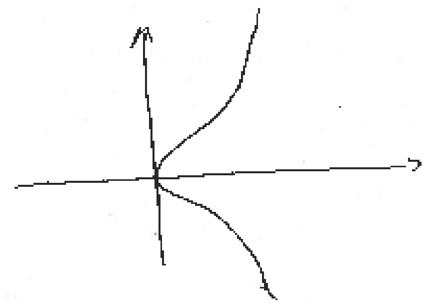
Differential Geometry of planar curves.

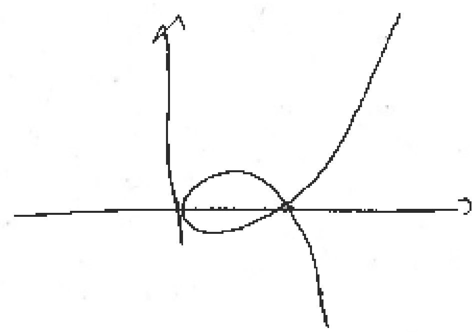
1. Humpy

$x(t) = t^2$
$y(t) = t^3 + t$



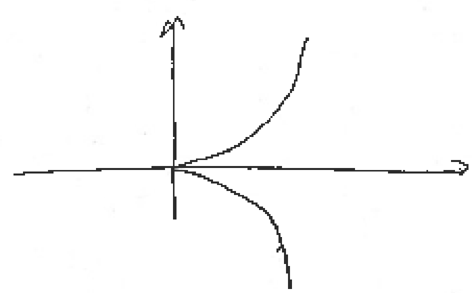
2. Loopy

$x(t) = t^2$
$y(t) = t^3 - t$



3. Pointy

$x(t) = t^2$
$y(t) = t^3$



In this lecture, we will only consider curves that are differentiable. I.e. $c'(t)$ exists.

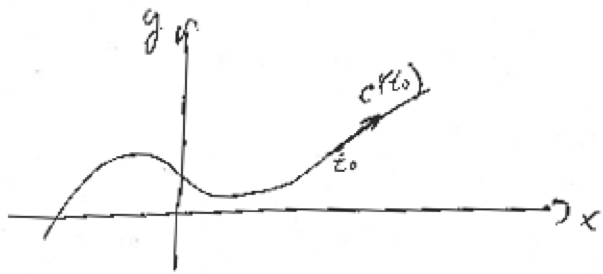
$$c = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \quad c'(t) = \frac{dc(t)}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Tangent Vectors

Def: Given a parametric curve $c(t)$, let $c'(t)$ be its <u>tangent vector</u>.

Tangent vectors approximate $c(t)$ to first order:
Taylor series approximating $c(t)$ around t_0:

$$c(t) = c(t_0) + c'(t_0)(t - t_0) + O((t-t_0)^2)$$



Around t_0, $c(t)$ looks like a line, ~~whose~~ in the direction $c'(t_0)$. Direction is much more important than magnitude.

For differential geometry it is important to define tangent vectors everywhere.

<u>Singular point</u>: a point t_0, such that $c'(t_0) = \underline{0}$.

Example pointy cubic with $t_0 = 0$.

$$c'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

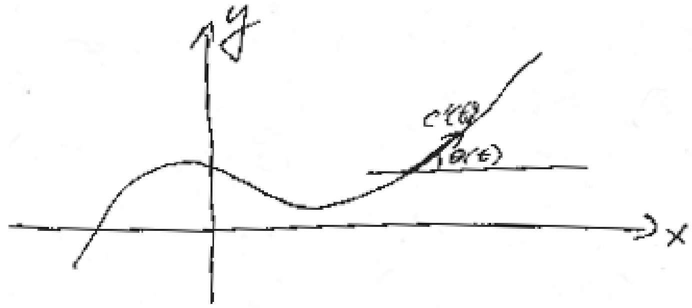
<u>Regular curve</u>: a curve without singular points.

$c'(t) \neq 0 \quad \forall t.$ Loopy $c'(t) = \begin{pmatrix} 2t \\ 2t-1 \end{pmatrix} \neq 0$

Humpy $c'(t) = \begin{pmatrix} 2t \\ 2t+1 \end{pmatrix} \neq 0$

Signed Curvature

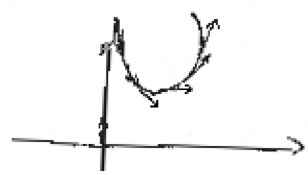
Let $\theta(t)$ be the angle between $c'(t)$ and x-axis:



Signed curvature $\kappa(t) = \frac{d\theta(t)}{d(t)} = \theta'(t)$.

