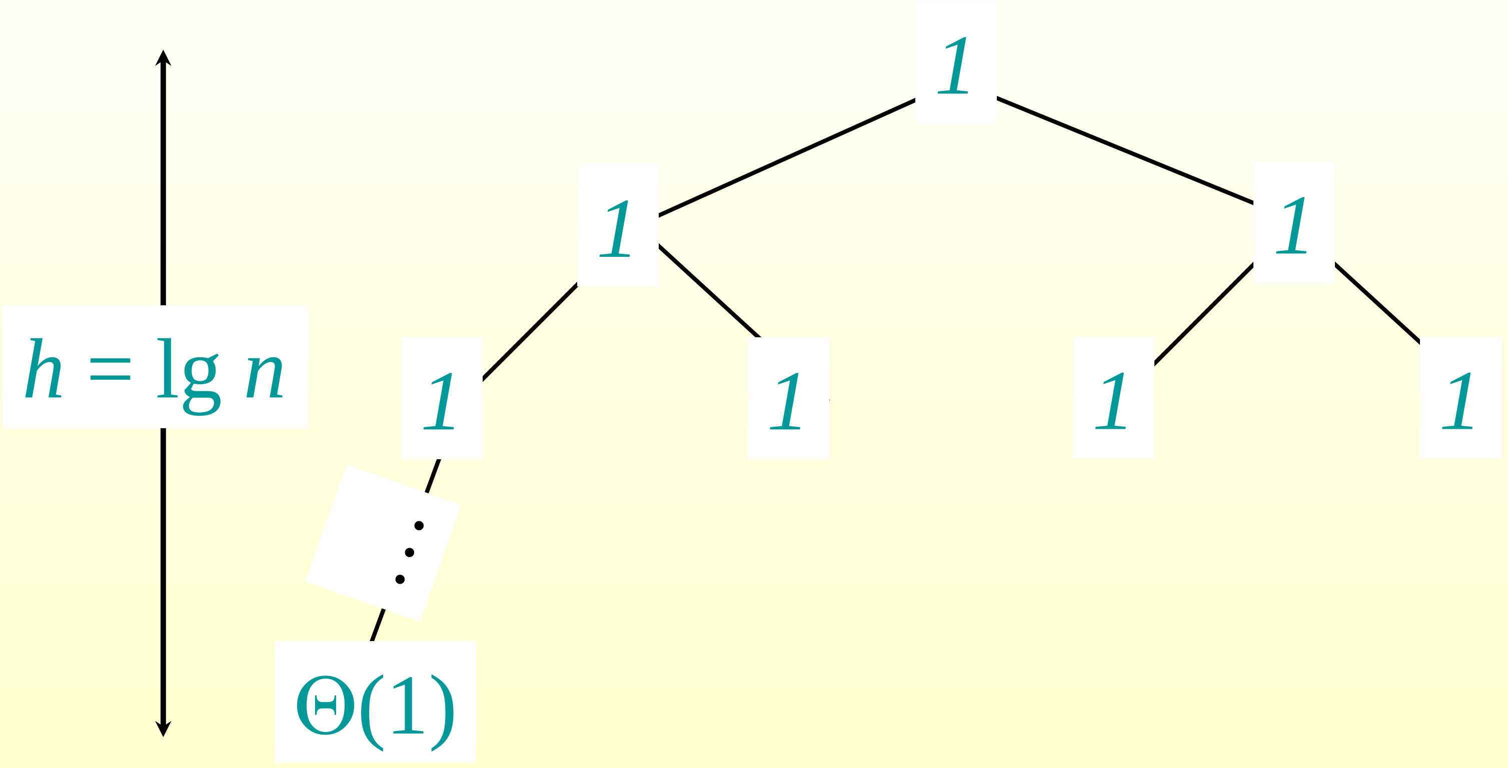

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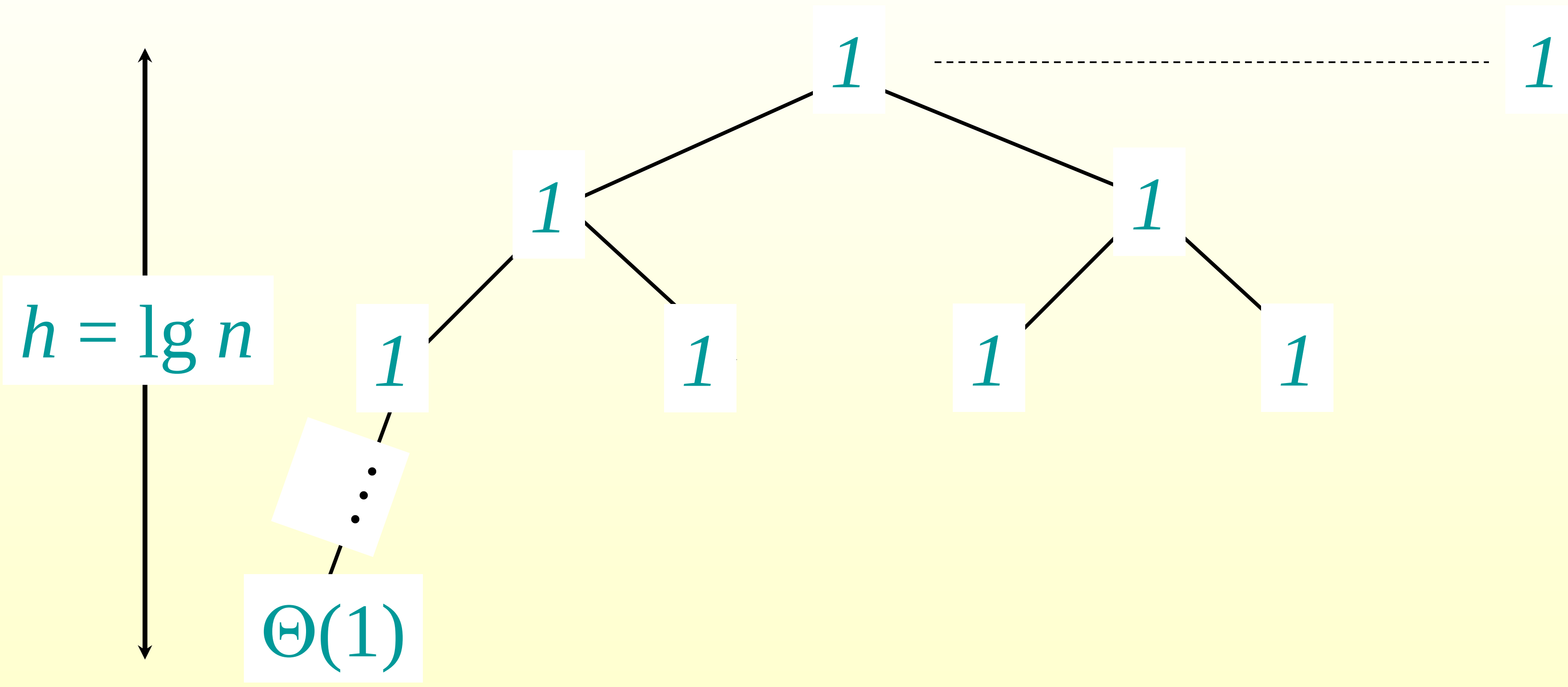

