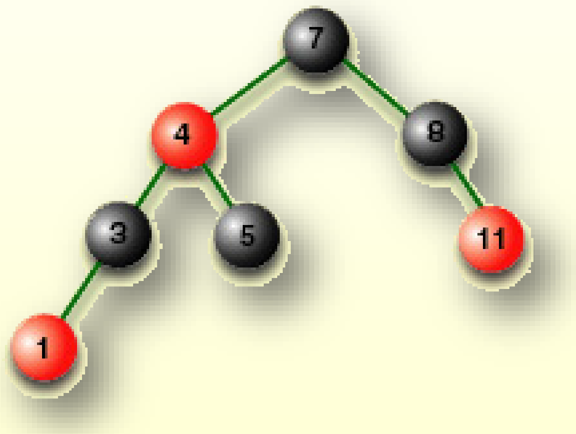
CS161:
Design and Analysis of Algorithms



Lecture 15
Leonidas Guibas

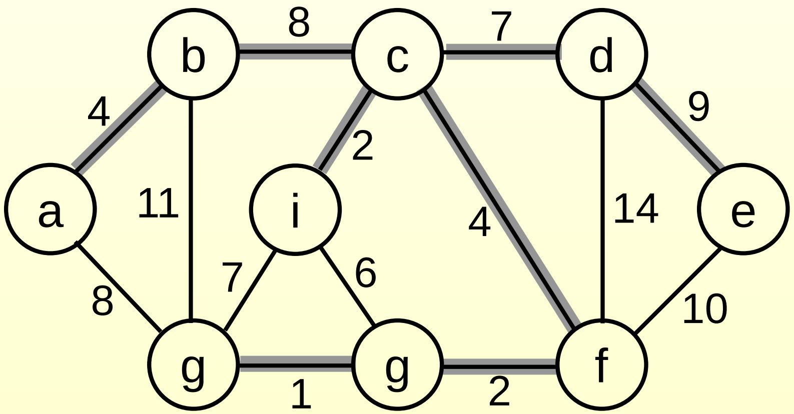
Outline

- Last lecture: Minimum spanning tree algorithms
- Today: Single source shortest path algorithms
  - shortest path properties; edge relaxation
  - Shortest paths on DAGs
  - Dijkstra's algorithm
  - Bellman-Ford algorithm

Slides modified from
- http://www.cs.bilkent.edu.tr/~atat/502/SingleSourceSP.ppt
- http://www.cs.unc.edu/.../comp122/

Minimum Spanning Trees

- Spanning Tree
  - A tree (i.e., connected, acyclic graph) which contains all the vertices of the graph
- Minimum Spanning Tree
  - Spanning tree with the **minimum sum of weights**



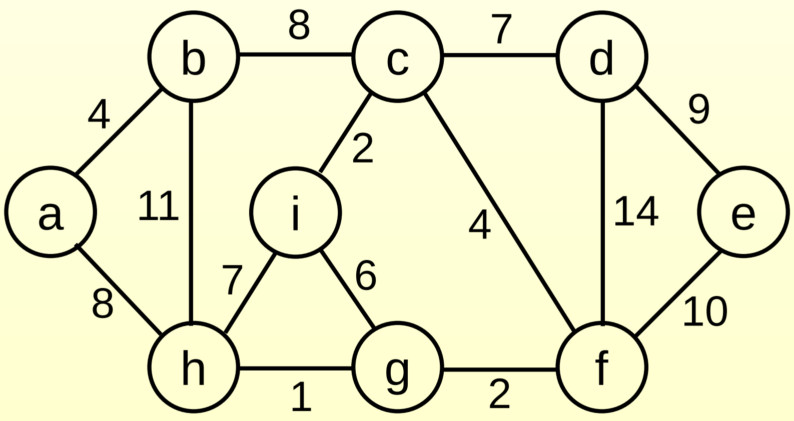
- Spanning forest
  - If a graph is not connected, then there is a spanning tree for each connected component of the graph

Greedy MST Algorithms

- Greedy algorithms
  - iteratively make “myopic” decisions – aimed at locally optimal choice
  - but somehow everything works out to yield the global optimum at the end
- While growing a partial MST, an edge not currently in the tree is safe, if it can be added while still being part of some MST

Generic MST algorithm

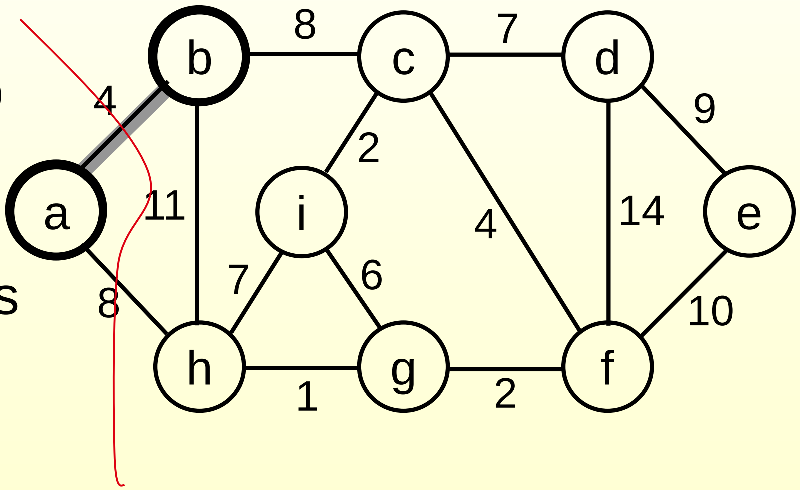
$A \leftarrow \varnothing$
while A is not a spanning tree
 do find an edge (u, v) that is safe for A
  $A \leftarrow A \cup \{(u, v)\}$
return A



- Key: how do we find safe edges?

Prim's Algorithm

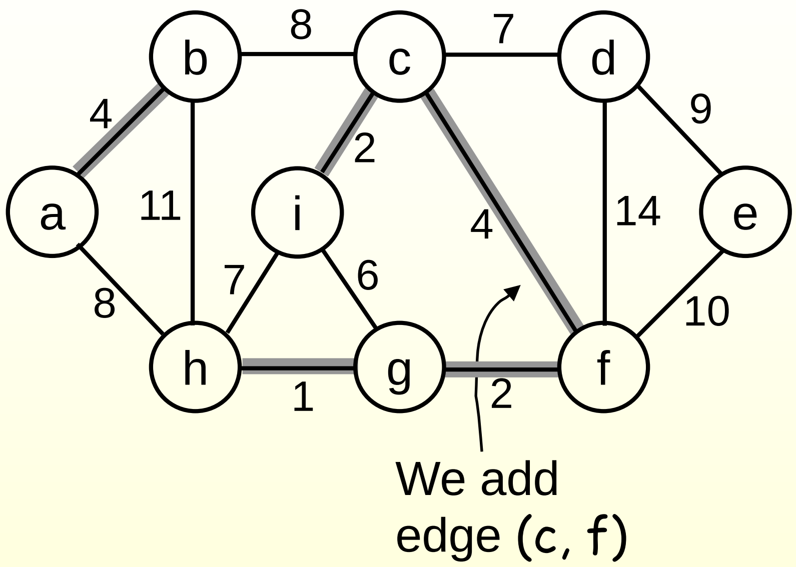
- The edges in set A always form a single tree
- Start from an arbitrary "root": $V_A = \{a\}$
- At each step:
  - Find a light edge crossing $(V_A, V - V_A)$
  - Add this edge to A
  - Repeat until the tree spans all vertices



Greedy approach

Kruskal's Algorithm

- Start with each vertex being its own component
- Repeatedly merge two components into one by choosing the **light** edge that connects them
- Which components to consider at each iteration?
  - Scan the set of edges in monotonically increasing order by weight (guarantees lightness)



Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once, but is more "parallel".

```
Algorithm BaruvkaMST(G)
    T ← V  {just the vertices of G}
  while T has fewer than V|-1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

- Each iteration of the while-loop halves the number of connected compontents in T.
- The running time of all three algorithms is basically O(E log V).

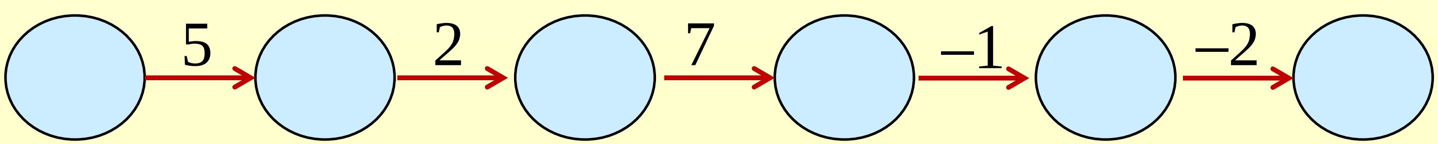
Introduction: Shortest Paths

Generalization of simple BFS to handle weighted graphs

- Direct Graph ***G*** = (***V***, ***E***), edge weight *function* w : ***E*** $\rightarrow$ ***R***
- In simple BFS, we have w(e)=1 for all e ϵ E

Weight of path p = $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$



Shortest Path

Shortest Path = Path of minimum weight between two vertices u and v

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\}; & \text{if there is a path from u to v,} \\ \infty & \text{otherwise.} \end{cases}$$

Distance from *u* to *v* = length of shortest path from *u* to *v*

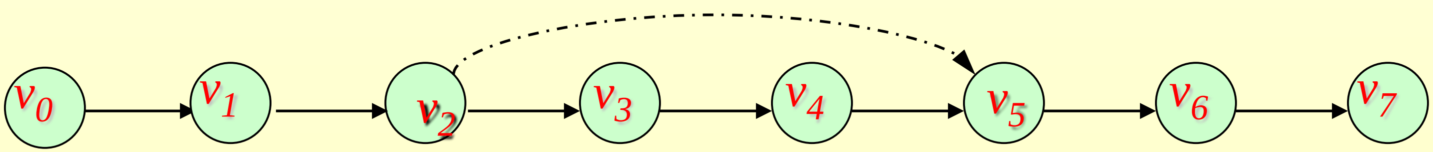
Shortest-Path Variants

- Shortest-Path problems
  - Single-source shortest-paths problem: Find the shortest path from *s* to each vertex *v*. (e.g. BFS)
  - Single-destination shortest-paths problem: Find a shortest path to a given *destination* vertex *t* from each vertex *v*.
  - Single-pair shortest-path problem: Find a shortest path from *u* to *v* for given vertices *u* and *v*.
  - All-pairs shortest-paths problem: Find a shortest path from *u* to *v* for every pair of vertices *u* and *v*.

Optimal Substructure Property

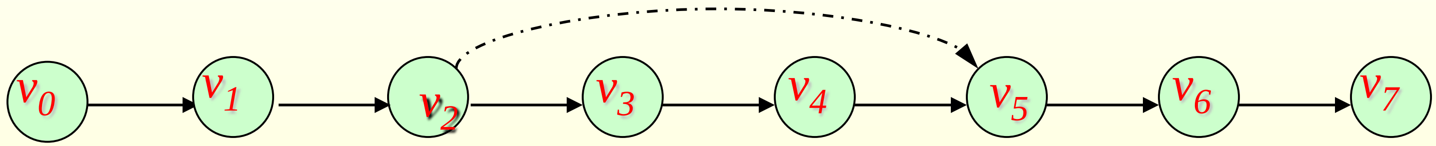
Theorem: Subpaths of shortest paths are also shortest paths

- Let $P_{1k} = <v_1, ... ,v_k>$ be a shortest path from v_1 to v_k
- Let $P_{ij} = <v_i, ... ,v_j>$ be subpath of P_{1k} from v_i to v_j for any $1 \le i \le j \le k$
- Then P_{ij} is itself a shortest path from v_i to v_j



Optimal Substructure Property

Proof: By cut and paste



- If some subpath *were not* a shortest path
- We could substitute a shorter subpath in the original path to create a *shorter total path*
- Hence, the original path would not be shortest path

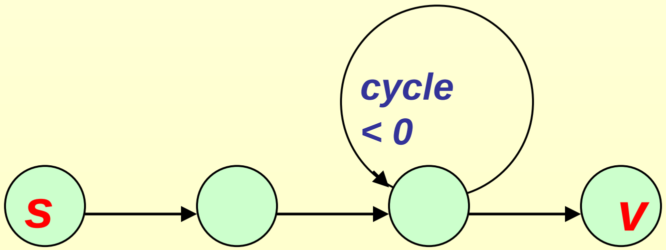
Negative Weight Cycles

Definition:

- $\delta(u,v)$ = weight of a shortest path(s) from u to v

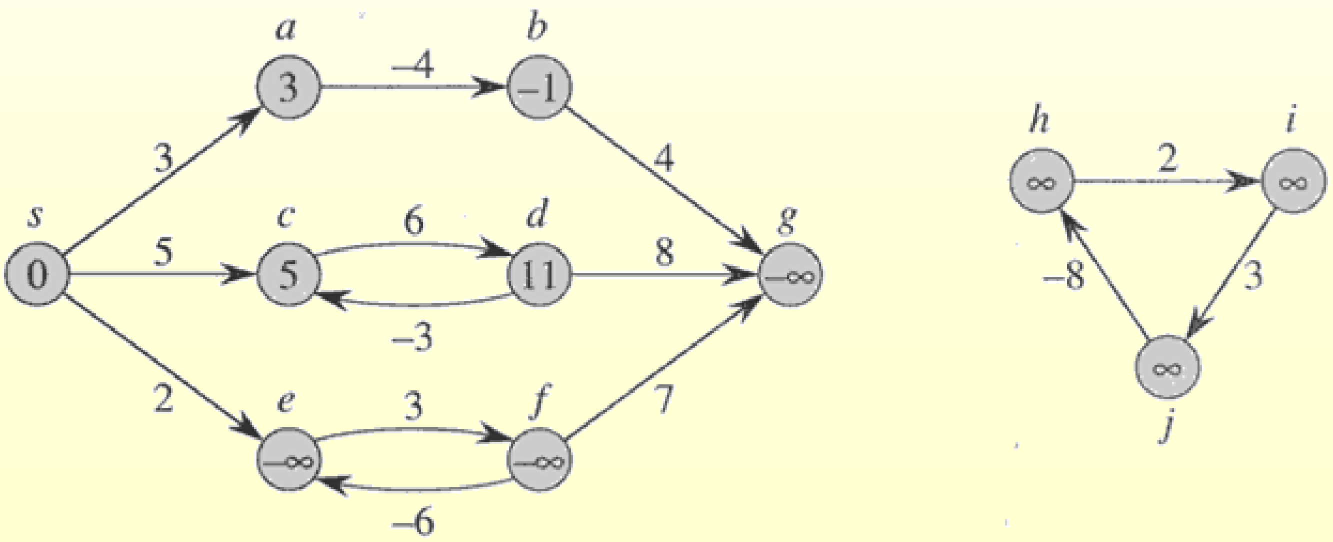
Not always well defined:

- *negative-weight cycle in graph*: Some shortest paths may not be defined
- *argument:* can always get a shorter path by going around the negative cycle again



Negative-Weight Edges

- No problem, as long as no negative-weight cycles are reachable from the source
- Otherwise, we can just keep going around it, and get w(s, v) = −∞ for all v on the cycle.



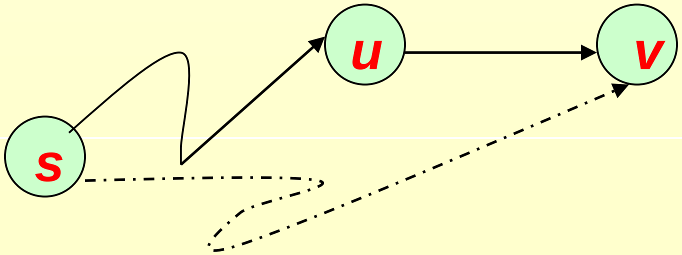
Triangle Inequality

Lemma 1: for a given vertex *s* ⇝ ***V*** *and for every edge (u,v)* ϵ *E,*

