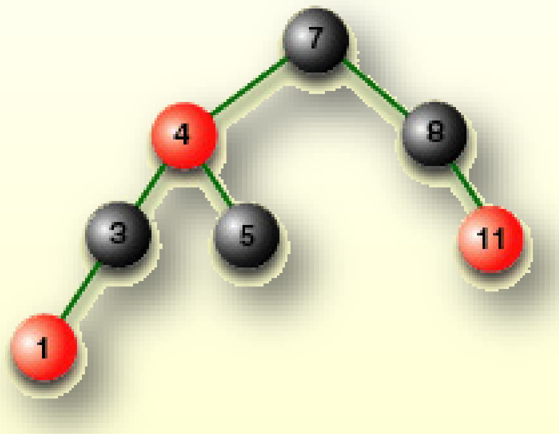
CS161: Design and Analysis of Algorithms



Lecture 16

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Outline

- Last lecture: Single source shortest path algorithms
- Today: All pairs shortest path algorithms
  - shortest paths and matrix multiplication
  - Floyd-Warshall algorithm
  - transitive closure of a DAG
  - Johnson's algorithm for sparse graphs

Slides modified from
- *http://www.cs.bilkent.edu.tr/~atat/502/AllPairsSP.ppt*

Shortest Path

Shortest Path = Path of minimum weight between two vertices u and v

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\}; & \text{if there is a path from u to v,} \\ \infty & \text{otherwise.} \end{cases}$$

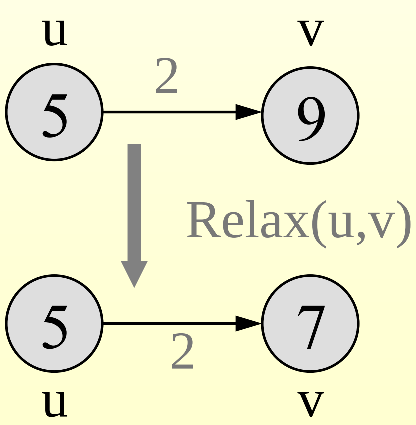
Distance from *u* to *v* = length of shortest path from *u* to *v*

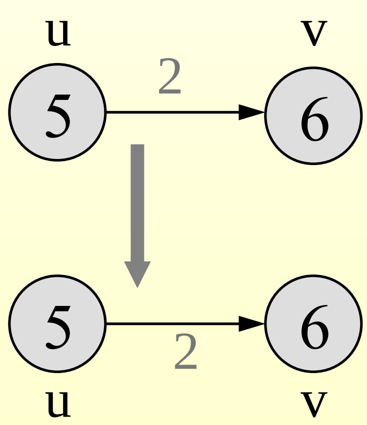
Shortest-Path Variants

- Shortest-Path problems
  - Single-source shortest-paths problem (SSSP): Find the shortest path from *s* to each vertex *v*. (e.g. BFS)
  - Single-destination shortest-paths problem (SDSP): Find a shortest path to a given *destination* vertex *t* from each vertex *v*.
  - Single-pair shortest-path problem (SPSP): Find a shortest path from *u* to *v* for given vertices *u* and *v*.
  - **All-pairs shortest-paths problem (APSP):** Find a shortest path from *u* to *v* for every pair of vertices *u* and *v*.

Edge Relaxation

RELAX(u, v)
    if d[v] > d[u]+w(u,v) **then**
        d[v] ← d[u]+w(u,v)
        π[v] ← u





d[u], d[v] denote our current estimates of their distances from s

Properties of Relaxation

Given:

- An edge weighted directed graph $G = (V, E)$ with *edge weight function* $(w: E \rightarrow R)$ and a source vertex $s \in V$
- G is initialized by ***INIT(G , s)***

Lemma 2: Immediately after relaxing edge (u,v),

$d[v] \leq d[u] + w(u,v)$

Lemma 3: For any sequence of relaxation steps over E,

(a) the invariant $d[v] \geq \delta(s,v)$ is maintained

(b) once d[v] achieves its lower bound, it never changes.

Single-Source Shortest Paths in DAGs

DAG-SHORTEST PATHS(G, s)

TOPOLOGICALLY-SORT the vertices of G

INIT(G, s)

for each vertex *u* taken in topologically sorted order do

for each *v* ⇝ Adj[*u*] do

RELAX(u, v)

Dijkstra's Algorithm For Shortest Paths

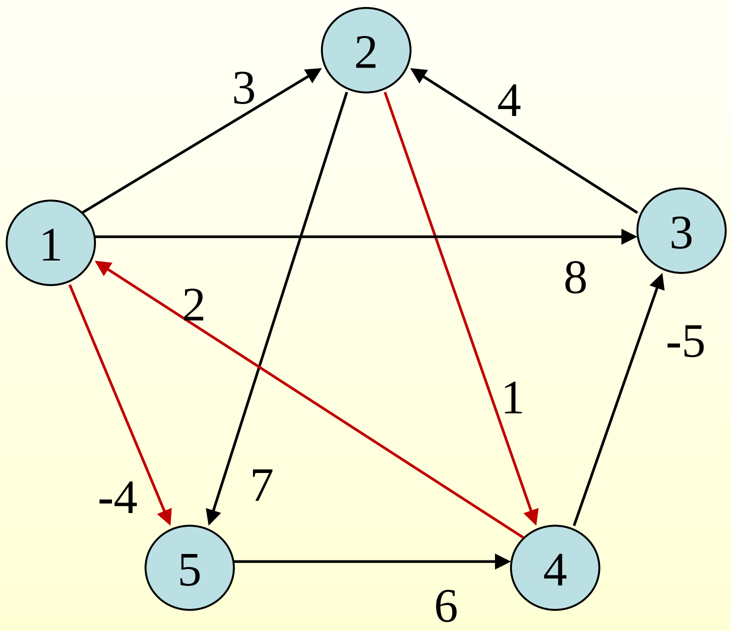
```
DIJKSTRA(G, s)
     INIT(G, s)
     S←Ø              > set of discovered nodes
     Q←V[G]
     while Q ≠Ø do
         u←EXTRACT-MIN(Q)
         S←S U {u}
         for each v ⇝ Adj[u] do
             RELAX(u, v) > May cause
                         > DECREASE-KEY(Q, v, d[v])
```

Bellman-Ford Algorithm for Single Source Shortest Paths

```
BELMAN-FORD( G, s )
    INIT( G, s )
    for i ←1 to |V|-1 do
        for each edge (u, v) ∈ E do
            RELAX( u, v )

    for each edge ( u, v ) ∈ E do
        if d[v] > d[u]+w(u,v) then
            return FALSE      > ∃ neg-weight cycle
    return TRUE
```

All Pairs Shortest Paths (APSP)



	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

D

Matrix representation of graphs

All Pairs Shortest Paths (APSP)

- given : directed graph $G = (V, E)$,
  weight function $\omega : E \rightarrow R$, $|V| = n$

- goal : create an $n \times n$ matrix $D = (d_{ij})$ of shortest path distances
  i.e., $d_{ij} = \delta (v_i , v_j)$

- trivial APSP solution : run a SSSP algorithm n times, using each vertex as the source.