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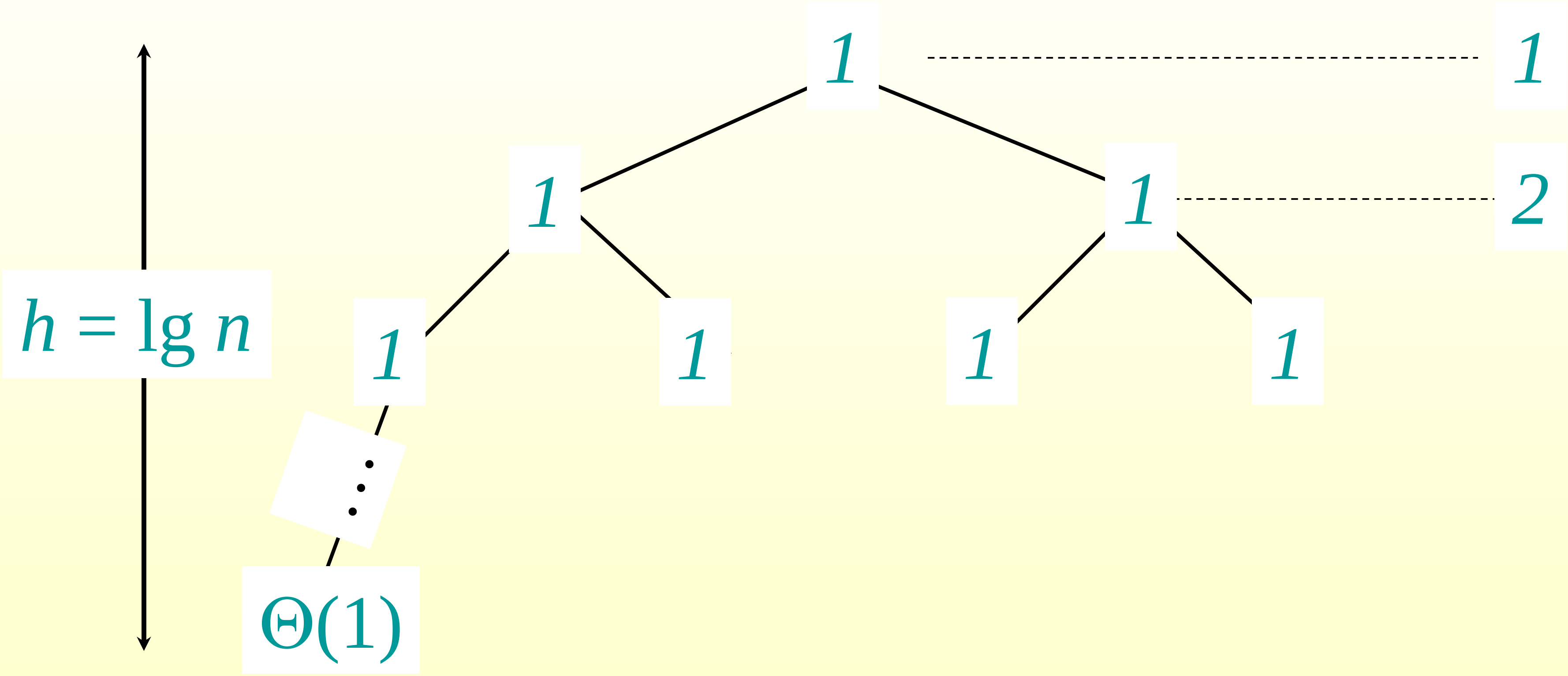