- $\delta(s,v) \leq \delta(s,u) + w(u,v)$

Proof: shortest path *s* ⇝ *v is not longer than any other path.*

- in particular the path that takes the shortest path *s* ⇝ *u* *and* then takes edge (u,v)



Edge Relaxation

- Maintain d[v] for each $v \rightsquigarrow V$
- d[v] is called a ***shortest-path weight estimate*** and is an *upper bound* on $\delta(s,v)$

INIT(G, s)
    for each $v \in V$ do
        d[v] $\leftarrow \infty$
        $\boldsymbol{\pi}$[v] $\leftarrow$ NIL
    d[s] $\leftarrow 0$

as before, predecessor on shortest path from s to v

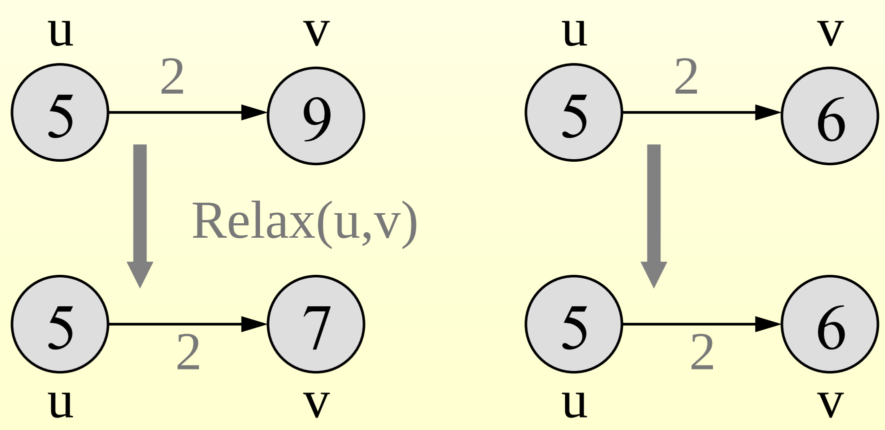
Edge Relaxation

RELAX(u, v)

if d[v] > d[u]+w(u,v) then

d[v] ← d[u]+w(u,v)

π[v] ← u



Properties of Relaxation

Shortest path algorithms work by relaxing edges. They differ in

- *how many times* they relax each edge, and
- *the order* in which they relax edges

Question: How many times each edge is relaxed in BFS?

Answer: Only once!

Properties of Relaxation

Given:

- An edge weighted directed graph $G = (V, E)$ with *edge weight function* $(w{:}E \rightarrow R)$ and a source vertex $s \in V$
- G is initialized by ***INIT(G , s)***

Lemma 2: Immediately after relaxing edge (u,v),

$d[v] \leq d[u] + w(u,v)$

Lemma 3: For any sequence of relaxation steps over E,

(a) the invariant $d[v] \geq \delta(s,v)$ is maintained

(b) once d[v] achieves its lower bound, it never changes.

Properties of Relaxation

Proof of (a): certainly true after

INIT(G,s) : d[s] = 0 = $\delta(s,s)$:d[v] = $\infty \geq \delta(s,v)\ \forall\ v \in$ V-{s}

- *Proof by contradiction:* Let v be the first vertex for which

  RELAX(u, v) causes d[v] < $\delta(s, v)$
- After ***RELAX(u , v)*** we have
  - d[u] + w(u,v) = d[v] < $\delta(s, v)$

    ≤ $\delta(s, u)$ + w(u,v) by *L2*
  - d[u]+w(u,v) < $\delta(s, u)$ + w(u, v) => d[u] < $\delta(s, u)$

    contradicting the assumption

Properties of Relaxation

Proof of (b):

- d[*v*] cannot decrease after achieving its lower bound; because d[*v*] $\geq \delta(s,v)$
- d[*v*] cannot increase since relaxations don't increase d values.

Properties of Relaxation

C1 : For any vertex v which is not reachable from s, we have the invariant $d[v] = \delta(s,v)$ that is maintained over any sequence of relaxations

Proof: By *L3(b)*, we always have $\infty = \delta(s,v) \leq d[v]$

$\Rightarrow d[v] = \infty = \delta(s,v)$

Properties of Relaxation

Lemma 4: Let $s \rightsquigarrow u \rightarrow v$ *be a shortest path from s to v for some u,v* $\rightsquigarrow$ *V*

- Suppose that a sequence of relaxations including ***RELAX(u,v)*** were performed on E
- If d[u] = $\delta(s, u)$ at any time prior to ***RELAX(u, v)***
- then d[v] = $\delta(s, v)$ at all times after ***RELAX(u, v)***

Properties of Relaxation

Proof: If d[u] = $\delta(s, v)$ prior to ***RELAX(u, v)***
d[u] = $\delta(s, v)$ hold thereafter by *L3*(*b*)

- After ***RELAX(u,v)***, we have d[v] ≤ d[u] + w(u, v) by *L2*
