3D Transformations

3D Rotations about an axis through the origin

if not: conjugation trick

Orientation vs. Rotation

- orientation is a <u>state</u>

  can be described by a rotation from a reference pose

- rotation is an action / motion / transformation

like

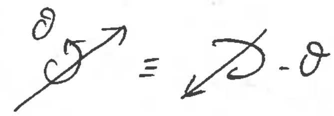
point	–	vector
orientation	–	rotation
state	–	movement

Many representations for 3D rotations, sometimes inconsistently handled

- axis & angle
- rotation matrix
- Euler angles
- quaternions

"not a linear object"

"circular"



$SO(3) \approx \mathbb{R}P^3$

Euler's Theorem: The general displacement of a rigid body with one point fixed is a rotation about some axis.

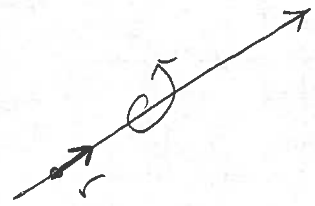
3 degrees of freedom (rot through origin)

5 degrees in general

4 = line + 1 = angle

Rotation representations

1. axis & angle — most intuitive



directional cosines

r a unit length vector at the origin

$$r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \qquad r_x^2 + r_y^2 + r_z^2 = 1$$

α an angle (radians)

sometimes $\omega = \alpha r$

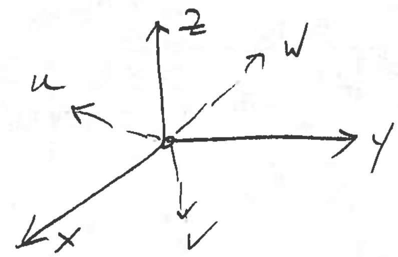
difficult to apply rotations to points or vectors ~~to~~ this form

2. rotation matrices — easiest for applying to points or vectors

$$M = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4 directional cosine matrix

cols of M are the unit vectors of the new frame after rotation



M is a real, orthogonal matrix

$u_x^2 + u_y^2 + u_z^2 = 1 \qquad u_x v_x + u_y v_y + u_z v_z = 0$

$v_x^2 + v_y^2 + v_z^2 = 1 \qquad v_x u_x + v_y w_y + v_z w_z = 0$ [col. relations]

$v_x^2 + w_y^2 + w_z^2 = 1 \qquad w_x v_x + w_y v_y + w_z v_z = 0$

9 uknowns − 6 relations = 3 degrees of freedom

Note that

$M^{-1} = M^{T}$ is also an orthogonal matrix,

so dual relations

$u_x^2 + v_x^2 + w_x^2 = 1,$ [row relations]

$MM^T = M^TM = I$ $\quad u_x u_y + v_x v_y + w_x w_y = 0$

also hold

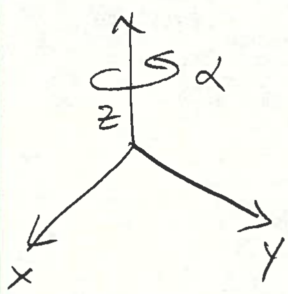
$\det(M) = +1 \quad \det^2(M) = 1$

$[\det(M) = -1$ are the reflections / glide reflections$]$

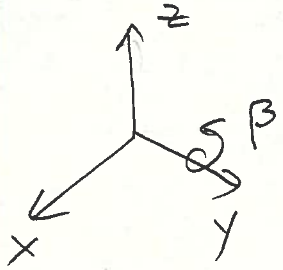
$O(3)$, $SO(3)$

Note that eigenvalues of M are $\{1, e^{\pm i\theta}\} = \{1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta\}$

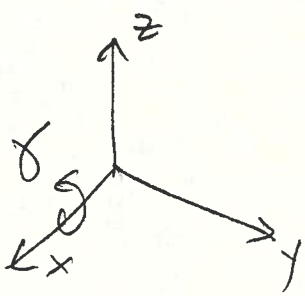
So what do these matrices look like? 3×3