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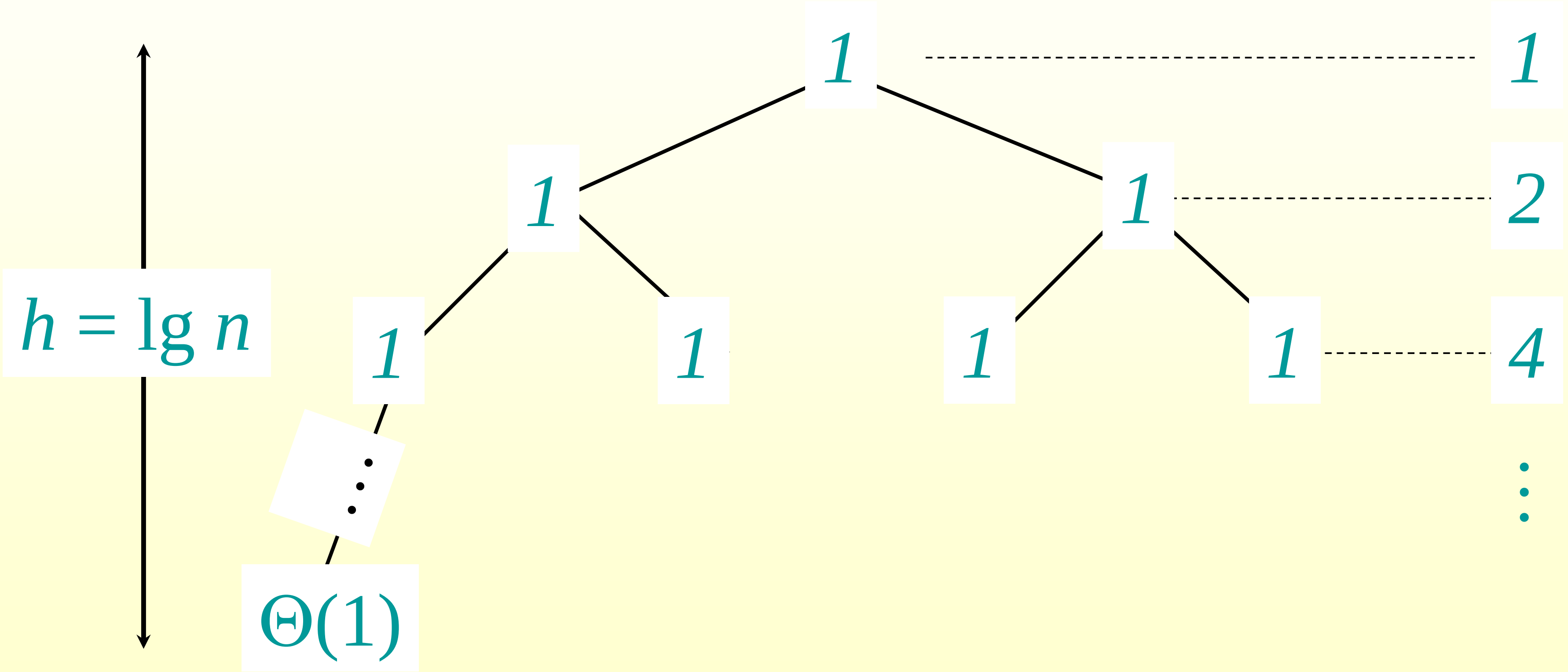

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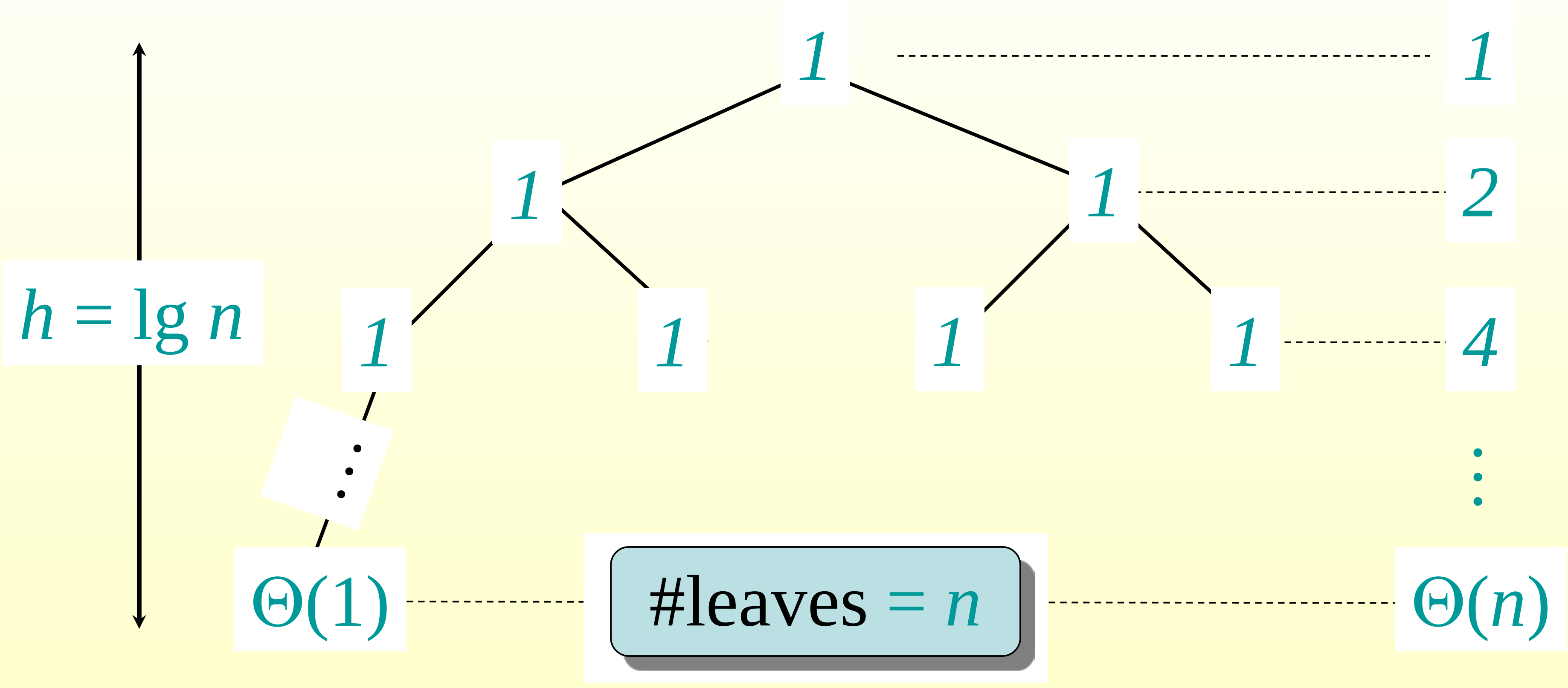