All Pairs Shortest Paths (APSP)

- all edge weights are nonnegative : use Dijkstra's algorithm
  - PQ = binary heap : $O(V^2 \lg V + EV \lg V) = O(V^3 \lg V)$ for dense graphs
  - PQ = Fibonacci heap : $O(V^2 \lg V + EV) = O(V^3)$ for dense graphs
- negative edge weights : use Bellman-Ford algorithm
  - $O(V^2 E) = O(V^4)$ on dense graphs

Adjacency Matrix Representation of Weighted Graphs

► n x n matrix W = (ω_{ij}) of edge weights :

$$\omega_{ij} = \begin{cases} \omega(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ \\ \infty & \text{if } (v_i, v_j) \notin E \end{cases}$$

► assume $\omega_{ii} = 0$ for all $v_i \in V$, because

- no neg-weight cycle

  $\Rightarrow$ shortest path to itself has no edge,

  i.e., $\delta(v_i, v_i) = 0$

Shortest Paths via Dynamic Programming

(1) Characterize the structure of an optimal solution.

(2) Recursively define the value of an optimal solution.

(3) Compute the value of an optimal solution in a bottom-up manner.

(4) Construct an optimal solution from information constructed in (3).

Shortest Paths and Matrix Multiplication

Assumption : negative edge weights may be present, but no negative weight cycles.

(1) Structure of a Shortest Path :

- Consider a shortest path p_{ij}^{m} from v_i to v_j such that $|p_{ij}^{m}| \leq m$
  - ► i.e., path p_{ij}^{m} has at most m edges.
- no negative-weight cycle $\Rightarrow$ all shortest paths are simple
  $\Rightarrow$ m is finite $\Rightarrow$ $m \leq |V| - 1$
- $i = j \Rightarrow |p_{ii}| = 0$ & $\omega(p_{ii}) = 0$
- $i \neq j \Rightarrow$ decompose path p_{ij}^{m} into p_{ik}^{m-1} & $v_k \rightarrow v_j$, where $|p_{ik}^{m-1}| \leq m - 1$
  - ► p_{ik}^{m-1} must be a shortest path from v_i to v_k by optimal substructure property.
  - ► Therefore, $\delta(v_i, v_j) = \delta(v_i, v_k) + \omega_{kj}$

Shortest Paths and Matrix Multiplication

(2) A Recursive Solution to All Pairs Shortest Paths Problem :

- d_{ij}^{m} = minimum weight of any path from v_i to v_j that contains at most "*m*" edges.

- *m = 0* : There exist a shortest path from v_i to v_j with no edges $\leftrightarrow$ $i = j$.

$$\blacktriangleright\ d_{ij}^{0} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

- $m \geq 1$: $d_{ij}^{m} = \min \{ d_{ij}^{m-1} , \min_{1\leq k\leq n \wedge k\neq j} \{ d_{ik}^{m-1} + \omega_{kj} \}\}$

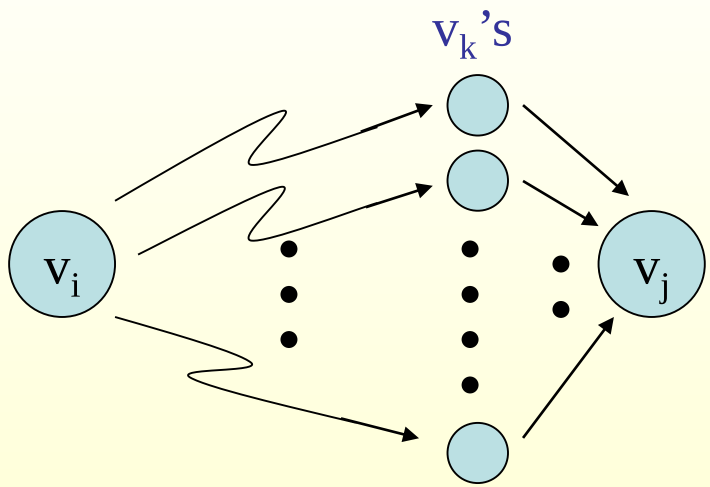
  $= \min_{1\leq k\leq n} \{d_{ik}^{m-1} + \omega_{kj} \}$ for all $v_k \in V$,

  since $\omega_{j\,j} = 0$ for all $v_j \in V$.

$$\boxed{d_{ij}^{m} = \min_{1\leq k\leq n} \{d_{ik}^{m-1} + \omega_{kj} \} \text{ for all } v_k \in V}$$

Shortest Paths and Matrix Multiplication

- to consider all possible shortest paths with $\leq m$ edges from v_i to v_j
  - ▶ consider shortest path with $\leq m$ -1 edges, from v_i to v_k, where $v_k \in R_{v_i}$ and $(v_k, v_j) \in E$



- note : $\delta(v_i, v_j) = d_{ij}^{n-1} = d_{ij}^{n} = d_{ij}^{n+1}$ …, since $m \leq n - 1 = |V| - 1$

$$\boxed{d_{ij}^{m} = \min_{1 \leq k \leq n} \{d_{ik}^{m-1} + \omega_{kj}\} \text{ for all } v_k \in V}$$

Shortest Paths and Matrix Multiplication

(3) Computing the shortest-path weights bottom-up:

- given $W = D^1$, compute a series of matrices D^2, D^3, ..., D^{n-1} , where $D^m = (d_{ij}^{m})$ for $m = 1, 2,\ldots, n-1$

  ► final matrix D^{n-1} contains actual shortest path weights, i.e., $d_{ij}^{n-1} = \delta (v_i ,v_j)$

- SLOW-APSP(W)

  $D^1 \leftarrow W$

  for $m \leftarrow 2$ to $n-1$ do

  $\quad D^m \leftarrow$ EXTEND(D^{m-1} , W)

  return D^{n-1}

Shortest Paths and Matrix Multiplication

EXTEND (D , W)

▶ **D** = (d_{ij}) is an n x n matrix

for $i \leftarrow 1$ **to** n **do**

  for $j \leftarrow 1$ **to** n **do**

    $d_{ij} \leftarrow \infty$

    for $k \leftarrow 1$ **to** n **do**

      $d_{ij} \leftarrow \mathbf{min}\{d_{ij}, d_{ik} + \omega_{kj}\}$

return D

MATRIX-MULT (A , B)

▶ **C** = (c_{ij}) is an n x n result matrix

for $i \leftarrow 1$ **to** n **do**

  for $j \leftarrow 1$ **to** n **do**

    $c_{ij} \leftarrow 0$

    for $k \leftarrow 1$ **to** n **do**

      $c_{ij} \leftarrow c_{ij} + a_{ik} \times b_{kj}$

return C

Shortest Paths and Matrix Multiplication

- relation to matrix multiplication $C = A \times B$: $\mathbf{c}_{ij} = \sum_{1 \le k \le n} \mathbf{a}_{ik} \text{ x } \mathbf{b}_{kj}$,
  - ► $D^{m-1} \leftrightarrow A$ & $\mathbf{W} \leftrightarrow B$ & $D^m \leftrightarrow C$
    "min" ↔ "+" & "+" ↔ "x" & "∞" ↔ "0"