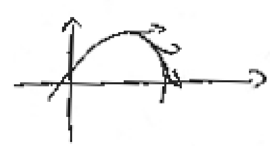
Curvature measures the change in the direction of tangent vectors. How much a curve bends.

~~High~~ Positive curvature:



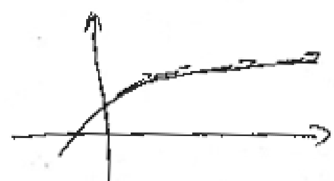
angle is increasing

Negative curvature:



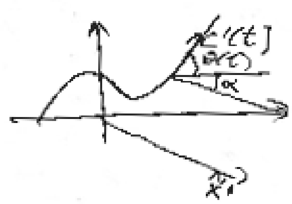
angle is decreasing

Small absolute value of curvature:



angle is almost constant. Curve is flat

Note that the axis can be chosen arbitrarily!



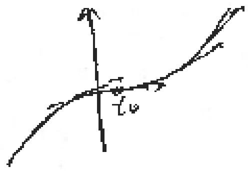
$$\tilde{\theta}(t) = \theta(t) + \alpha$$

$$\tilde{\theta}'(t) = \theta'(t)$$

Orientation changes the sign.

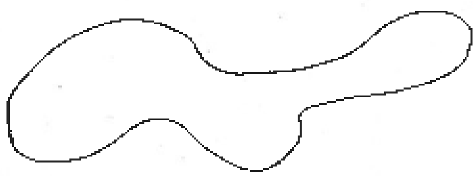
Inflection Points:

Points at which curvature changes sign.



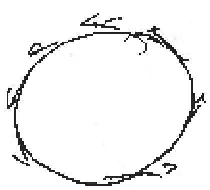
Total curvature.

Suppose $c(t)$ is a closed regular curve.
$c(t_0) = c(t_1)$ for some t_0, t_1

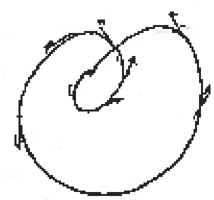


Then $\int_{t_0}^{t_1} \kappa(t)\,dt = 2\pi I$ for some integer I

$$\int_{t_0}^{t_1} \kappa(t)\,dt = \int_{t_0}^{t_1} \theta'(t)\,dt = \theta(t_1) - \theta(t_0) = 2\pi I.$$



$I = 1$

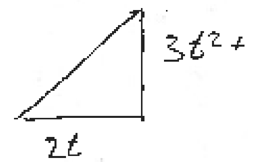


$I = 2$.

Example:

Inflection points of Loopy, Humpy cubics:

$$c(t) = \begin{pmatrix} t^2 \\ t^3+1 \end{pmatrix} \qquad c'(t) = \begin{pmatrix} 2t \\ 3t^2+1 \end{pmatrix}$$



$$\theta(t) = \arctan\left(\frac{3t^2+1}{2t}\right)$$

$$\theta'(t) = \frac{1}{1+\left(\frac{3t^2+1}{2t}\right)^2}\left(\frac{6t\cdot 2t - 2(3t^2+1)}{4t^2}\right)$$

$$= \frac{1}{1+\left(\frac{3t^2+1}{2t}\right)}\left(\frac{12t^2-6t^2-2}{4t^2}\right)$$

$$= \underbrace{\frac{1}{1+\left(\frac{3t^2+1}{2t}\right)}}_{\geqslant 0}\underbrace{\left(\frac{3t^2-1}{2t^2}\right)}_{\frac{(\sqrt{3}t-1)(\sqrt{3}t+1)}{2t^2}}$$

$t = \pm\frac{1}{\sqrt{3}}$ is an inflection point.

Loopy

$$\theta'(t) = \frac{1}{1+\left(\frac{3t^2-1}{2t}\right)}\left(\frac{3t^2+1}{2t^2}\right) \geqslant 0 \quad \forall t$$

Fundamental Theorem of Planar curves

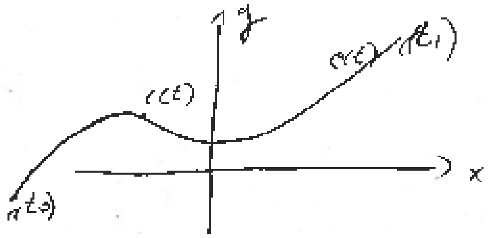
Given a differentiable function $\kappa(s)$ there exists a curve $c(t)$ whose curvature is $\kappa(s)$. Moreover any other such curve can be obtained by rotation and translation of $c(t)$.

Curvature determines $c(t)$ up to <u>a rigid motion</u>.
Second order property.