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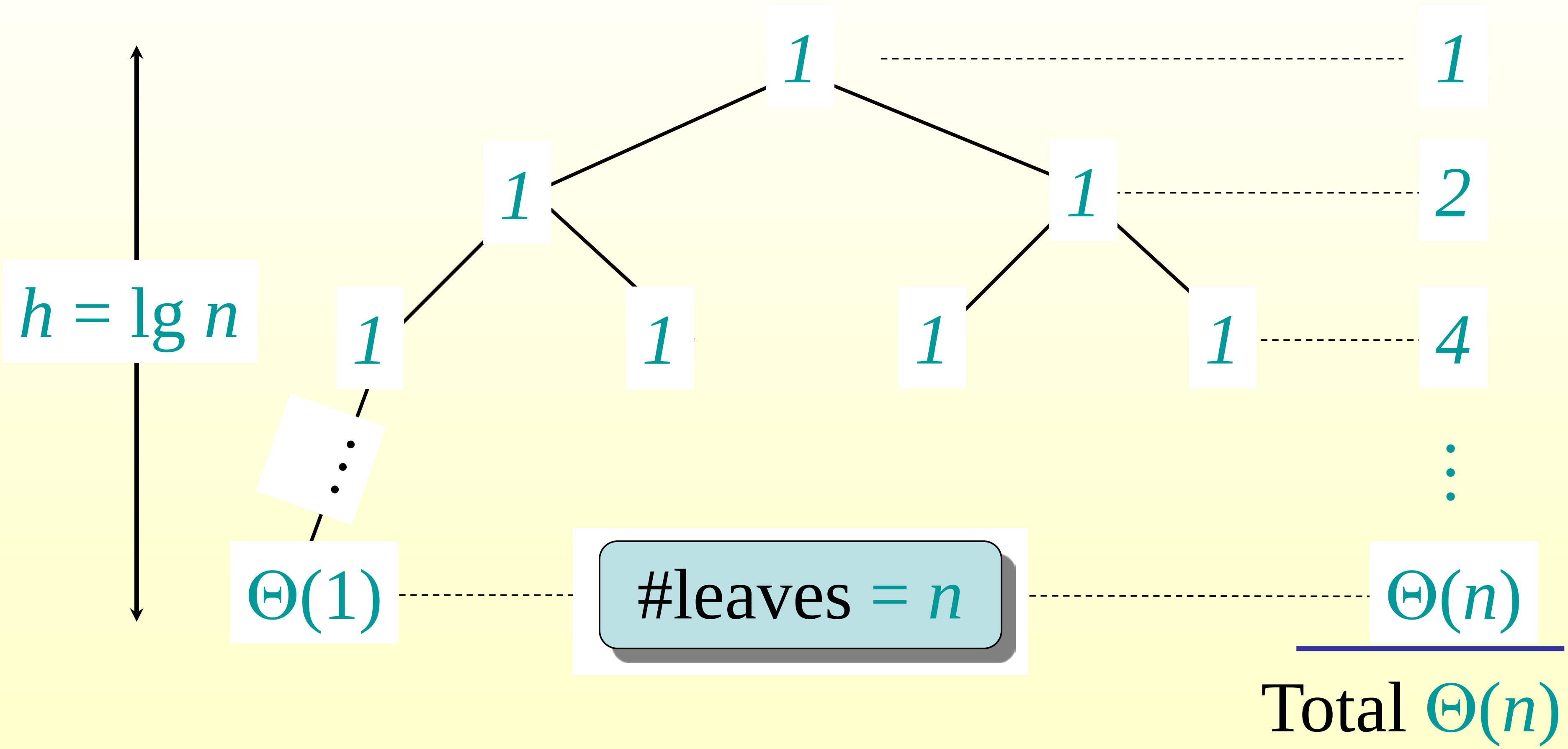


$1 + 2 + 4 + \ldots + 2^k = 2^{k+1} - 1$

More Iteration Method Examples

- $T(n) = T(n-1) + 1$

$$= T(n-2) + 1 + 1$$

$$= T(n-3) + 1 + 1 + 1$$

$$= T(1) + \underbrace{1 + 1 + \ldots + 1}_{n - 1}$$

$$= \Theta(n)$$

More Iteration Method Examples

- $T(n) = T(n-1) + n$

  $= T(n-2) + (n-1) + n$

  $= T(n-3) + (n-2) + (n-1) + n$

  $= T(1) + 2 + 3 + \ldots + n$

  $= \Theta(n^2)$

Saw the same sum in InsertionSort

3-Way-MergeSort

```
3-way-merge-sort (A[1..n])
    If (n <= 1) return;
    3-way-merge-sort(A[1..n/3]);
    3-way-merge-sort(A[n/3+1..2n/3]);
    3-way-merge-sort(A[2n/3+1.. n]);
    Merge A[1..n/3] and A[n/3+1..2n/3];
    Merge A[1..2n/3] and A[2n/3+1..n];
```

- Is this algorithm correct?
- What's the recurrence function for the running time?
- What does the recurrence function solve to?

Unbalanced-MergeSort

```
ub-merge-sort (A[1..n])
    if (n<=1) return;
    ub-merge-sort(A[1..n/3]);
    ub-merge-sort(A[n/3+1.. n]);
    Merge A[1.. n/3] and A[n/3+1..n].
```

- Is this algorithm correct?
- What’s the recurrence function for the running time?
- What does the recurrence function solve to?