  = $\delta(s, u)$ + w(u, v) hypothesis
  = $\delta(s, v)$ by optimal subst.property
- Thus d[v] ≤ $\delta(s, v)$
- But d[v] ≥ $\delta(s, v)$ by *L3(a)* => d[v] = $\delta(s, v)$

Q.E.D.

Single-Source Shortest Paths in DAGs

- Shortest paths are always *well-defined* in *dags*
  - no cycles => no negative-weight cycles even if there are negative-weight edges
- **Idea:** If we were lucky
  - To process vertices on each shortest path from left to right, we would be done in 1 pass due to *L4*

Single-Source Shortest Paths in DAGs

In a DAG:

- Every path is a subsequence of the topologically sorted vertex order
- If we do topological sort and process edges in the order of their origins
- We will process each path in forward order
  - Never relax edges out of a vertex until have processed all edges into the vertex
- Thus, just one pass is sufficient

Single-Source Shortest Paths in DAGs

DAG-SHORTEST PATHS(G, s)

TOPOLOGICALLY-SORT the vertices of G

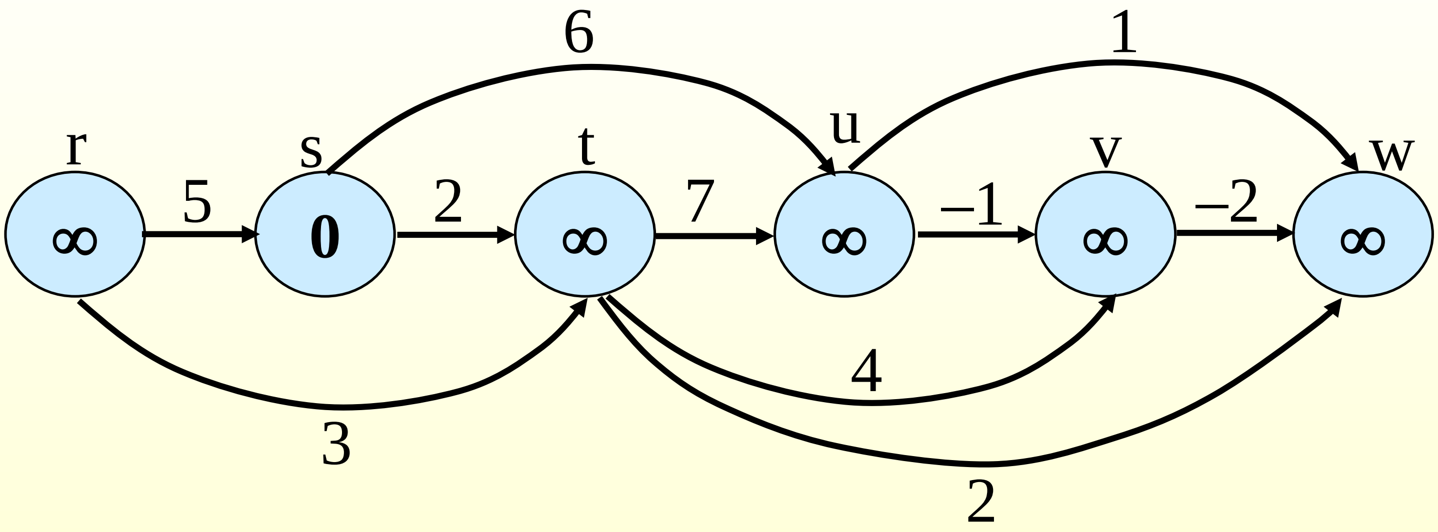
INIT(G, s)

for each vertex *u* taken in topologically sorted order do

for each *v* ⇝ Adj[*u*] do

RELAX(u, v)

Example



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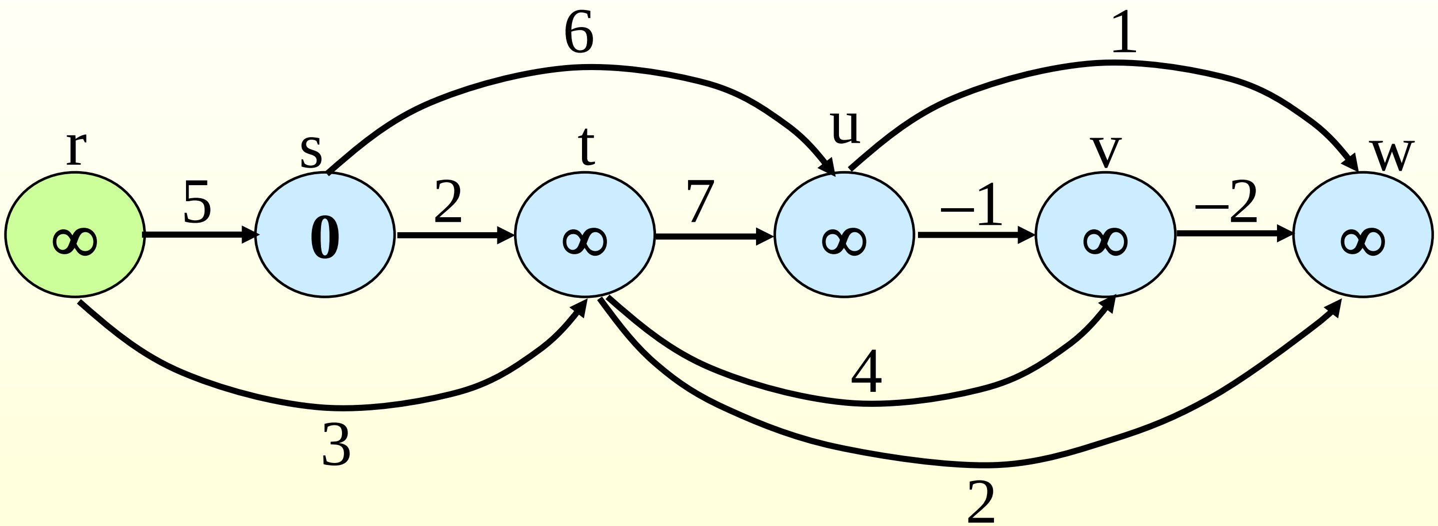

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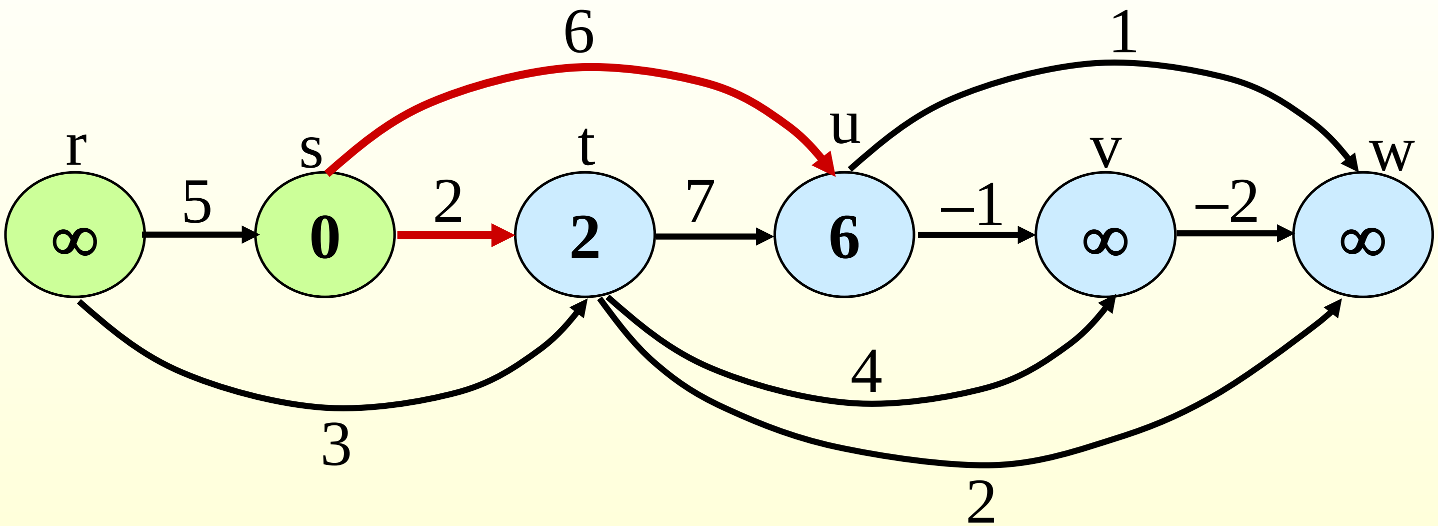

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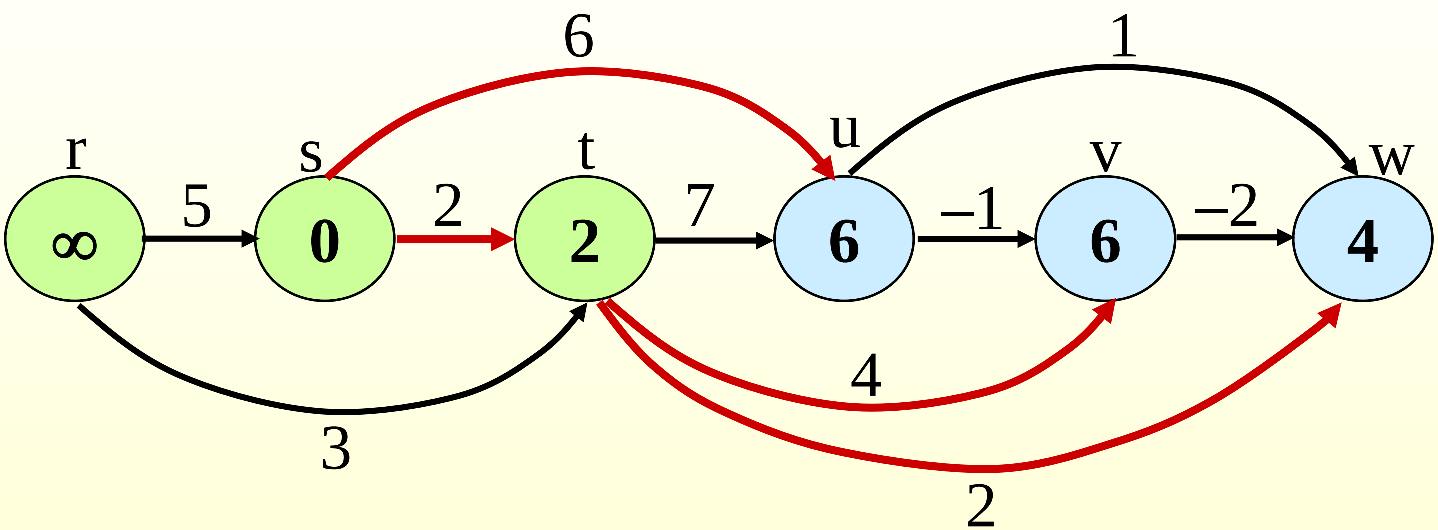

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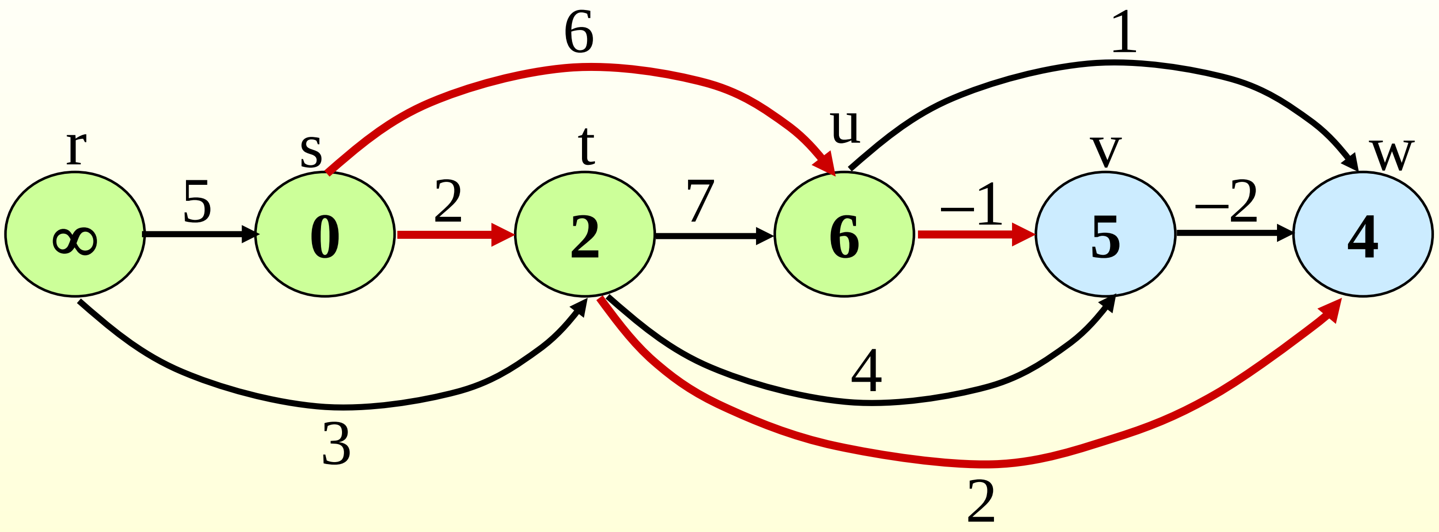

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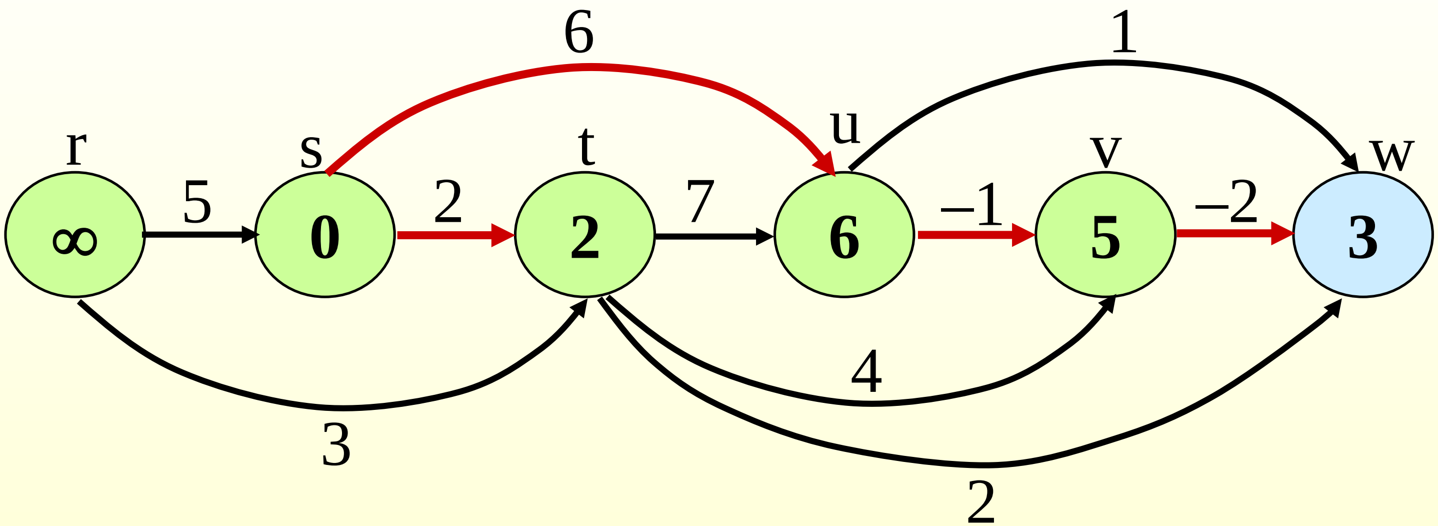

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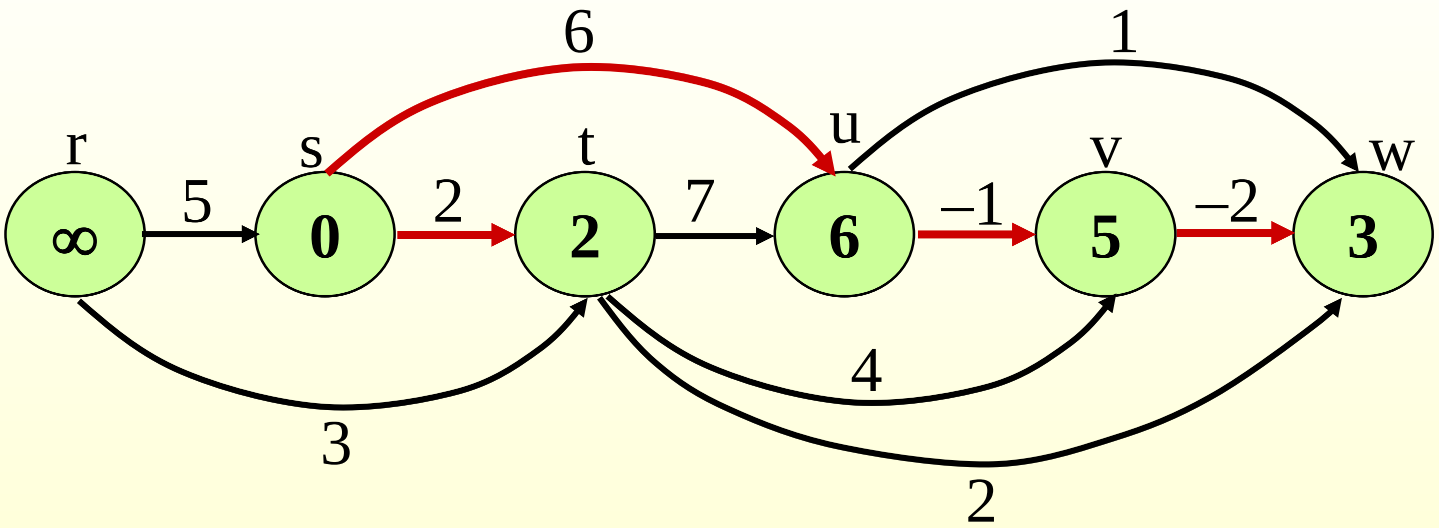


Single-Source Shortest Paths in DAGs

Runs in linear time: $\Theta(V+E)$

- topological sort: $\Theta(V+E)$
- initialization: $\Theta(V+E)$
- *for-loop:* $\Theta(V+E)$

  each vertex processed exactly once

  => each edge processed exactly once: $\Theta(V+E)$

Single-Source Shortest Paths in DAGs

Thm: (Correctness of *DAG-SHORTEST-PATHS*):

At termination of *DAG-SHORTEST-PATHS* procedure

$d[v] = \delta(s, v)$ for all v $\rightsquigarrow$ V

Single-Source Shortest Paths in DAGs

Proof: If v ϵ R_s , *then* d[v] = $\delta(s, v)$ $\forall v \rightsquigarrow V$

- If v ϵ R_s , so ∃a shortest path

  p = **$<v_0=s, v_1, v_2, \ldots, v_k=v>$**
- Because we process vertices in topologically sorted order
  - Edges on p are relaxed in the order

    $(u_0, u_1),(u_1, u_2),\ldots,(u_{k-1}, u_k)$
- A simple induction on k using *L4* shows that
  - $d[v_i] = \delta(s, v)$ at termination for i = 0,1,2,…,k

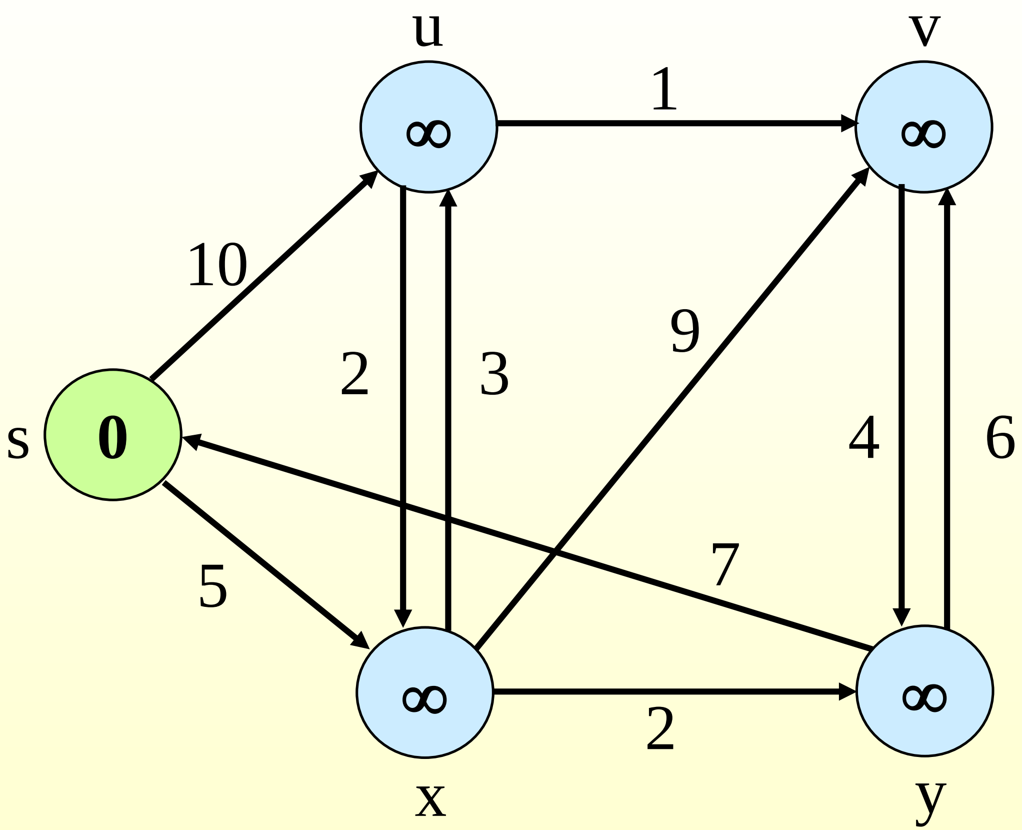
Dijkstra’s Algorithm For Shortest Paths

- Non-negative edge weights
- Like BFS: If all edge weights are equal, then use BFS, otherwise use this algorithm
- Use Q = priority queue keyed on d[v] values
  (note: BFS uses FIFO)

Dijkstra's Algorithm For Shortest Paths