- Thus, we compute the sequence of matrix products

$$D^1 = D^0 \text{ x } W = W \;;\; \text{note } D^0 = \text{identity matrix, i.e., } d_{ij}^0 = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

$$D^2 = D^1 \text{ x } W = W^2$$

$$D^3 = D^2 \text{ x } W = W^3$$

$$\vdots$$

$$D^{n-1} = D^{n-2} \text{ x } W = W^{n-1}$$

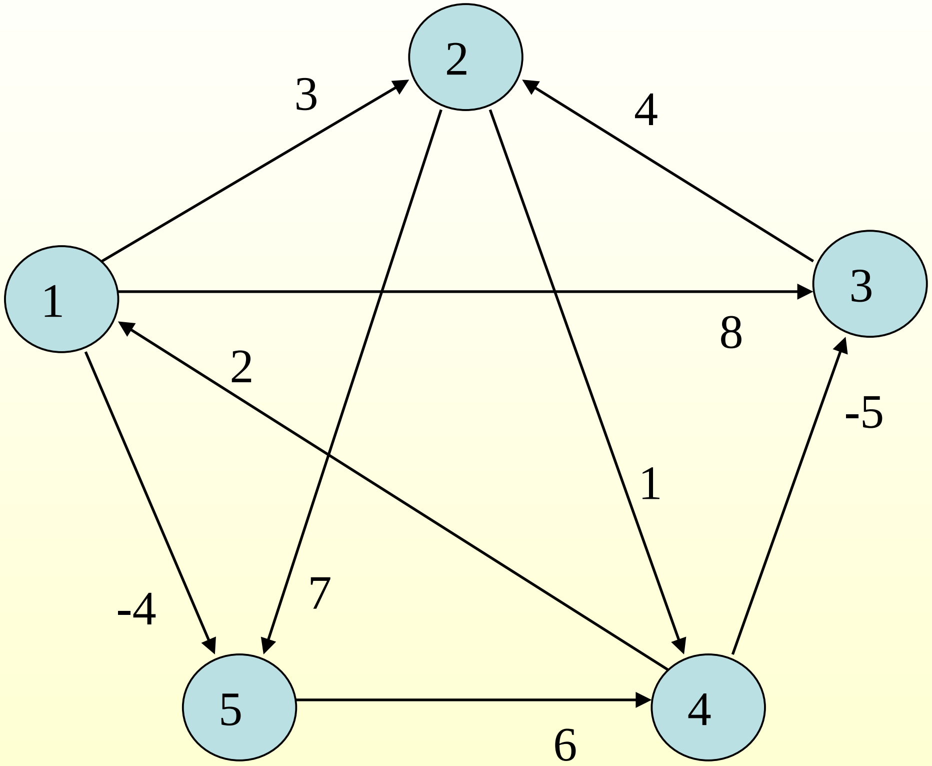
- running time : $\Theta(n^4) = \Theta(V^4)$
  - ► each matrix product : $\Theta(n^3)$
  - ► number of matrix products : n-1

$$\mathbf{c}_{ij} = \sum_{1 \le k \le n} \mathbf{a}_{ik} \text{ x } \mathbf{b}_{kj}$$

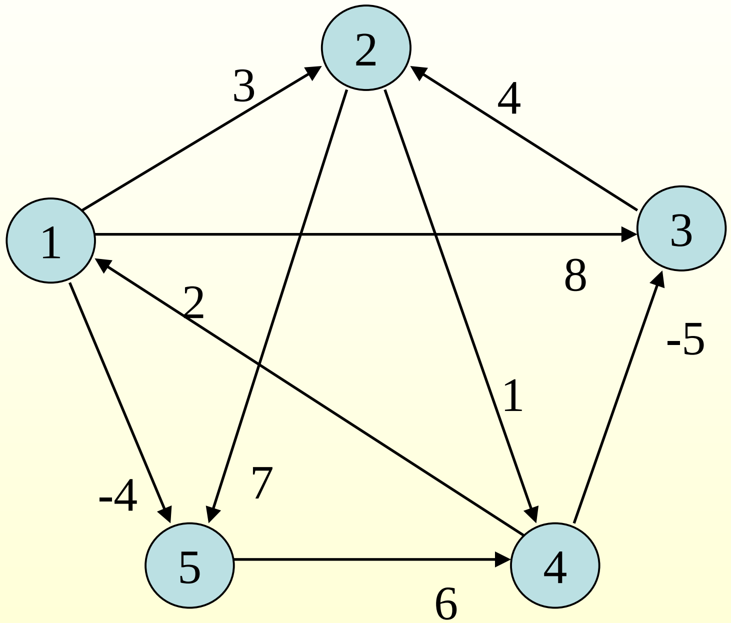
$$d_{ij}^m = \min_{1 \le k \le n} \{ d_{ik}^{m-1} + \omega_{kj} \} \text{ for all } v_k \in V$$

Shortest Paths and Matrix Multiplication

- Example



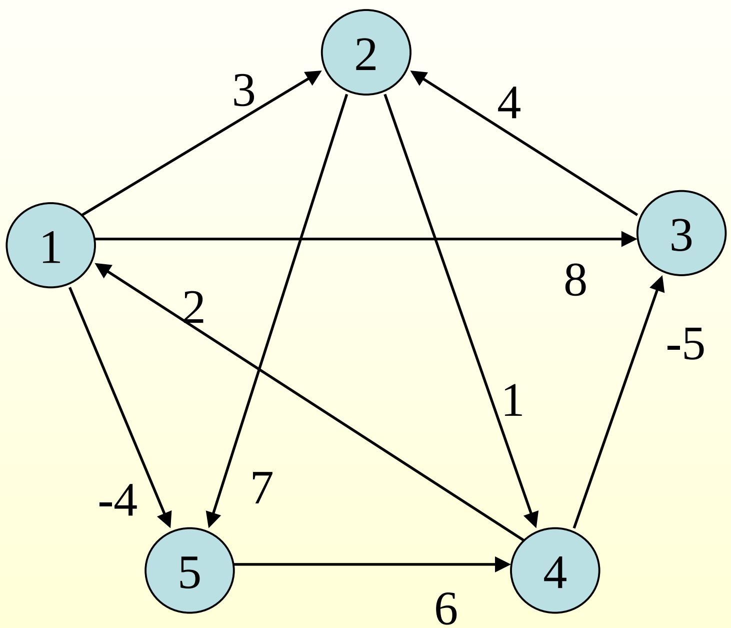
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$$D^1 = D^0 W$$

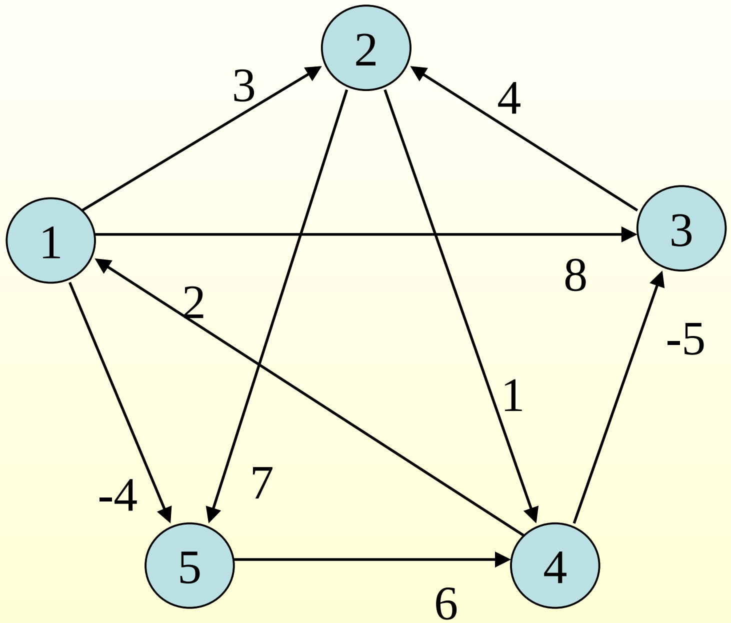
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	8	2	-4
2	3	0	-4	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	8	∞	1	6	0