More Recursion Tree Examples

- $T(n) = 3T(n/3) + n$ [3-Way MergeSort]
- $T(n) = T(n/3) + T(2n/3) + n$ [ub-MergeSort]
- $T(n) = 3T(n/4) + n$
- $T(n) = 3T(n/4) + n^2$

The Master Method

The Master Method

The master method applies to recurrences of the form

$T(n) = a\,T(n/b) + f(n)$,

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

1. ***Divide*** the problem into a subproblems, **each** of size n/b
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions

   Divide + combine takes $f(n)$ time.

Master Theorem

$$T(n) = a\,T(n/b) + f(n)$$

Key: compare $f(n)$ with $n^{\log_b a}$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$

.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $\underline{a\,f(n/b) \le c\,f(n)}$

Regularity Condition

$\Rightarrow T(n) = \Theta(f(n))$.

$$n^{\log_b a} = a^{\log_b n}$$

Case 1

$f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

Alternatively: $n^{\log_b a} / f(n) = \Omega(n^{\varepsilon})$

Intuition: $f(n)$ grows polynomially slower than $n^{\log_b a}$

Or: $n^{\log_b a}$ dominates $f(n)$ by an n^{ε} factor for some $\varepsilon > 0$

Solution: $T(n) = \Theta(n^{\log_b a})$

$T(n) = 4T(n/2) + n$
$b = 2, a = 4, f(n) = n$
$\log_2 4 = 2$
$f(n) = n = O(n^{2-\varepsilon})$, or
$n^2 / n = n^1 = \Omega(n^{\varepsilon})$, for $\varepsilon = 1$
$\therefore T(n) = \Theta(n^2)$

$T(n) = 2T(n/2) + n/\log n$
$b = 2, a = 2, f(n) = n / \log n$
$\log_2 2 = 1$
$f(n) = n/\log n \notin O(n^{1-\varepsilon})$, or
$n^1 / f(n) = \log n \notin \Omega(n^{\varepsilon})$, for any $\varepsilon > 0$
$\therefore$ CASE 1 does not apply

Case 2

$f(n) = \Theta\ (n^{\log_b a})$.

Intuition: $f(n)$ and $n^{\log_b a}$ have the same asymptotic order.

Solution: $T(n) = \Theta(n^{\log_b a} \log n)$

e.g. $T(n) = T(n/2) + 1$ $\qquad \log_b a = 0$

$T(n) = 2\ T(n/2) + n$ $\qquad \log_b a = 1$

$T(n) = 4T(n/2) + n^2$ $\qquad \log_b a = 2$

$T(n) = 8T(n/2) + n^3$ $\qquad \log_b a = 3$

Case 3

$f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.

Alternatively: $f(n) / n^{\log_b a} = \Omega(n^{\varepsilon})$

Intuition: $f(n)$ grows polynomially faster than $n^{\log_b a}$

Or: $f(n)$ dominates $n^{\log_b a}$ by an n^{ε} factor for some $\varepsilon > 0$