```
DIJKSTRA(G, s)
    INIT(G, s)
    S←Ø              > set of discovered nodes
    Q←V[G]
    while Q ≠Ø do
        u←EXTRACT-MIN(Q)
        S←S U {u}
        for each v ⇝ Adj[u] do
            RELAX(u, v) > may cause
                        > DECREASE-KEY(Q, v, d[v])
```

Example



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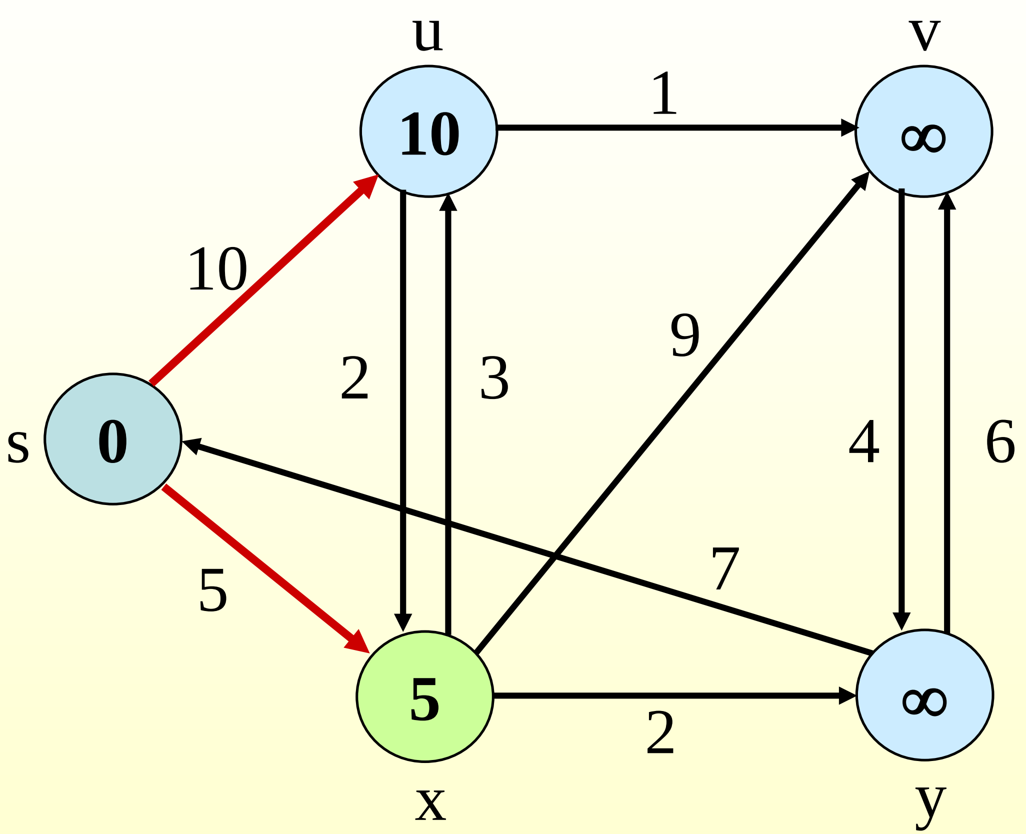

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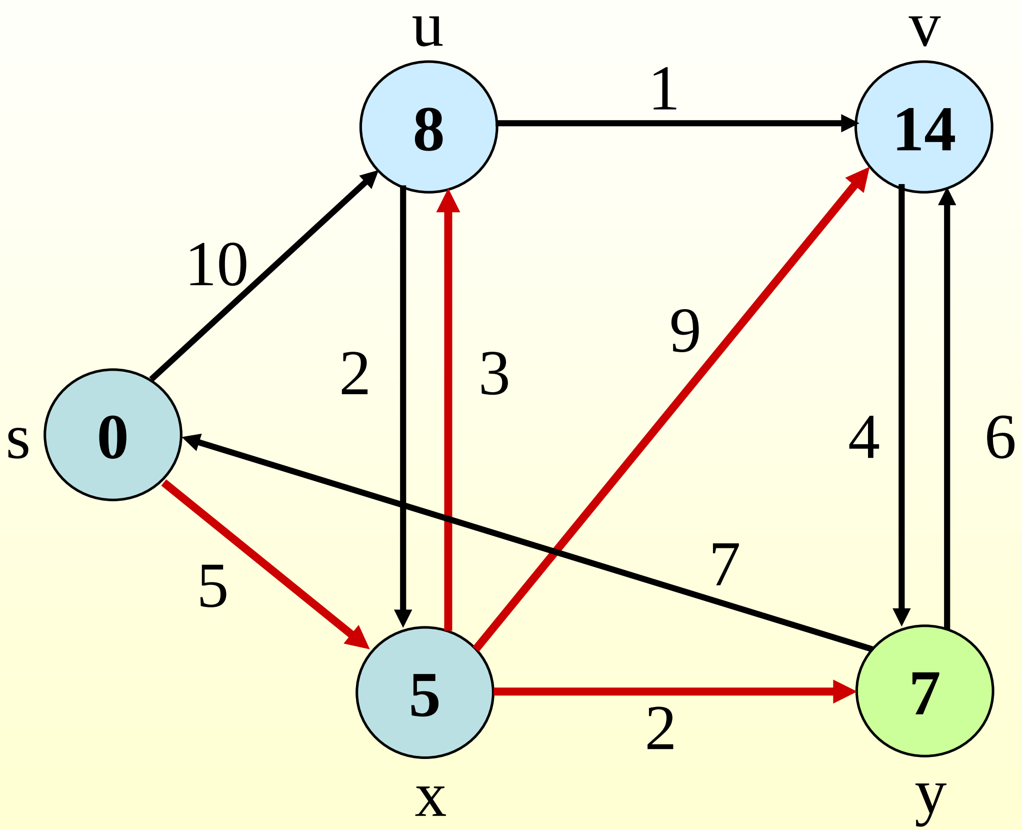

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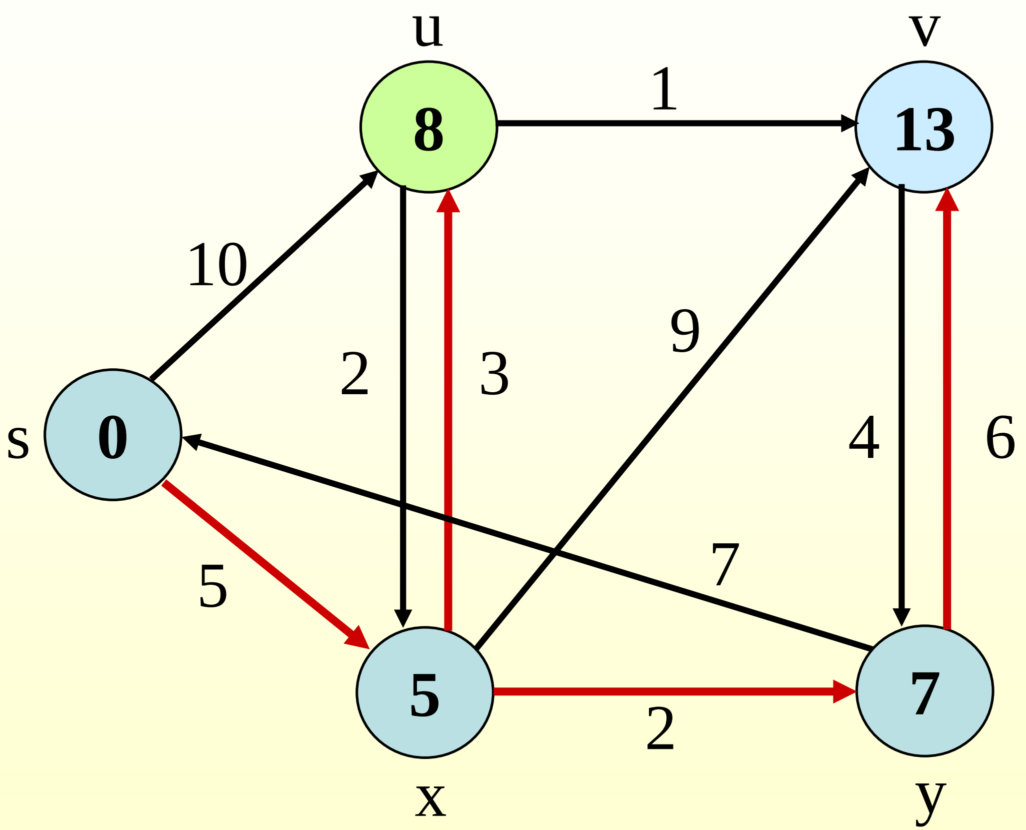

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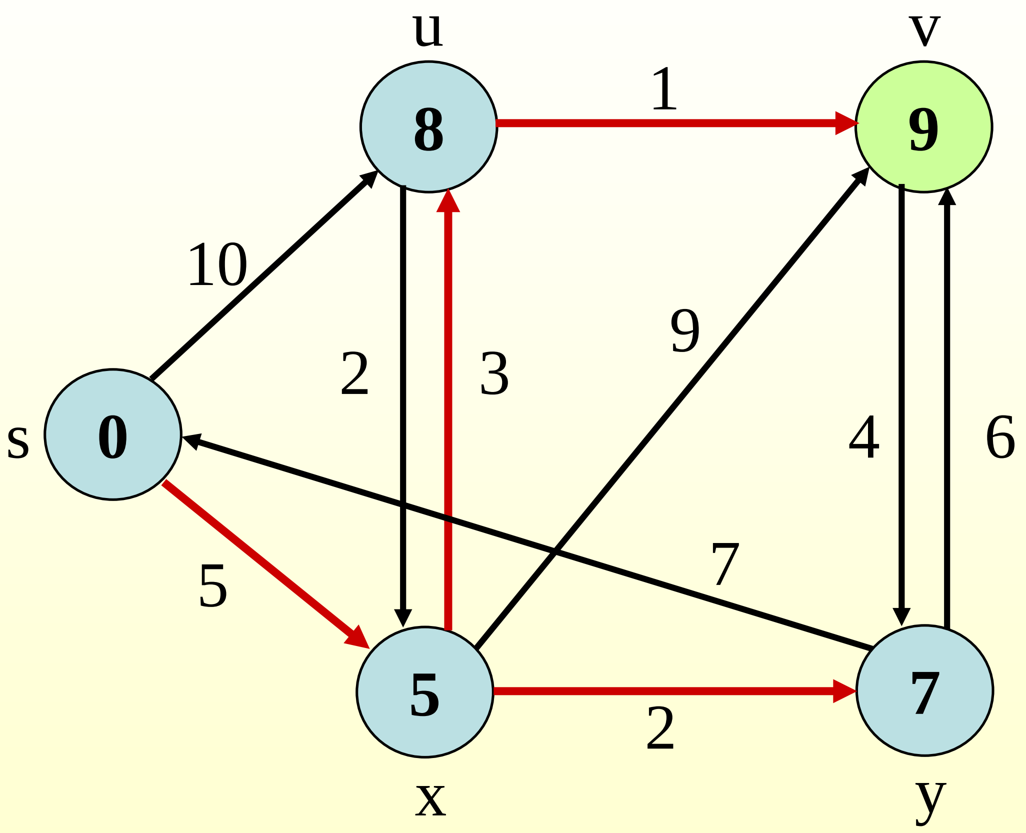
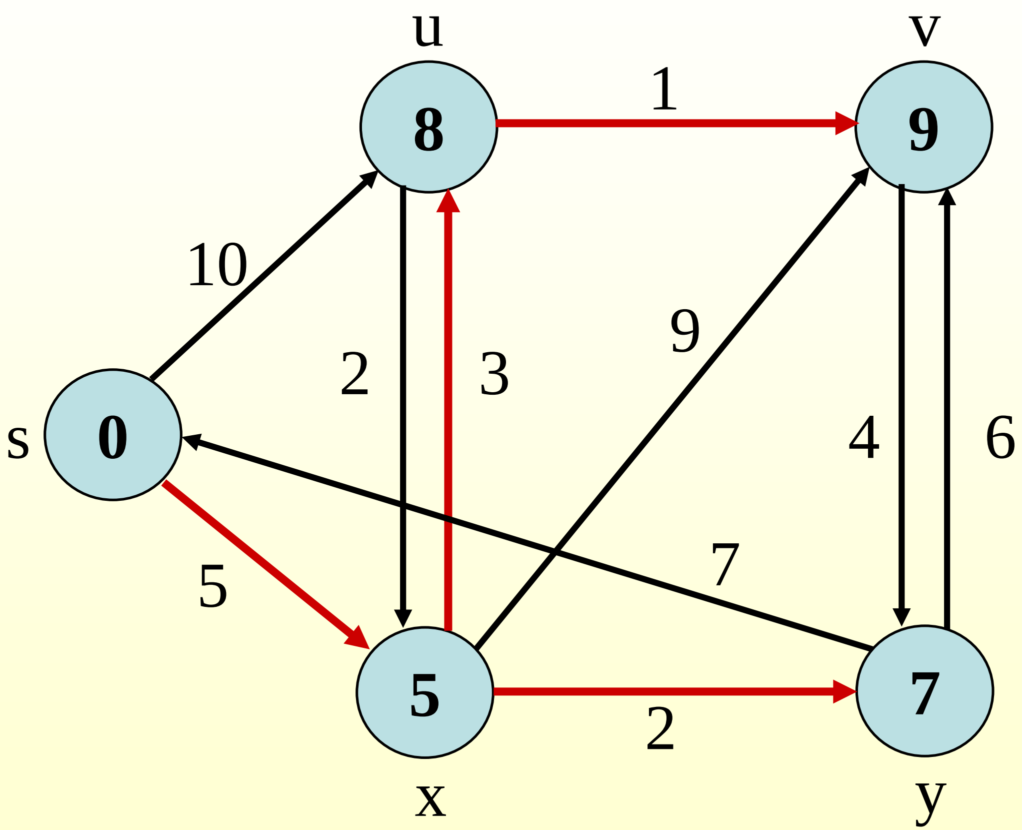


Example



Dijkstra’s Algorithm For Shortest Paths

Observe :

- Each vertex is extracted from Q and inserted into S exactly once
- Each edge is relaxed exactly once
- S = set of vertices whose final shortest paths have already been determined
  - i.e. , $S = \{v \rightsquigarrow V: d[v] = \delta(s, v) \neq \infty \}$

Dijkstra’s Algorithm For Shortest Paths

- *Similar to BFS algorithm:* S corresponds to the set of black vertices in BFS which have their correct breadth-first distances already computed

- *Greedy strategy:* Always chooses the closest (lightest) vertex in Q = V-S to insert into S

- Relaxation may reset d[v] values thus updating Q = ***DECREASE-KEY*** operation.

Dijkstra's Algorithm For Shortest Paths

- Similar to Prim's MST algorithm: Both algorithms use a priority queue to find the lightest vertex outside a given set S

- Insert this vertex into the set

- Adjust weights of remaining adjacent vertices outside the set accordingly

Correctness

Theorem : Upon termination, $d[u] = \delta(s, u)$ for all u in V (assuming non-negative weights).

Proof:

By Lemma 3(b), once $d[u] = \delta(s, u)$ holds, it continues to hold.

We prove: For each u in V, $d[u] = \delta(s, u)$ when u is inserted in S.

Suppose not. Let u be the first vertex such that $d[u] \neq \delta(s, u)$ when inserted in S.

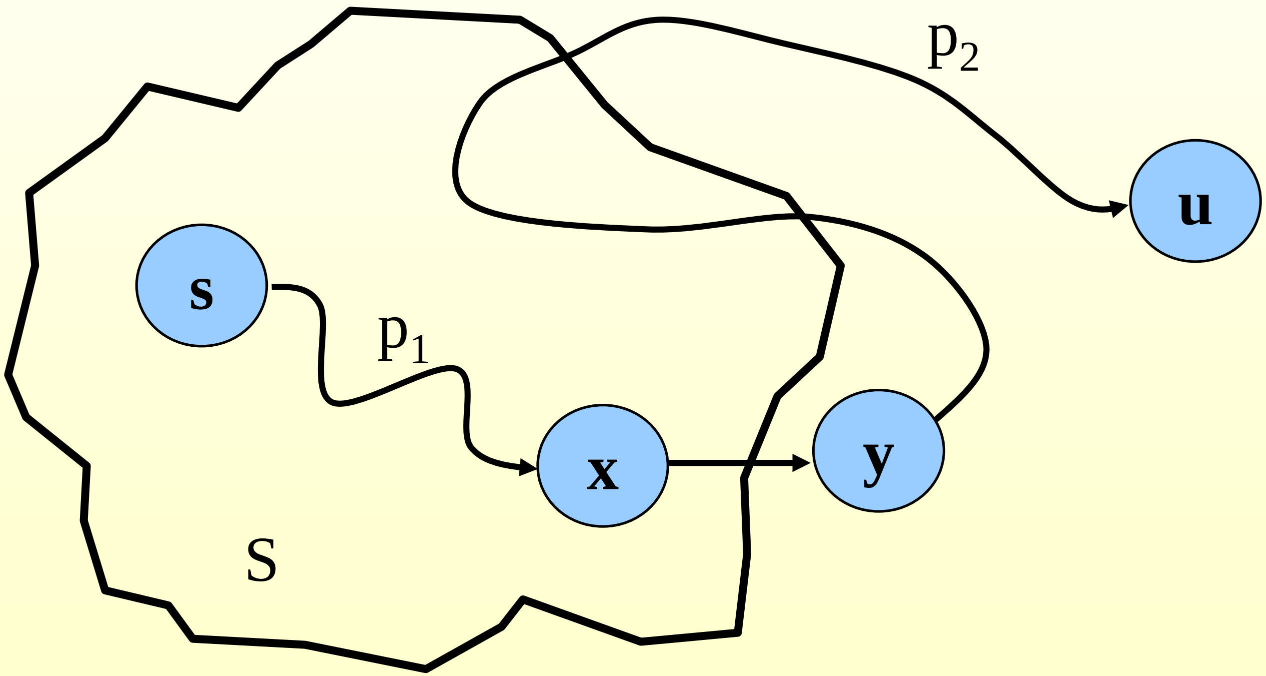
Note that $d[s] = \delta(s, s) = 0$ when s is inserted, so $u \neq s$.

$\Rightarrow$ $S \neq \emptyset$ just before u is inserted (in fact, $s \in S$).