$D^2 = D^1 W$

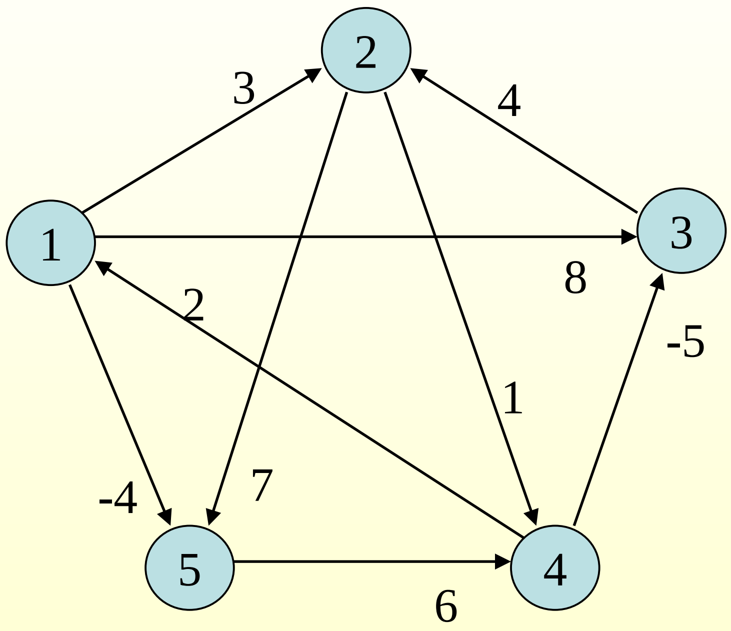
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	11
4	2	-1	-5	0	-2
5	8	5	1	6	0

$$D^3 = D^2 W$$

Shortest Paths and Matrix Multiplication

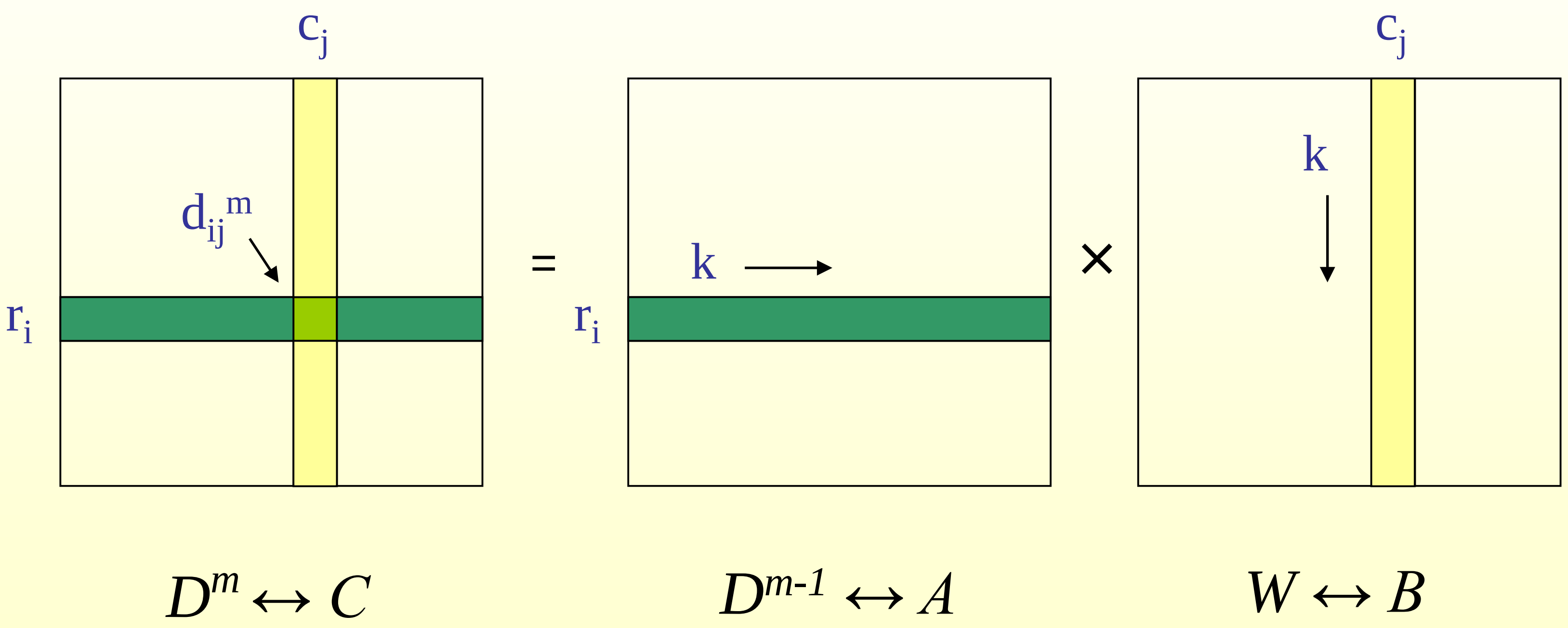


	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

$D^4 = D^3W$

SSSP and Matrix-Vector Multiplication

- relation of APSP to one step of matrix multiplication



SSSP and Matrix-Vector Multiplication

- $d_{ij}^{\,n-1}$ at row r_i and column c_j of product matrix $= \delta\,(\,v_i{=}s,\, v_j\,)$ for $j = 1, 2, 3, \ldots , n$

- row r_i of the product matrix = solution to single-source shortest path problem for $s = v_i$.

► r_i of C = matrix B multiplied by r_i of A

$$\Rightarrow D_i^{\,m} = D_i^{\,m-1} \times W$$

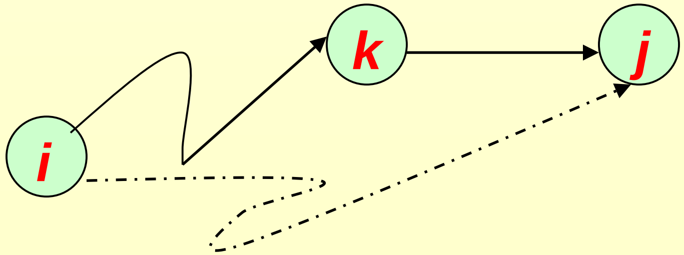
SSSP and Matrix-Vector Multiplication

- let $D_i^0 = d^0$, where $d_j^0 = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$
- we compute a sequence of n-1 "matrix-vector" products

$$d_i^1 = d_i^0 \text{ x } W$$
$$d_i^2 = d_i^1 \text{ x } W$$
$$d_i^3 = d_i^2 \text{ x } W$$
$$\vdots$$
$$d_i^{n-1} = d_i^{n-2} \text{ x } W$$

$$d_{ij}^{m} = \min_{1 \le k \le n} \{d_{ik}^{m-1} + \omega_{kj}\} \text{ for all } v_k \in V$$

Relaxing the edge (k,j)



SSSP and Matrix-Vector Multiplication

- this sequence of matrix-vector products
  - ► same as Bellman-Ford algorithm.
  - ► vector $d_i^m \Rightarrow$ d values of Bellman-Ford algorithm after m-th relaxation pass.
  - ► $d_i^m \leftarrow d_i^{m-1} x\ W$

    $\Rightarrow$ *m-th* relaxation pass over all edges.