Solution: $T(n) = \Theta(f(n))$

$T(n) = T(n/2) + n$

$b = 2, a = 1, f(n) = n$

$n^{\log_2 1} = n^0 = 1$

$f(n) = n = \Omega(n^{0+\varepsilon})$, or

$n / 1 = n = \Omega(n^{\varepsilon})$

$\therefore T(n) = \Theta(n)$

$T(n) = T(n/2) + \log n$

$b = 2, a = 1, f(n) = \log n$

$n^{\log_2 1} = n^0 = 1$

$f(n) = \log n \notin \Omega(n^{0+\varepsilon})$, or

$f(n) / n^{\log_2 1} = \log n \notin \Omega(n^{\varepsilon})$

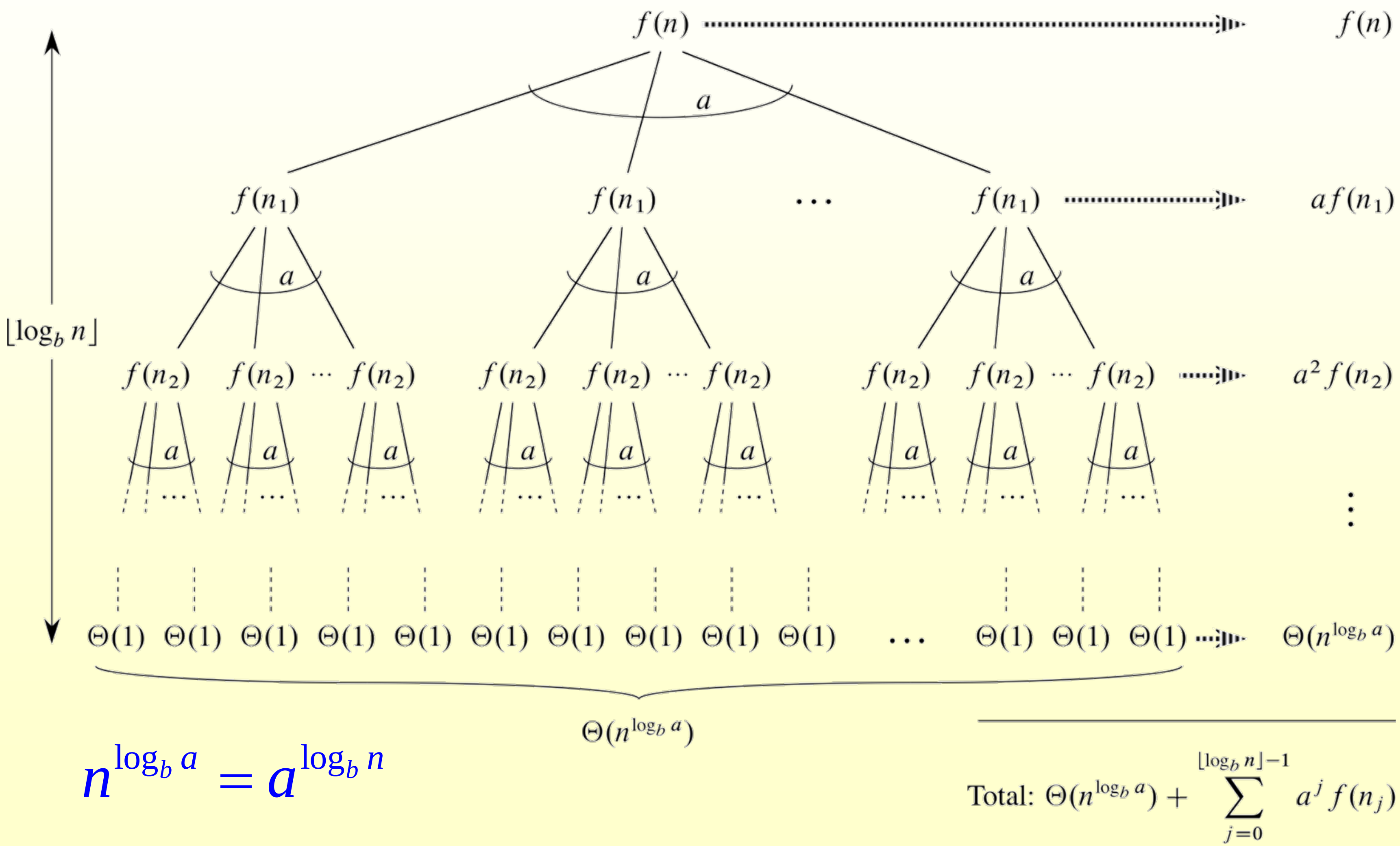
$\therefore$ *CASE 3 does not apply*

Regularity Condition

- $af(n/b) \le cf(n)$ for some $c < 1$ and all sufficiently large n
- This is needed for the master method to be mathematically correct.
  - to deal with some non-converging functions such as sine or cosine functions
- For most *f(n)* you'll see (e.g., polynomial, logarithm, exponential), you can safely ignore this condition, because it is implied by the first condition $f(n) = \Omega(n^{\log_b a + \varepsilon})$

Proof by Picture

$n_i = n / b^i$



$$n^{\log_b a} = a^{\log_b n}$$

Examples

$T(n) = 4T(n/2) + n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

$T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

CASE 2: $f(n) = \Theta(n^2).$

$\therefore\ T(n) = \Theta(n^2 \log n).$

Examples

$T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore T(n) = \Theta(n^3)$.

$T(n) = 4T(n/2) + n^2/\log n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\log n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\log n)$.

Examples

$T(n) = 4T(n/2) + n^{2.5}$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^{2.5}$.

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 0.5$

and $4(n/2)^{2.5} \le cn^{2.5}$ (reg. cond.) for $c = 0.75$.

$\therefore\ T(n) = \Theta(n^{2.5})$.

$T(n) = 4T(n/2) + n^2 \log n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2 \log n$.

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\log n)$.

How do I know which case to use? Do I need to try all three cases one by one?

- Compare $f(n)$ with $n^{\log_b a}$

check if $n^{\log_b a} / f(n) \in \Omega(n^{\varepsilon})$

- $f(n) \in \begin{cases} o(n^{\log_b a}) & \text{Possible CASE 1} \\ \Theta(n^{\log_b a}) & \text{CASE 2} \\ \omega(n^{\log_b a}) & \text{Possible CASE 3} \end{cases}$

check if $f(n) / n^{\log_b a} \in \Omega(n^{\varepsilon})$

Examples

a. $T(n) = 4T(n/2) + n;$ — $\log_b a = 2$. $n = o(n^2)$ => Check case 1

b. $T(n) = 9T(n/3) + n^2;$ — $\log_b a = 2$. $n^2 = \Theta(n^2)$ => Check case 2

c. $T(n) = 6T(n/4) + n;$ — $\log_b a = 1.3$. $n = o(n^{1.3})$ => Check case 1

d. $T(n) = 2T(n/4) + n;$ — $\log_b a = 0.5$. $n = \omega(n^{0.5})$ => Check case 3

e. $T(n) = T(n/2) + n \log n;$ — $\log_b a = 0$. $n\log n = \omega(n^0)$ => Check case 3

f. $T(n) = 4T(n/4) + n \log n.$ — $\log_b a = 1$. $n\log n = \omega(n)$ => Check case 3

Some Tricks

- Changing variables
- Obtaining upper and lower bounds
  - Make a guess based on the bounds
  - Prove using the substitution method

Changing Variables

$$T(n) = 2T(n-1) + 1$$

- Let $n = \lg m$, i.e., $m = 2^n$

$\Rightarrow T(\lg m) = 2\,T(\lg (m/2)) + 1$

- Let $S(m) = T(\lg m) = T(n)$

$\Rightarrow S(m) = 2S(m/2) + 1$

$\Rightarrow S(m) = \Theta(m)$

$\Rightarrow T(n) = S(m) = \Theta(m) = \Theta(2^n)$

Changing Variables

$$T(n) = T(\sqrt{n}) + 1$$

- Let n $=2^m$

=> sqrt(n) = $2^{m/2}$

- We then have $T(2^m) = T(2^{m/2}) + 1$
- Let $T(n) = T(2^m) = S(m)$

=> $S(m) = S(m/2) + 1$

$\Rightarrow S(m) = \Theta(\log m) = \Theta(\log \log n)$

$\Rightarrow T(n) = \Theta(\log \log n)$

Changing Variables

- $T(n) = 2T(n-2) + 1$
- Let $n = \lg m$, i.e., $m = 2^n$

=> $T(\lg m) = 2\,T(\lg m/4) + 1$

- Let $S(m) = T(\lg m) = T(n)$

=> $S(m) = 2S(m/4) + 1$

=> $S(m) = m^{1/2}$

=> $T(n) = S(m) = (2^n)^{1/2} = (\mathrm{sqrt}(2))^{n} \approx 1.4^n$

Obtaining Bounds

Solve the Fibonacci variant:

$$T(n) = T(n-1) + T(n-2) + 1$$

- $T(n) >= 2T(n-2) + 1$ [1]
- $T(n) <= 2T(n-1) + 1$ [2]