Proof (Continued)

Note that there exists a path from s to u, for otherwise $d[u] = \delta(s, u) = \infty$ by Corollary 24.12.

$\Rightarrow$ there exists a SP from s to u. Say SP looks like this:



Proof (Continued)

Claim: $d[y] = \delta(s, y)$ when u is inserted into S.

We had $d[x] = \delta(s, x)$ when x was inserted into S.

Edge (x, y) was relaxed at that time.

By Lemma 3(b), this implies the claim.

Now, we have: $d[y] = \delta(s, y)$, by Claim.
$\leq \delta(s, u)$, nonnegative edge weights.
$\leq d[u]$, by Lemma 3(a).

Because u was added to S before y, $d[u] \leq d[y]$.

Thus, $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Contradiction.

Computing Paths (not just Distances)

- Maintain for each node v a predecessor node π(v)
- π(v) is initialized to be null
- Whenever an edge (u,v) is relaxed such that d(v) improves, then π(v) can be set to be u
- Paths can be generated from this data structure

Running Time Analysis of Dijkstra's Algorithm

- Look at different Q implementation, as we did for Prim's algorithm
- Initialization (INIT) : Θ(V) time
- While-loop:
  - ***EXTRACT-MIN*** executed |V| times
  - ***DECREASE-KEY*** executed |E| times

- Time $T = |V| \times T_{\text{E-MIN}} + |E| \times T_{\text{D-KEY}}$

Running Time Analysis of Dijkstra's Algorithm

- Look at different Q implementation, as did for Prim's algorithm

Q	$T_{E\text{-}MIN}$	$T_{D\text{-}KEY}$	TOTAL
Linear Unsorted Array:	O(V)	O(1)	$O(V^2+E)$
Binary Heap:	O(lgV)	O(logV)	O(VlgV+ElgV) = O(ElgV)
Fibonacci heap:	O(lgV) (Amortized)	O(1) (Amortized)	O(VlgV+E) (Worst Case)

Running Time Analysis of Dijkstra's Algorithm

Q = unsorted-linear array:

- Scan the whole array for EXTRACT-MIN
- Joint index for DECREASE-KEY

Q = Fibonacci heap: note advantage of amortized analysis

- Can use amortized Fibonacci heap bounds per operation in the analysis as if they were worst-case bound
- Still get (real) worst-case bounds on aggregate running time

Bellman-Ford Algorithm for Single Source Shortest Paths

- More general than Dijkstra’s algorithm:
  - Allows edge-weights can be negative

- As a by-product, it detects the existence of negative-weight cycle(s) reachable from s.

Bellman-Ford Algorithm for Single Source Shortest Paths