SSSP and Matrix-Vector Multiplication

BELLMAN-FORD (G , v_i)

► perform RELAX (u , v) for

► every edge (u , v) ∈ E

for $j \leftarrow 1$ to n do

for $k \leftarrow 1$ to n do

RELAX (v_k , v_j)

RELAX (u , v)

$d_v = \min \{ d_v , d_u + \omega_{uv} \}$

EXTEND (d_i , W)

► d_i is an n-vector

for $j \leftarrow 1$ to n do

$d_j \leftarrow \infty$

for $k \leftarrow 1$ to n do

$d_j \leftarrow \min \{ d_j , d_k + \omega_{kj} \}$

Improving Running Time Through Repeated Squaring

- idea : goal is not to compute all D^m matrices
  - ► we are interested only in matrix D^{n-1}
- recall : no negative-weight cycles $\Rightarrow D^m = D^{n-1}$ for all $m \geq n-1$
- we can compute D^{n-1} with only $\lceil \lg(n-1) \rceil$ matrix products as

$$D^1 = W$$
$$D^2 = W^2 = W \times W$$
$$D^4 = W^4 = W^2 \times W^2$$
$$D^8 = W^8 = W^4 \times W^4$$
$$\vdots$$
$$D^{2^{\lceil \lg(n-1) \rceil}} = W^{2^{\lceil \lg(n-1) \rceil}} = W^{2^{\lceil \lg(n-1) \rceil - 1}} \times W^{2^{\lceil \lg(n-1) \rceil - 1}}$$

- This technique is called repeated squaring.

Improving Running Time Through Repeated Squaring

- FASTER-APSP (W)

    $D^1 \leftarrow W$

    $m \leftarrow 1$

    while $m < n\text{-}1$ do

        $D^{2m} \leftarrow$ EXTEND (D^m , D^m)

        $m \leftarrow 2m$

    return D^m

- final iteration computes D^{2m} for some $n\text{-}1 \leq 2m \leq 2n\text{-}2 \Rightarrow D^{2m} = D^{n\text{-}1}$

- running time : $\Theta(n^3 \lg n) = \Theta(V^3 \lg V)$

    ► each matrix product : $\lceil \Theta(n^3) \rceil$
    ► # of matrix products : $\lg(n\text{-}1)$
    ► simple code, no complex data structures, small hidden constants in Θ-notation.

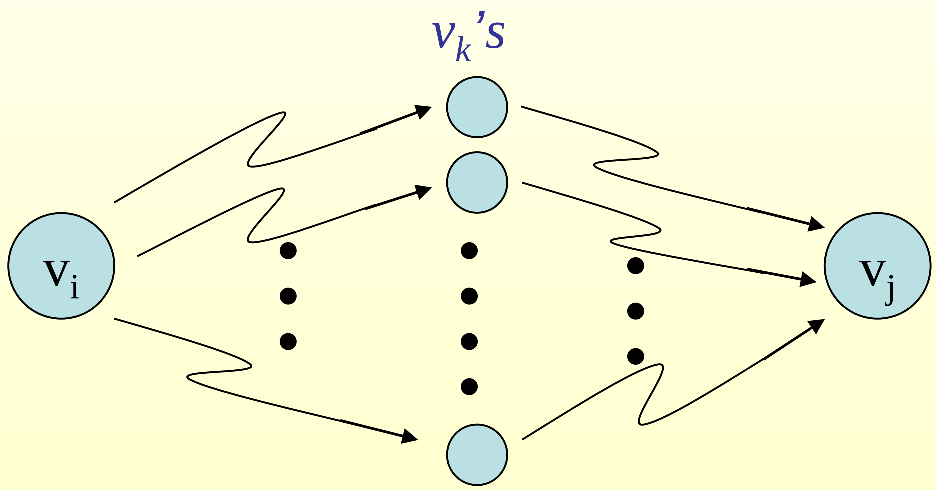
Idea Behind Repeated Squaring

- decompose p_{ij}^{2m} as p_{ik}^{m} & p_{kj}^{m}, where

p_{ij}^{2m} : $v_i \rightsquigarrow v_j$

p_{ik}^{m} : $v_i \rightsquigarrow v_k$

p_{kj}^{m} : $v_k \rightsquigarrow v_j$



A Different Way: the Floyd-Warshall Algorithm

- assumption : negative-weight edges, but no negative-weight cycles

(1) The Structure of a Shortest Path :

- Definition : intermediate vertex of a path $p = < v_1 , v_2 , v_3 , \ldots , v_k >$
  - ► any vertex of p other than v_1 or v_k .
- p_{ij}^{m} : a shortest path from v_i to v_j with all intermediate vertices from $V_m = \{ v_1 , v_2 , \ldots , v_m \}$
- relationship between p_{ij}^{m} and p_{ij}^{m-1}
  - ► depends on whether v_m is an intermediate vertex of p_{ij}^{m}

- case 1: v_m is not an intermediate vertex of p_{ij}^{m}

$\Rightarrow$ all intermediate vertices of p_{ij}^{m} are in V_{m-1}

$\Rightarrow p_{ij}^{m} = p_{ij}^{m-1}$

Floyd-Warshall Algorithm

- case 2 : v_m is an intermediate vertex of p_{ij}^{m}

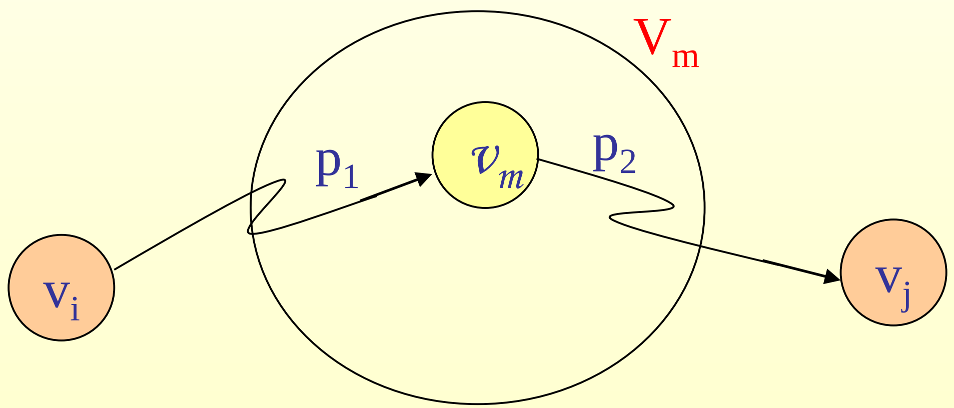
  - decompose path as $v_i \rightsquigarrow v_m \rightsquigarrow v_j$

    $\Rightarrow p_1 : v_i \rightsquigarrow v_m$ & $p_2 : v_m \rightsquigarrow v_j$

  - by opt. structure property both p_1 & p_2 are shortest paths.

  - v_m is not an intermediate vertex of p_1 & p_2

    $\Rightarrow p_1 = p_{im}^{m-1}$ & $p_2 = p_{mj}^{m-1}$



Floyd-Warshall Algorithm

(2) A Recursive Solution to APSP Problem :

- $d_{ij}^{m} = \omega(p_{ij})$: weight of a shortest path from v_i to v_j with all intermediate vertices from

  $V_m = \{ v_1, v_2, \dots, v_m \}$.