- Solving [1], we obtain $T(n) >= 1.4^n$
- Solving [2], we obtain $T(n) <= 2^n$
- Actually, $T(n) \approx 1.62^n$

Obtaining Bounds

- $T(n) = T(n/2) + \log n$
- $T(n) \in \Omega(\log n)$
- $T(n) \in O(T(n/2) + n^\varepsilon)$
- Solving $T(n) = T(n/2) + n^\varepsilon$,

  we obtain $T(n) = O(n^\varepsilon)$, for any $\varepsilon > 0$
- So: $T(n) \in O(n^\varepsilon)$ for any $\varepsilon > 0$
  - T(n) is unlikely polynomial
  - Actually, $T(n) = \Theta(\log^2 n)$ by extended case 2

Extended Case 2

CASE 2: $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$.

Extended CASE 2: (k >= 0)

$f(n) = \Theta(n^{\log_b a} \, log^k n) \Rightarrow T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Solving Recurrences

1. Recursion tree / iteration method
   - Good for guessing an answer
   - Need to **verify** guess
2. Master method
   - Easy to learn, useful in **limited cases** only
   - Some tricks may help in other cases
3. Substitution method
   - Generic method, rigid, may be hard

The Substitution Method

Substitutions

For	*Use*
Buttermilk - 1 cup	1 TB lemon juice + enough milk to = 1 cup
Whole Milk - 1 cup	½ c. evaporated milk + ½ c. water
Unsweetened Chocolate - 1 oz	1 TB fat + 3 TB cocoa
Honey - 1 cup	¼ c. liquid + 1 ¼ c. sugar
Shortening (for baking) - 1 cup	1 1/8 c. butter or margine less ½ tsp of salt in recipe
Corn Syrup - 1 cup	1 c. sugar + ¼ c. of liquid
Cornstarch - 1 ½ tsp	1 TB flour
1 whole egg	2 egg yolks + 1 TB water
Peppermint extract - 1 TB	¼ c. fresh mint, chopped
Cream ½ & ½ - 1 cup	3 TB oil + milk to = 1 cup
Cream, heavy for baking & cooking - 1 cup	3/4 c. milk + ½ c. butter or margarine
Marshmallow Creme - 1 cup (jar = 2 1/8 cups)	16 lg (160 sm) marshmallows + 2 TB corn syrup (melted in double broiler)
Catsup	1 c. tomato sauce, ½ c. sugar, 2 TB vinegar

Substitution Method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution
   (e.g. by recursion tree / iteration method)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for O/Ω -constants n_0 and c (base cases of induction)

Substitution Method

By log we mean lg

- Recurrence: $T(n) = 2T(n/2) + n$.
- Guess: $T(n) = O(n \log n)$. (e.g., by recursion tree method)
- To prove, have to show $T(n) \leq c\, n \log n$ for some $c > 0$ and for all $n > n_0$
- Proof by induction: assume it is true for $T(n/2)$, prove that it is also true for $T(n)$. This means:
- Given: $T(n) = 2T(n/2) + n$
- Need to Prove: $T(n) \leq c\, n \log (n)$
- Assuming: $T(n/2) \leq cn/2 \log (n/2)$

Proof

- Given: $T(n) = 2T(n/2) + n$
- Need to Prove: $T(n) \leq c\, n \log(n)$
- Assuming**:** $T(n/2) \leq cn/2 \log(n/2)$

- *Proof:*

  Substituting $T(n/2) \leq cn/2 \log(n/2)$ into the recurrence, we get

  $$
  \begin{aligned}
  T(n) &= 2\,T(n/2) + n \\
  &\leq cn \log(n/2) + n \\
  &\leq c\,n \log n - c\,n + n \\
  &\leq c\,n \log n - (c-1)\,n \\
  &\leq c\,n \log n \text{ for all } n > 0 \text{ (if } c \geq 1).
  \end{aligned}
  $$

Therefore, by definition, $T(n) = O(n \log n)$.

Substitution method – Example 2

- Recurrence: $T(n) = 2T(n/2) + n$.
- Guess: $T(n) = \Omega(n \log n)$.
- To prove, have to show $T(n) \geq c\, n \log n$ for some $c > 0$ and for all $n > n_0$
- Proof by induction: assume it is true for $T(n/2)$, prove that it is also true for $T(n)$. This means:
- Given: $T(n) = 2T(n/2) + n$
- Need to Prove: $T(n) \geq c\, n \log (n)$
- Assuming: $T(n/2) \geq cn/2 \log (n/2)$

Proof

- Given: $T(n) = 2T(n/2) + n$
- Need to Prove: $T(n) \geq c\, n \log (n)$
- Assuming**:** $T(n/2) \geq cn/2 \log (n/2)$

- *Proof:*

  Substituting $T(n/2) \geq cn/2 \log (n/2)$ into the recurrence, we get

$$
\begin{aligned}
T(n) &= 2\, T(n/2) + n \\
&\geq cn \log (n/2) + n \\
&\geq c\, n \log n - c\, n + n \\
&\geq c\, n \log n + (1 - c)\, n \\
&\geq c\, n \log n \quad \text{for all } n > 0 \ (\text{if } c \leq 1).
\end{aligned}
$$

Therefore, by definition, $T(n) = \Omega(n \log n)$.