```
BELMAN-FORD( G, s )
    INIT( G, s )
    for i ←1 to |V|-1 do
        for each edge (u, v) ∈ E do
            RELAX( u, v )

    for each edge ( u, v ) ∈ E do
        if d[v] > d[u]+w(u,v) then
            return FALSE      > neg-weight cycle
    return TRUE
```

Bellman-Ford Algorithm for Single Source Shortest Paths

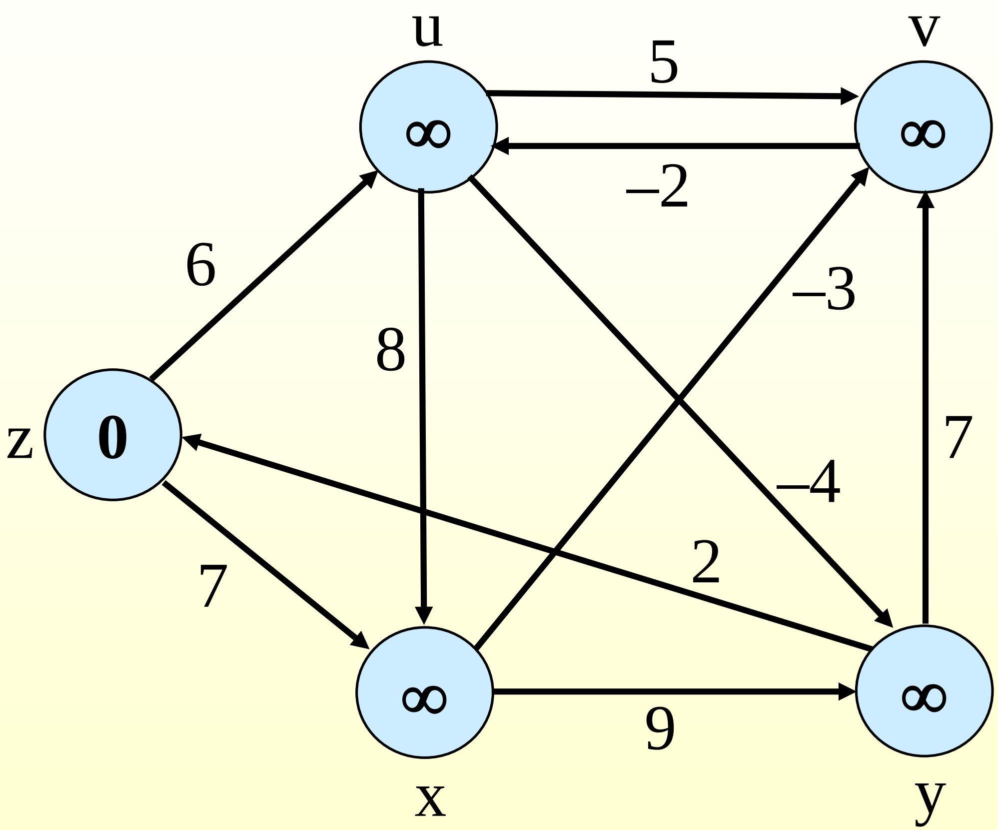
Observe:

- First nested for-loop performs |V|-1 relaxation passes; relax every edge at each pass
- Last for-loop checks the existence of a negative-weight cycle reachable from s

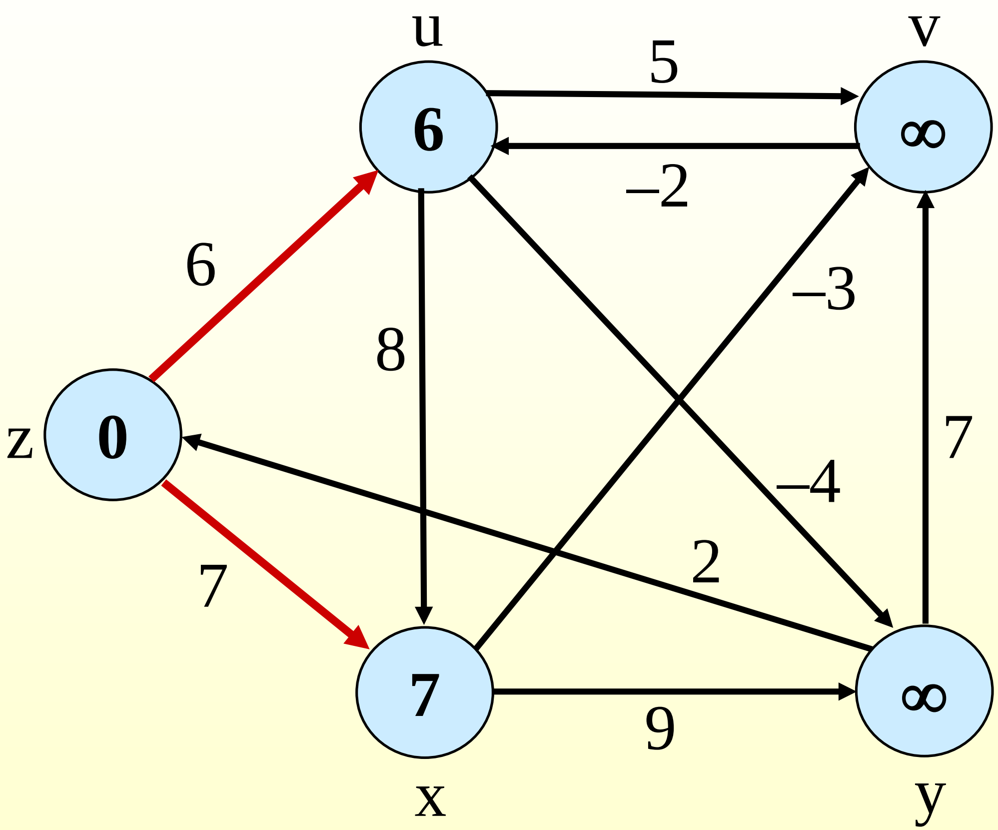
Bellman-Ford Algorithm for Single Source Shortest Paths

- *Running time* = O(V E)
  Constants are good; it's simple, short code(very practical)
- *Example:* Run algorithm on a sample graph with no negative weight cycles.

Example



Example



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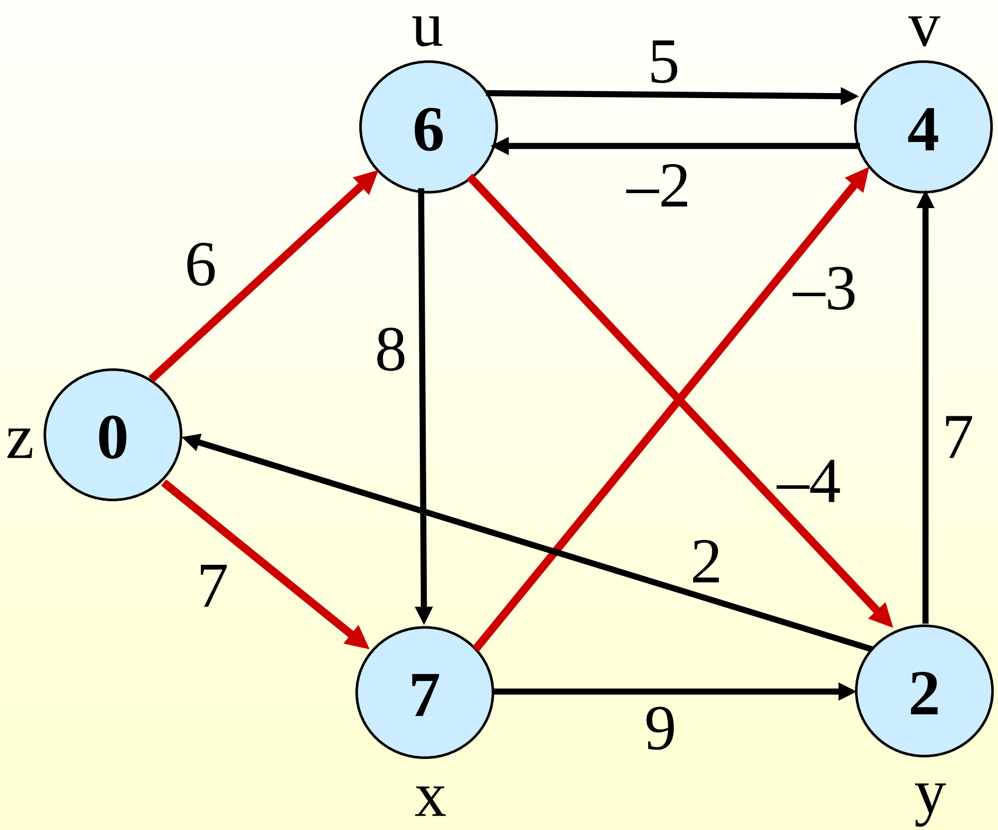
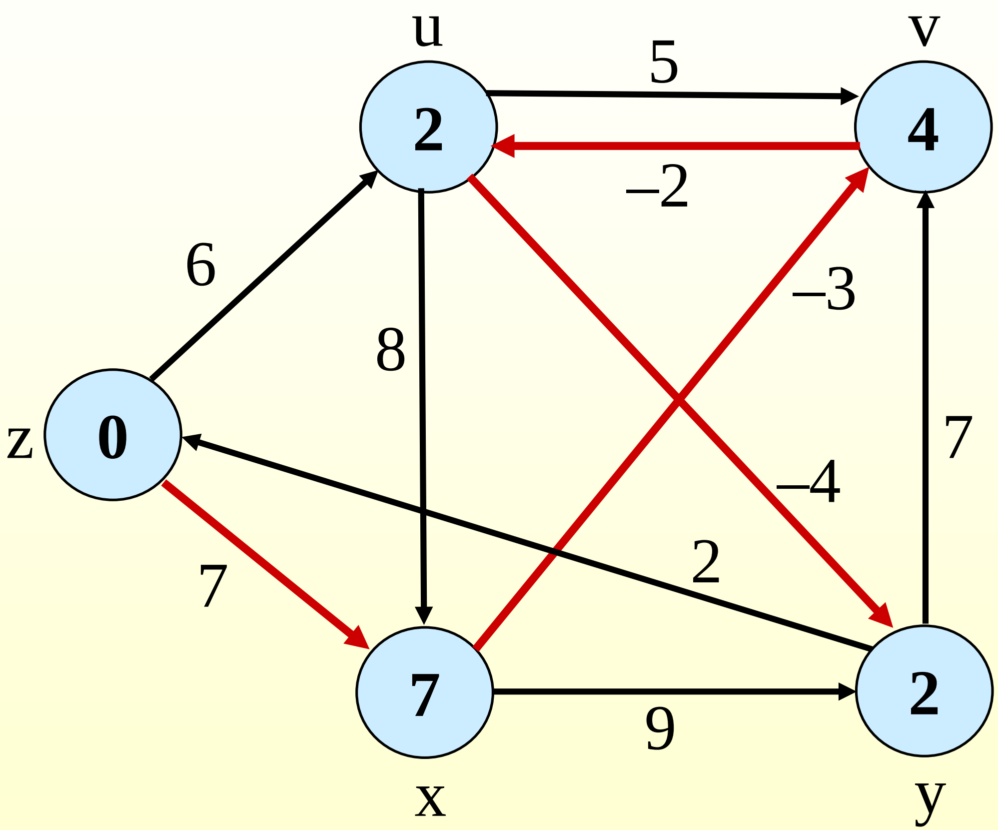
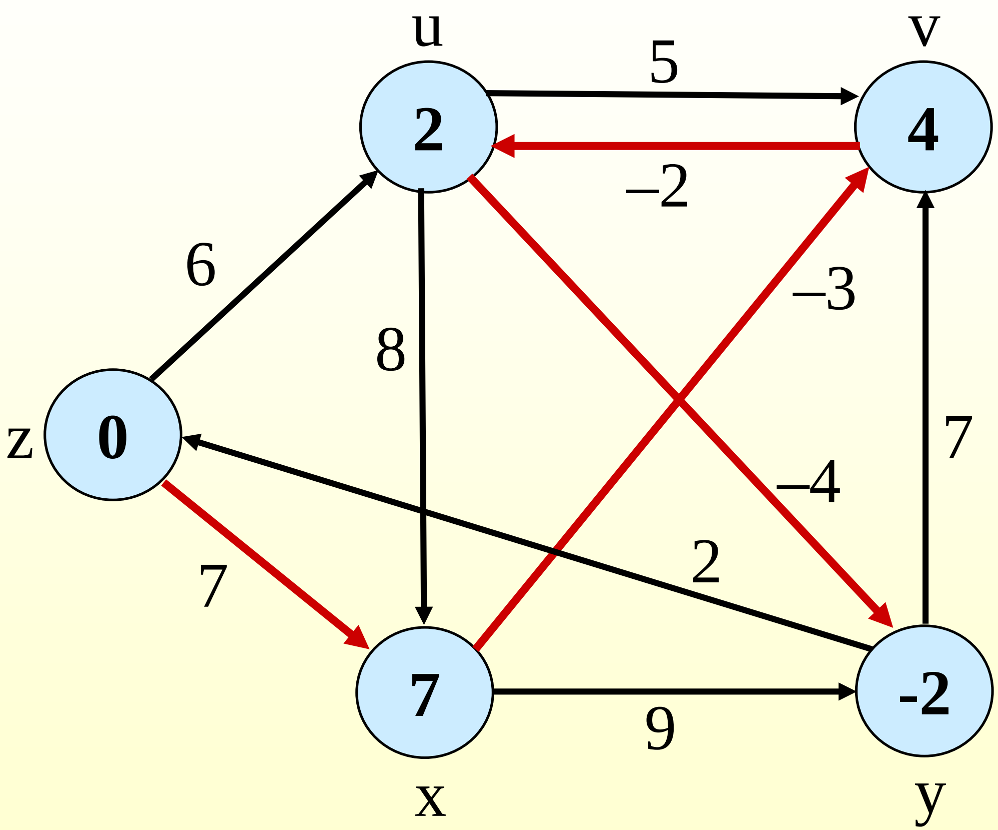


Example



Example



Bellman-Ford Algorithm for Single Source Shortest Paths

- Converges in just 2 relaxation passes
- Values you get on each pass & how early converges depend on edge process order
- d value of a vertex may be updated more than once in a pass

Bellman-Ford Correctness

Lemma: Assuming no negative-weight cycles reachable from s, $d[v] = \delta(s, v)$ holds upon termination for all vertices v reachable from s.