- note : $d_{ij}^{n} = \delta(v_i, v_j)$ since $V_n = V$
  - ► i.e., all vertices are considered for being intermediate vertices of p_{ij}^{n}.

Floyd-Warshall Algorithm

- compute d_{ij}^{m} in terms of d_{ij}^{k} with smaller $k < m$
- $m = 0$: V_0 = empty set

  $\Rightarrow$ path from v_i to v_j with no intermediate vertex.

  i.e., v_i to v_j paths with at most one edge

  $\Rightarrow d_{ij}^{0} = \omega_{i\,j}$

- $m \geq 1$: $d_{ij}^{m} = \min \{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$

Floyd-Warshall Algorithm

(3) Computing Shortest Path Weights Bottom Up :

FLOYD-WARSHALL(W)

► $D^0, D^1, \ldots, D^n$ are n x n matrices

for $m \leftarrow 1$ to n do

for $i \leftarrow 1$ to n do

for $j \leftarrow 1$ to n do

$d_{ij}^{m} \leftarrow \min\{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$

return D^n

Floyd-Warshall Algorithm

- maintaining *n* D matrices can be avoided by dropping all superscripts.
  - *m-th* iteration of outermost for-loop

    begins with $D = D^{m-1}$

    ends with $D = D^{m}$
  - computation of d_{ij}^{m} depends on d_{im}^{m-1} and d_{mj}^{m-1} .

    no problem if d_{im} & d_{mj} are already updated to d_{im}^{m} & d_{mj}^{m}
    since $d_{im}^{m} = d_{im}^{m-1}$ & $d_{mj}^{m} = d_{mj}^{m-1}$.

- running time : $\Theta(n^3) = \Theta(V^3)$

  simple code, no complex data structures, small hidden constants

Floyd-Warshall Algorithm

```
FLOYD-WARSHALL ( W )
      ► D  is an n x n matrix
      D ← W
      for m ← 1 to n do
         for i ← 1 to n do
            for j ← 1 to n do
               if  d_ij > d_im + d_mj  then
                  d_ij ← d_im + d_mj
      return D
```

Transitive Closure of a Directed Graph

- $G' = (V, E')$: transitive closure of $G = (V, E)$, where
  - ► $E' = \{ (v_i, v_j)$: there exists a path from v_i to v_j in G $\}$

- trivial solution : assign W such that

$$\omega_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \in E \\ \infty \quad \text{otherwise} \end{cases}$$

  - ► run Floyd-Warshall algorithm on W
  - ► $d_{ij}^{n} < n \Rightarrow$ there exists a path from v_i to v_j , i.e., $(v_i, v_j) \in E'$
  - ► $d_{ij}^{n} = \infty \Rightarrow$ no path from v_i to v_i , i.e., $(v_i, v_j) \notin E'$
  - ► running time : $\Theta(n^3) = \Theta(V^3)$

Transitive Closure of a Directed Graph

- Slightly better $\Theta(V^3)$ algorithm : saves time and space.
  - ► W = adjacency matrix : $\omega_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$
  - ► run Floyd-Warshall algorithm by replacing "min" → "$\vee$" & "+" → "$\wedge$"
- define $t_{ij}^{m} = \begin{cases} 1 & \text{if } \exists \text{ a path from } v_i \text{ to } v_j \text{ with all intermediate vertices from } V_m \\ 0 & \text{otherwise} \end{cases}$
  - ► $t_{ij}^{n} = 1 \Rightarrow (v_i, v_j) \in E'$ & $t_{ij}^{n} = 0 \Rightarrow (v_i, v_j) \notin E'$
- recursive definition for $t_{ij}^{m} = t_{ij}^{m-1} \vee (t_{im}^{m-1} \wedge t_{mj}^{m-1})$ with $t_{ij}^{0} = \omega_{ij}$

Transitive Closure of a Directed Graph

```
T-CLOSURE (G )
  ▶ T = ( t_ij ) is an n x n boolean matrix
    for i ← 1 to n do
      for j ← 1 to n do
        if  i = j  or ( v_i , v_j ) ∈ E  then
          t_ij ← 1
        else
          t_ij ← 0
    for  m ← 1 to n do
      for i ← 1 to n do
        for j ← 1 to n do
          t_ij  ← t_ij ∨ ( t_im ∧  t_mj )
```

Johnson's Algorithm for Sparse Graphs

- For sparse graphs it is attractive to think of running SSSP Dijkstra from on every vertex to solve APSP

$$V \text{ x } O(V\lg V + E) = O(V^2\lg V + EV)$$

if $E = O(V)$, then the above is $O(V^2\lg V)$

But Dijkstra requires non-negative edge weights …

Can we make all weights non-negative, while preserving the shortest path structure of the original graph?

Johnson's Algorithm for Sparse Graphs

(1) Preserving shortest paths by edge re-weighting :

- L1 : given $G = (V, E)$ with $\omega : E \rightarrow R$
  - ► let $h : V \rightarrow R$ be any weighting function on the vertex set
  - ► define $\hat{\omega}(\omega, h) : E \rightarrow R$ as $\hat{\omega}(u, v) = \omega(u, v) + h(u) - h(v)$
  - ► let $p_{0k} = < v_0, v_1, \ldots, v_k >$ be a path from v_0 to v_k

(a) $\hat{\omega}(p_{0k}) = \omega(p_{0k}) + h(v_0) - h(v_k)$

(b) $\omega(p_{0k}) = \delta(v_0, v_k)$ in $(G, \omega) \Leftrightarrow \hat{\omega}(p_{0k}) = \hat{\delta}(v_0, v_k)$ in $(G, \hat{\omega})$

(c) (G, ω) has a neg-wgt cycle $\Leftrightarrow$ $(G, \hat{\omega})$ has a neg-wgt cycle

Johnson's Algorithm for Sparse Graphs

- proof (a): $\hat{\omega}(p_{0k}) = \sum_{1 \le i \le k} \hat{\omega}(v_{i-1} ,v_i)$
  $= \sum_{1 \le i \le k} (\omega(v_{i-1} ,v_i) + h (v_0) - h (v_k))$
  $= \sum_{1 \le i \le k} \omega(v_{i-1} ,v_i) + \sum_{1 \le i \le k} (h (v_0) - h (v_k))$
  $= \omega(p_{0k}) + h (v_0) - h (v_k)$

- proof (b): ($\Rightarrow$) show $\omega(p_{0k}) = \delta (v_0 , v_k) \Rightarrow \hat{\omega}(p_{0k}) = \hat{\delta} (v_0 , v_k)$ by contradiction.
  - ► Suppose that $\exists$ a shorter path p_{0k}' from v_0 to v_k in $(G , \hat{\omega})$, then $\hat{\omega}(p_{0k}') < \hat{\omega}(p_{0k})$

- due to (a) we have

  - $\omega(p_{0k}') + h (v_0) - h (v_k) = \hat{\omega}(p_{0k}') < \hat{\omega}(p_{0k}) = \omega(p_{0k}) + h (v_0) - h (v_k)$
    $\omega(p_{0k}') + h (v_0) - h (v_k) < \omega(p_{0k}) + h (v_0) - h (v_k)$
    $\omega(p_{0k}') < \omega(p_{0k}) \Rightarrow$ contradicts that p_{0k} is a shortest path in (G , ω)

Johnson's Algorithm for Sparse Graphs

- proof (b): (<=) similar

- proof (c): ($\Leftrightarrow$) consider a cycle $c = < v_0 , v_1 , \ldots , v_k = v_0 >$.