More Substitution Examples [1]

- Prove that $T(n) = 3T(n/3) + n = O(n \log n)$
- Need to show that $T(n) \leq c\, n \log n$ for some c, and sufficiently large n
- Assume above is true for $T(n/3)$, i.e.

  $T(n/3) \leq cn/3 \log (n/3)$

3-way Merge Sort

$T(n) = 3\,T(n/3) + n$

$\leq 3\,cn/3 \log (n/3) + n$

$\leq cn \log n - cn \log 3 + n$

$\leq cn \log n - (cn \log 3 - n)$

$\leq cn \log n$ (if $cn \log 3 - n \geq 0$)

$cn \log 3 - n \geq 0$

=> $c \log 3 - 1 \geq 0$ (for $n > 0$)

=> $c \geq 1/\log 3$

=> $c \geq \log_3 2$

Therefore, $T(n) = 3\,T(n/3) + n \leq cn \log n$ for $c = \log_3 2$ and $n > 0$. By definition, $T(n) = O(n \log n)$.

More Substitution Examples [2]

- Prove that $T(n) = T(n/3) + T(2n/3) + n = O(n \log n)$
- Need to show that $T(n) \leq c\, n \log n$ for some c, and sufficiently large n
- Assume above is true for $T(n/3)$ and $T(2n/3)$, i.e.

$T(n/3) \leq cn/3 \log (n/3)$

$T(2n/3) \leq 2cn/3 \log (2n/3)$

Unbalanced Merge Sort

$$
\begin{aligned}
T(n) &= T(n/3) + T(2n/3) + n \\
&\leq cn/3 \log(n/3) + 2cn/3 \log(2n/3) + n \\
&\leq cn \log n + n - cn (\log 3 - 2/3) \\
&\leq cn \log n + n(1 - c\log 3 + 2c/3) \\
&\leq cn \log n, \text{ for all } n > 0 \text{ (if } 1 - c \log 3 + 2c/3 \leq 0)
\end{aligned}
$$

$$c \log 3 - 2c/3 \geq 1$$

$\Rightarrow c \geq 1 / (\log 3 - 2/3) > 0$

Therefore, $T(n) = T(n/3) + T(2n/3) + n \leq cn \log n$ for $c = 1 / (\log 3 - 2/3)$ and $n > 0$. By definition, $T(n) = O(n \log n)$.

More Substitution Examples [3]

- Prove that $T(n) = 3T(n/4) + n^2 = O(n^2)$
- Need to show that $T(n) \leq c\, n^2$ for some c, and sufficiently large n
- Assume above is true for $T(n/4)$, i.e.

  $T(n/4) \leq c(n/4)^2 = cn^2/16$

$$T(n) = 3T(n/4) + n^2$$

$$\leq 3\,c\,n^2 / 16 + n^2$$

$$\leq (3c/16 + 1)\, n^2$$

$$\overset{?}{\leq} cn^2$$

$3c/16 + 1 \leq c$ implies that $c \geq 16/13$

Therefore, $T(n) = 3(n/4) + n^2 \leq cn^2$ for $c = 16/13$ and all n. By definition, $T(n) = O(n^2)$.

Avoiding Pitfalls

- Guess $T(n) = 2T(n/2) + n = O(n)$ [really $O(n \log n)$]
- Need to prove that $T(n) \le c\, n$
- Assume $T(n/2) \le cn/2$
- $T(n) \le 2 * cn/2 + n = cn + n = O(n)$
- What's wrong?
- Need to prove $T(n) \le cn$, not $T(n) \le cn + n = (c+1)n$

Subtleties

- Prove that $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 = O(n)$
- Need to prove that $T(n) \leq cn$
- Assume above is true for $T(\lfloor n/2 \rfloor)$ & $T(\lceil n/2 \rceil)$

$T(n) <= c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1$

$\leq cn + 1$

Is it a correct proof?

No! have to prove $T(n) <= cn$

However we can prove $T(n) = O(n - 1)$

may be easier to prove a stronger induction hypothesis

Making a Good Guess

$T(n) = 2T(n/2 + 17) + n$

When n approaches infinity, $n/2 + 17$ are not too different from $n/2$
Therefore can guess $T(n) = \Theta(n \log n)$
Prove Ω:
Assume $T(n/2 + 17) \geq c\,(n/2+17) \log (n/2 + 17)$
Then we have

$$\begin{aligned} T(n) &= n + 2T(n/2+17) \\ &\geq n + 2c\,(n/2+17) \log (n/2 + 17) \\ &\geq n + c\,n \log (n/2 + 17) + 34\,c \log (n/2+17) \\ &\geq c\,n \log (n/2 + 17) + 34\,c \log (n/2+17) \\ &\ldots \end{aligned}$$

Maybe can guess $T(n) = \Theta((n-17) \log (n-17))$ (trying to get rid of the +17).
Details skipped.

Summary: Solving Recurrences

1. Recursion tree / iteration method
    - Good for guessing an answer
2. Master method
    - Easy to learn, useful in limited cases only
    - Some tricks may help in other cases
3. Substitution method
    - Generic method, rigid, may be hard

$T(n) = 3T(\frac{n}{2}) + n$

subproblems at level i of recursion tree

3^i

size of subproblems $\frac{n}{2^i}$

total work $= \sum_{i=0}^{\log_2 n} 3^i \frac{n}{2^i} = n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i$

$= n\, O\left(\left(\frac{3}{2}\right)^{\log_2 n}\right) = \frac{n}{n} O\left(3^{\log_2 n}\right)$

$= O\left(n^{\log_2 3}\right)$

$a^{\log_b n} = n^{\log_b a}$