$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

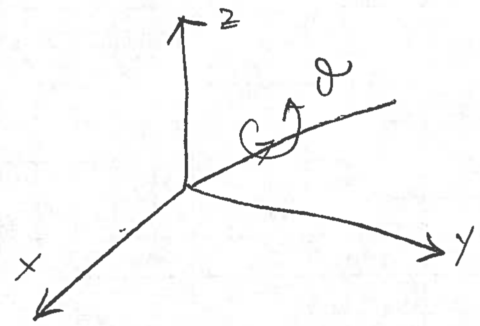


$$\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

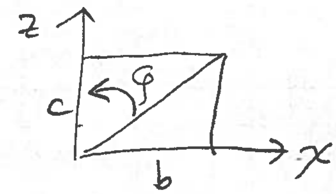


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

arbitrary axis case



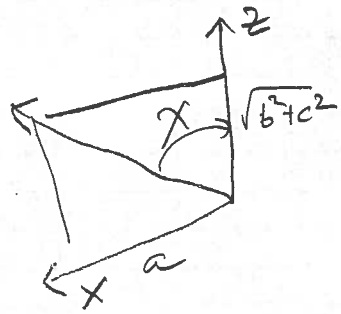
a, b, c $\quad a^2+b^2+c^2$



put axis on xz plane
by an x rotation
R_x^{φ}

$$\cos\varphi = \frac{c}{\sqrt{b^2+c^2}}$$

$$\sin\varphi = \frac{b}{\sqrt{b^2+c^2}}$$



$$\cos\chi = \frac{\sqrt{b^2+c^2}}{\sqrt{\ }} \qquad \sin\chi = \frac{a}{\sqrt{\ }}$$

align axis with z
by an y rotation
R_y^{χ}

$$R_x^{-\varphi} R_y^{-\chi} R_z^{\vartheta} R_y^{\chi} R_x^{\varphi} \qquad \leftarrow R_x(\varphi)$$

General formula (Rodrigues)

$c = \cos\theta \qquad s = \sin\theta$

$$R = c I + (1-c)\, r r^T + s\, r^{\wedge}$$

↑ identity

$$r^{\wedge} = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$

skew symmetric
$[\, r^{\wedge T} = -r^{\wedge} \,]$

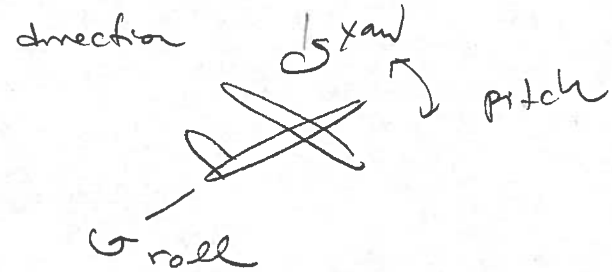
$$r^{\wedge} v = r \times v = \begin{pmatrix} r_y v_z - r_z v_y \\ r_z v_x - r_x v_z \\ r_x v_y - r_y v_x \end{pmatrix}$$

Euler angles

It is possible to specify arbitrary rotations about an axis through the origins as compositions of $\underline{3}$ axis-aligned (x, y, z) rotations

e.g. roll, pitch, yaw for aircraft

↑
ascend
descend



several variations : order of axes

fixed or moving axes

XYZ fixed

$\circlearrowleft x$	γ
$\circlearrowleft y$	β
$\circlearrowleft z$	α

$$R_z(\alpha)\, R_y(\beta)\, R_x(\gamma)$$

$$\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} =$$

$$\begin{bmatrix} c\alpha\, c\beta & c\alpha\, s\beta\, s\gamma - s\alpha\, c\gamma & c\alpha\, s\beta\, c\gamma + s\alpha\, s\gamma \\ s\alpha\, c\beta & s\alpha\, s\beta\, s\beta + c\alpha\, c\gamma & s\alpha\, s\beta\, c\gamma - c\alpha\, s\gamma \\ -s\beta & c\beta\, s\gamma & c\beta\, c\gamma \end{bmatrix}$$

how to compute α, β, γ ? $\begin{bmatrix} & & \\ & & \end{bmatrix}$

From matrices to Euler angles

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$X - Y - Z$ fixed
$\alpha \quad \beta \quad \gamma$

$$\beta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$\alpha = \mathrm{Atan2}\left(r_{21}/c\beta, r_{11}/s\beta\right)$$
$$\gamma = \mathrm{Atan2}\left(r_{32}/c\beta, r_{33}/c\beta\right)$$

$\mathrm{Atan2}(y, x)$ $\quad$ $\tan^{-1}\left(\frac{y}{x}\right)$

$\uparrow \uparrow$ uses both signs

single solution $\quad -90 \le \beta \le 90$

Another Euler theorem

Any rotation about the origin can be expressed as the product of 3 rotations around the axes k, ℓ, m, where $k \neq \ell$, $\ell \neq m$

Z-Y-X Euler

Rot ∂z α

∂y β

∂x γ

moving system

mostly used by animation software

XYZ fixed = ZYX Euler

Z-Y-Z Euler

∂z α

∂Y β

∂z γ

24 different angle conventions

fixed / Euler

dmy dmy

different