  Due to (a)

  ► $\hat{\omega}(c) = \sum_{1 \leq i \leq k} \hat{\omega}(v_{i-1}, v_i) = \omega(c) + h(v_0) - h(v_k)$

  $= \omega(c) + h(v_0) - h(v_0) = \omega(c)$ since $v_k = v_0$

  ► $\hat{\omega}(c) = \omega(c)$.

QED

Johnson's Algorithm for Sparse Graphs

(2) Producing nonnegative edge weights by reweighting :

- given (G, ω) with $G = (V, E)$ and $\omega : E \to R$ construct an augmented graph (G', ω') with $G' = (V', E')$ and $\omega' = E' \to R$
  - ► $V' = V \cup \{ s \}$ for some new vertex $s \notin V$
  - ► $E' = E \cup \{ (s, v) : v \in V \}$
  - ► $\omega'(u,v) = \omega(u,v)\ \forall (u,v) \in E$ and $\omega'(s,v) = 0$, $\forall v \in V$

- vertex s has no incoming edges $\Rightarrow$
  - ► no shortest paths from $u \neq s$ to v in G' contains vertex s
  - ► (G', ω') has no neg-wgt cycle $\Leftrightarrow$ (G, ω) has no neg-wgt cycle

Johnson's Algorithm for Sparse Graphs

- suppose that G and G' have no neg-wgt cycle

- L2 : if we define $h(v) = \delta(s, v) \quad \forall v \in V$ in G' and $\hat{\omega}$ according to L1.

  ► we will have $\hat{\omega}(u,v) = \omega(u,v) + h(u) - h(v) \geq 0 \quad \forall\ v \in V$

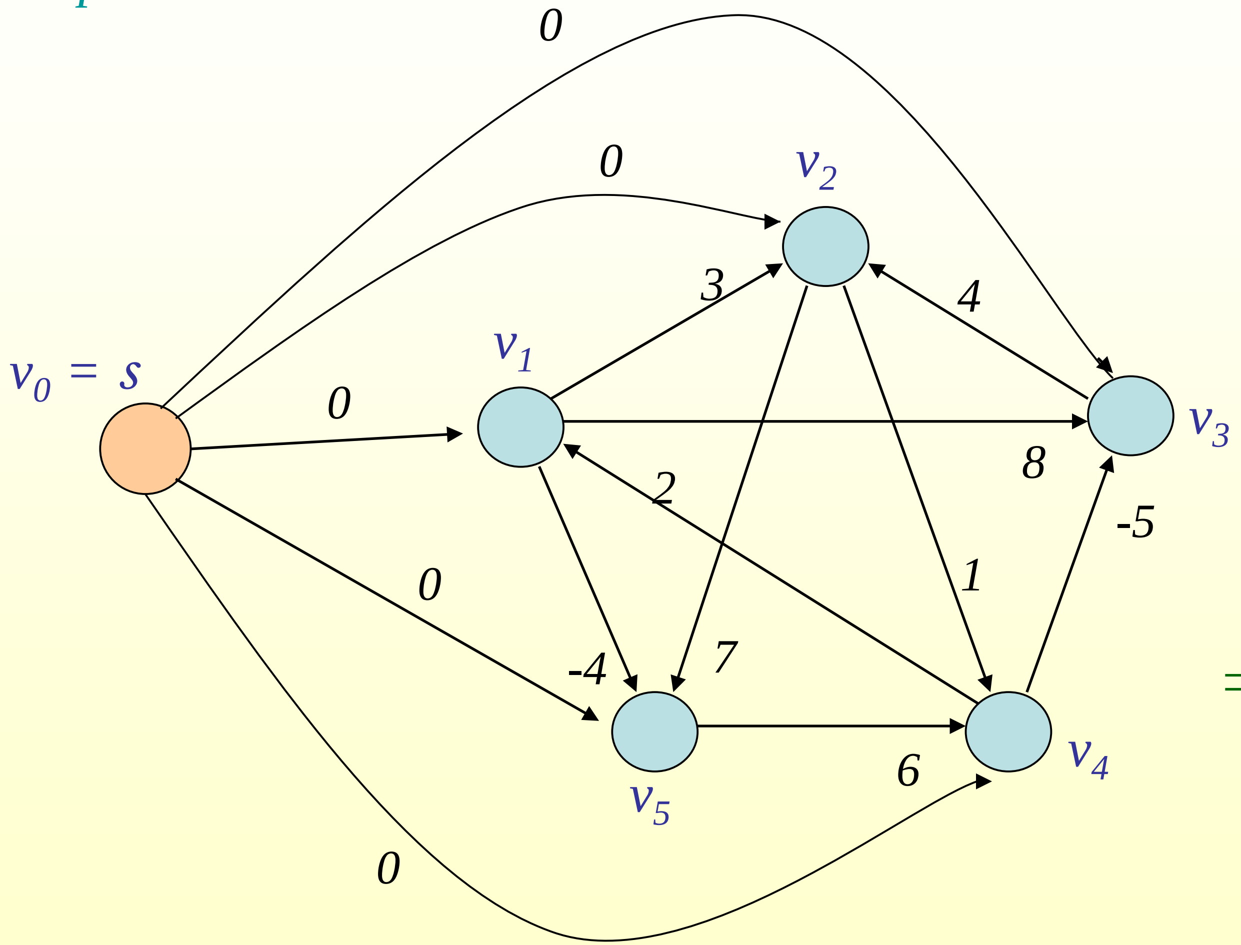
proof : for every edge $(u, v) \in E$

$$\delta(s, v) \leq \delta(s, u) + \omega(u, v) \text{ in } G' \text{ due to triangle inequality}$$

$$h(v) \leq h(u) + \omega(u, v) \Rightarrow 0 \leq \omega(u, v) + h(u) - h(v) = \hat{\omega}(u, v)$$