Note that the curve is also unique up to reparametrization. Is there a natural, "geometric" parametrization of a curve?

Arc-Length Parametrization

Arc length of a curve:



$c(t)$; $t_0 \le t \le t_1$

Length of a curve between t_0 and t.

$$l(t) = \int_{t_0}^{t} \| c'(s) \|_2 \, ds = \int_{t_0}^{t} \left(c'(s)^T c'(s) \right)^{\frac{1}{2}} ds$$

$$= \int_{t_0}^{t} \left(x'(s)^2 + y'(s)^2 \right)^{\frac{1}{2}} ds$$

If $c'(s) \neq 0 \quad \forall s$

$\ell(t)$ is a monotonically increasing function. Has an inverse:

$$\ell^{-1}(\alpha) = t \quad \text{s.t.} \quad \int_{t_0}^{t} \| c'(s) \|_2 \, ds = \alpha \quad , \quad \begin{matrix} 0 \le \alpha \le \ell(t_1) \\ t_0 \le t \le t_1 \end{matrix}$$

Define:

$$\tilde{c}(\alpha) = c(\ell^{-1}(\alpha)) \quad , \quad 0 \le \alpha \le \ell(t_1)$$

Note $\tilde{c}(\alpha) \in c(\cdot)$ - lies on the trace of c

Conversely, $\forall t \; \exists \alpha$ s.t. $\ell^{-1}(\alpha) = t$ so

$$\text{Trace}(c) = \text{Trace}(\tilde{c})$$

Also remember:

$$\int_{t_0}^{\ell^{-1}(\alpha)} \| c'(s) \|_2 \, ds = \alpha$$

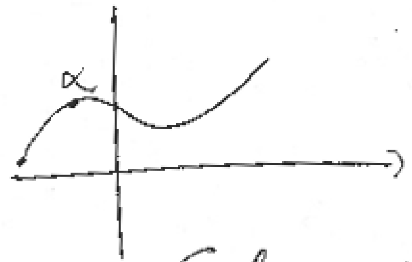
Differentiate both sides: w.r.t. α.

$$\| c'(\ell^{-1}(\alpha)) \|_2 = 1 \quad \forall \alpha$$

Or $\| \tilde{c}'(\alpha) \|_2 = 1$.

And $\displaystyle\int_0^{\alpha} \| \tilde{c}'(s) \|_2 \, ds = \int_0^{\alpha} ds = \alpha$

I.e. α is the length of $\tilde{c}$, between 0 and α.



Each point now gets a name α, which is the length of the curve between 0 and itself.

Arc length parametrization $\tilde{c}$ exists for any regular curve c

Curvature $K(s)$ is defined as:

$$\boxed{\|\tilde{c}''(s)\|}$$

Purely geometric definition.

Extends to $\mathbb{R}^3$. Has very many nice properties

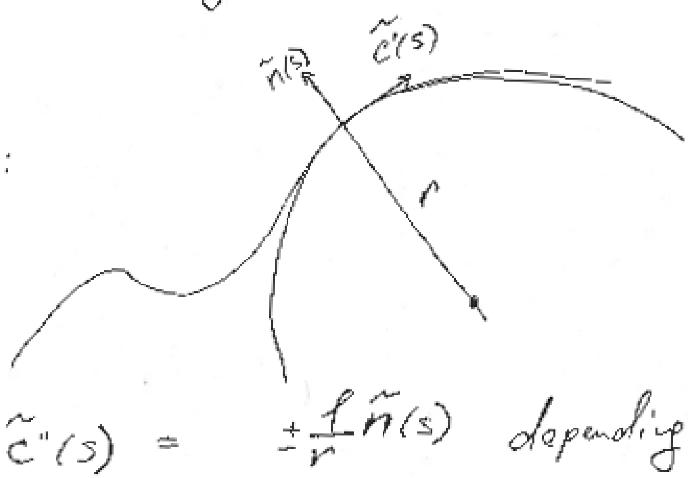
$\tilde{c}'(s)^T \tilde{c}'(s) = 1$ differentiate both sides w.r.t. s:

$\Rightarrow \tilde{c}''(s)^T \tilde{c}'(s) = 0$ or

$\tilde{c}''(s)$ is perpendicular to $\tilde{c}'(s)$.

In the plane, the only vector perp. to $\tilde{c}'(s)$ is the normal. $\tilde{n}(s)$

Osculating circle:



$$r = \frac{1}{\|K(s)\|} \quad \text{or}$$

$K(s) = \frac{1}{r} \Rightarrow \tilde{c}''(s) = \pm\frac{1}{r}\tilde{n}(s)$ depending on normal orientation.