Proof:

Consider a SP p, where $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$.

Assume $k \leq |V| - 1$, otherwise p has a cycle.

Claim: $d[v_i] = \delta(s, v_i)$ holds after the i^{th} pass over edges.
Proof follows by induction on i.

By Lemma 3(b), once $d[v_i] = \delta(s, v_i)$ holds, it continues to hold.

Correctness

Claim: Algorithm returns the correct value.

(Part of Theorem 24.4. Other parts of the theorem follow easily from earlier results.)

Case 1: There is no reachable negative-weight cycle.

Upon termination, we have for all (u, v):

$d[v] = \delta(s, v)$, if v is reachable;
$d[v] = \delta(s, v) = \infty$ otherwise.

$\leq \delta(s, u) + w(u, v)$

$= d[u] + w(u, v)$

So, algorithm returns **true**.

Case 2

Case 2: There exists a reachable negative-weight cycle $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$.

We have $\Sigma_{i = 1, \ldots, k} w(v_{i-1}, v_i) < 0.$ (*)

Suppose algorithm returns true. Then, $d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1, \ldots, k$. (because Relax didn't change any $d[v_i]$). Thus,

$$\Sigma_{i = 1, \ldots, k} d[v_i] \leq \Sigma_{i = 1, \ldots, k} d[v_{i-1}] + \Sigma_{i = 1, \ldots, k} w(v_{i-1}, v_i)$$

But, $\Sigma_{i = 1, \ldots, k} d[v_i] = \Sigma_{i = 1, \ldots, k} d[v_{i-1}]$.

Can show no $d[v_i]$ is infinite. Hence, $0 \leq \Sigma_{i = 1, \ldots, k} w(v_{i-1}, v_i)$.

Contradicts (*). Thus, algorithm returns **false**.