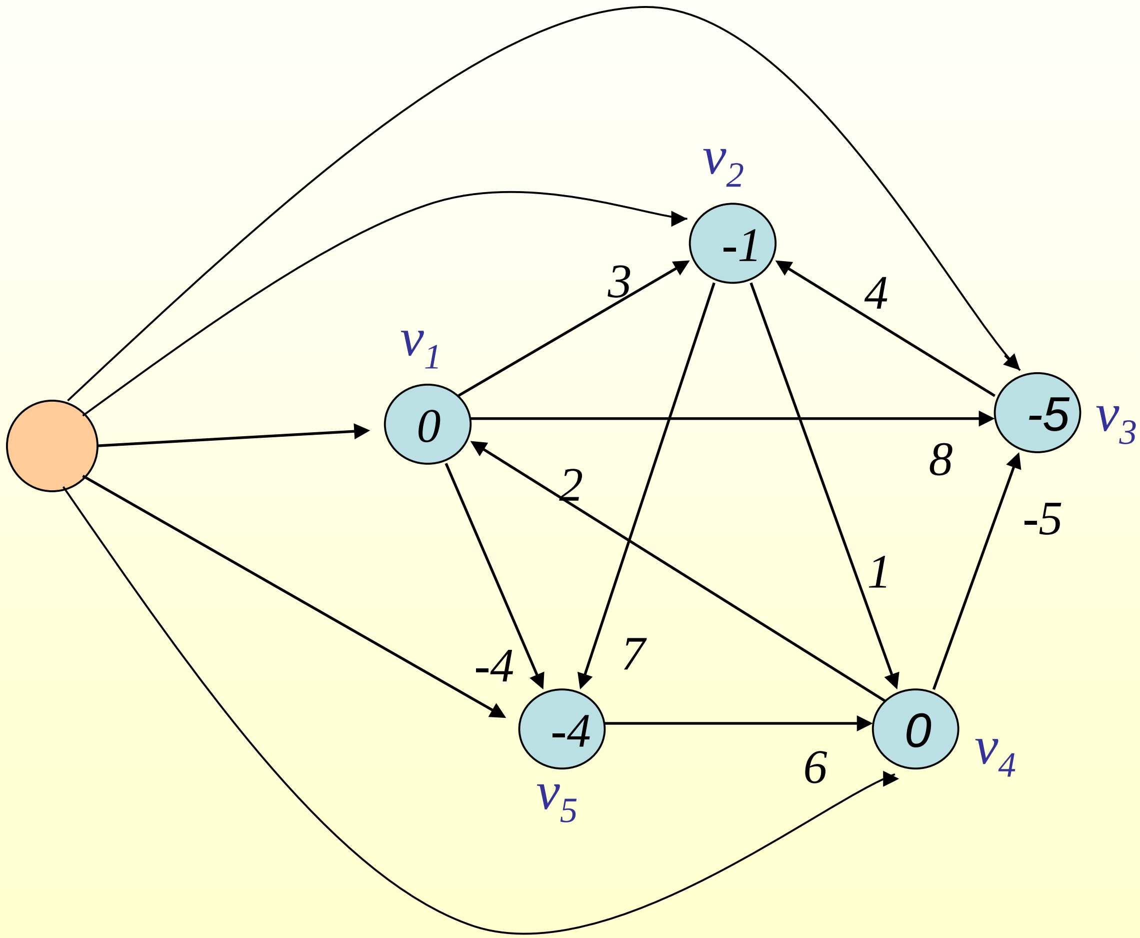
Johnson's Algorithm for Sparse Graphs

example :



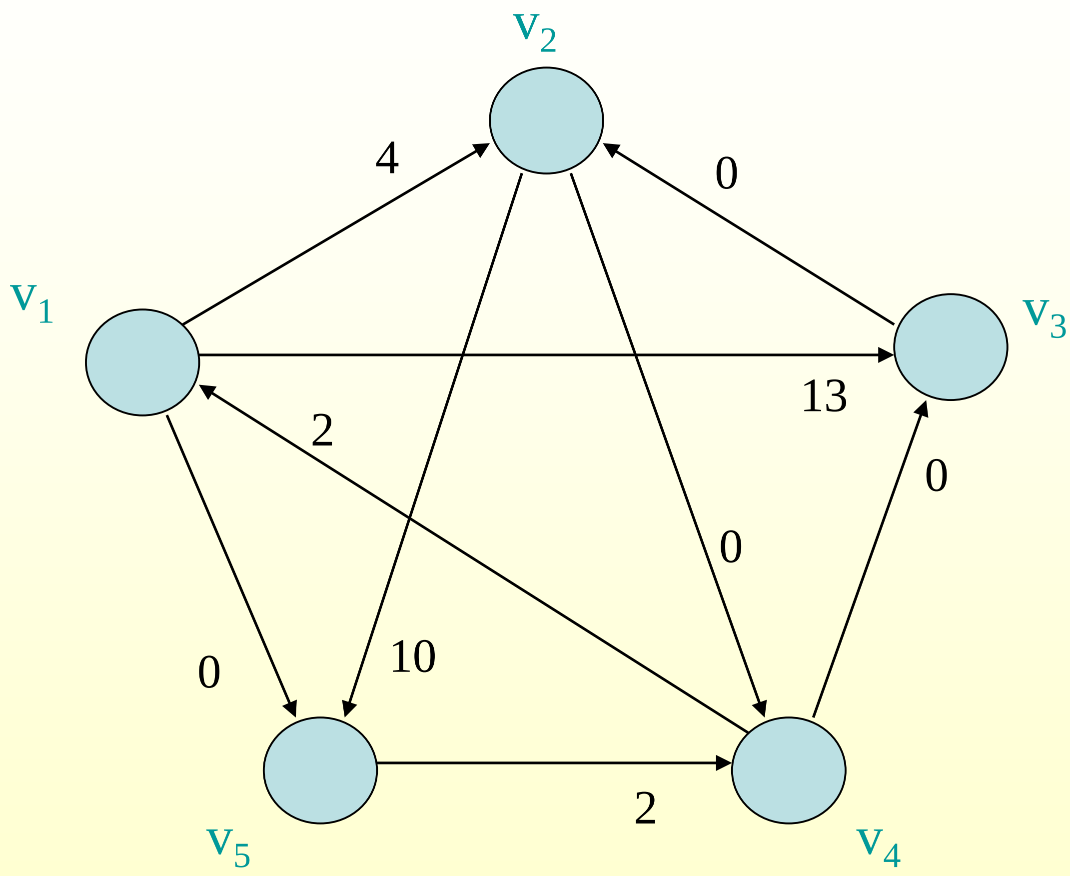
$\Rightarrow (G', \omega')$

Johnson's Algorithm for Sparse Graphs
Edge Reweighting



$\Rightarrow (G', \omega')$
with $h(v)$

Johnson's Algorithm for Sparse Graphs
Edge Reweighting



$\Rightarrow (G, \hat{\omega})$

Johnson's Algorithm for Sparse Graphs

Computing All-Pairs Shortest Paths

- adjacency list representation of G.
- returns n x n matrix D = (d_{ij}) where

  $d_{ij} = \delta_{ij}$,

  or reports the existence of a neg-wgt cycle.

Johnson's Algorithm for Sparse Graphs

- JOHNSON(G,ω)

  ► $D=(d_{ij})$ is an nxn matrix

  ► construct ($G' = (V', E')$, ω') s.t. $V' = V \cup \{s\}$; $E' = E \cup \{ (s,v) : \forall v \in V \}$

  ► $\omega'(u,v) = \omega(u,v)$, $\forall (u,v) \in E$ & $\omega'(s,v) = 0$ $\forall v \in V$

  if BELLMAN-FORD(G', ω', s) = FALSE then

      return "negative-weight cycle"

  else

      for each vertex $v \in V' - \{s\} = V$ do

          $h[v] \leftarrow d'[v]$ ► $d'[v] = \delta'(s,v)$ computed by BELLMAN-FORD(G', ω', s)

      for each edge $(u,v) \in E$ do

          $\hat{\omega}(u,v) \leftarrow \omega(u,v) + h[u] - h[v]$ ► edge reweighting

      for each vertex $u \in V$ do

          run DIJKSTRA$(G, \hat{\omega}, u)$ to compute $\hat{d}[v] = \hat{\delta}(u,v)$ for all v in $V \in (G,\hat{\omega})$

          for each vertex $v \in V$ do

              $d_{uv} = \hat{d}[v] - (h[u] - h[v])$

  return D

Johnson's Algorithm for Sparse Graphs

- running time : $O(V^2 \lg V + EV)$
  - ► edge reweighting
    - BELLMAN-FORD(G', ω', s) : $O(EV)$
    - computing $\hat{\omega}$ values : $O(E)$
  - ► |V| runs of DIJKSTRA : $|V| \times O(V \lg V + EV)$
    - $= O(V^2 \lg V + EV)$;
    - PQ = Fibonacci heap