CS164: Voronoi/Delaunay Diagrams, Distance Functions

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Distance Functions

Object Matching Queries

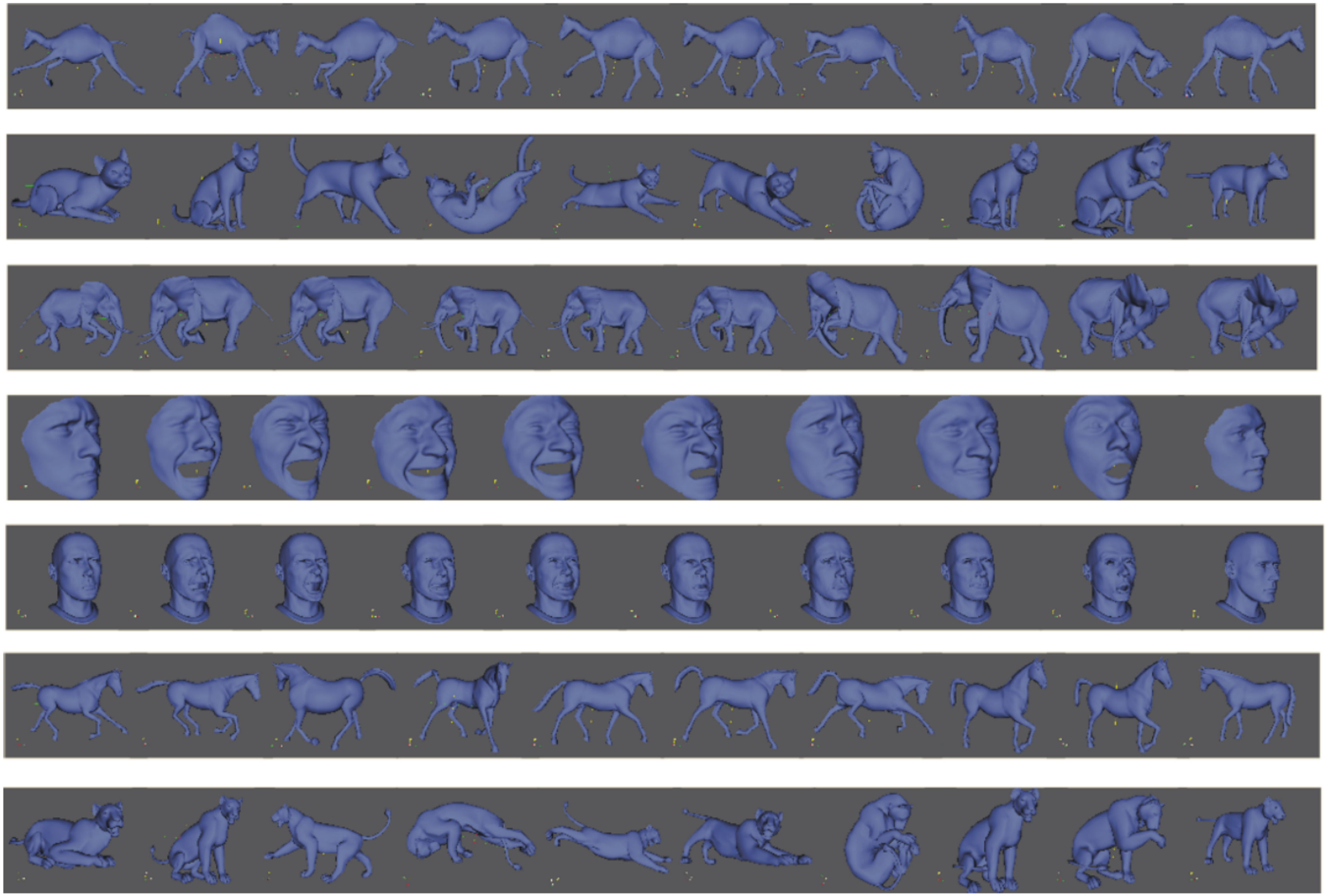
- assume you have database $\mathcal{D}$ of objects.
- assume $\mathcal{D}$ is composed by several objects, and that each of these objects belongs to one of n classes $C_1, \ldots, C_n$.
- imagine you are given a new object o, not in your database, and you are asked to determine whether o belongs to one of the classes. If yes, you also need to point to the class.
- One simple procedure is to say that you will assign object o the class of the *closest* object in $\mathcal{D}$:

$$\text{class}(o) = \text{class}(z)$$

$$\text{where } z \in \mathcal{D} \text{ minimizes } \mathbf{dist}(o, z)$$

- in order to do this, one first needs to define a notion **dist** of *distance* or *dis-similarity between objects.*

Useful for Object/Shape Classification

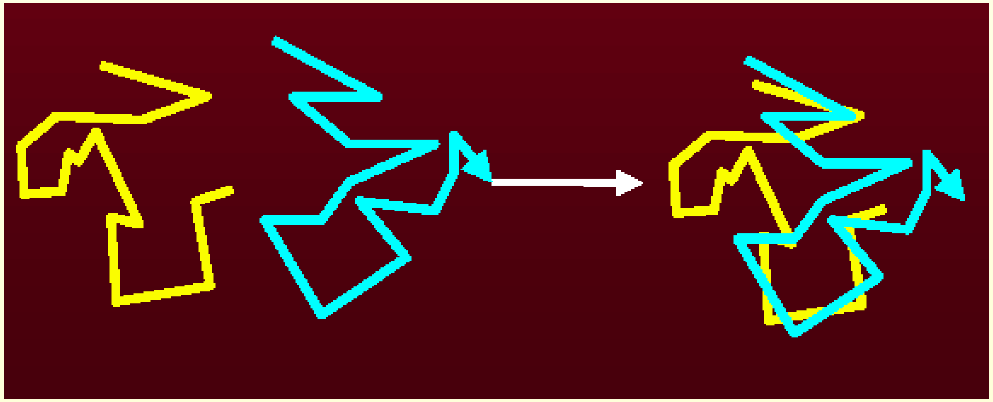


More Shape Similarity Methods

- Hausdorff distance
- Fréchet distance
- Morphing metrics

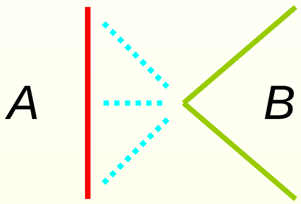
Proteins are defined as having a common fold if they have the same major secondary structures in the same arrangement and with the same topological connections

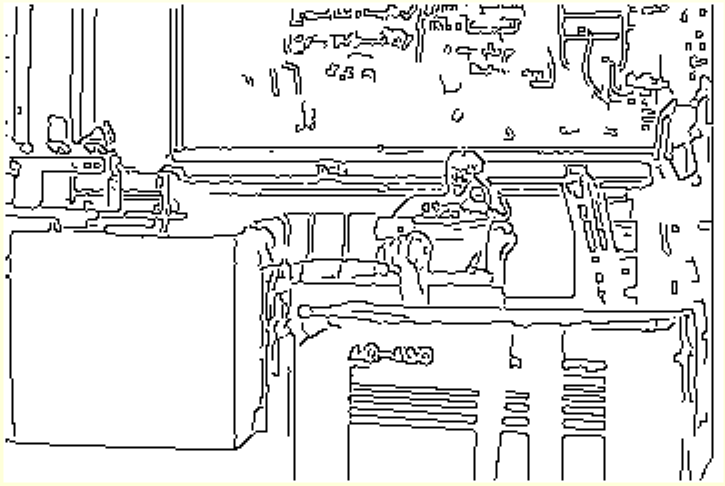
(SCOP)



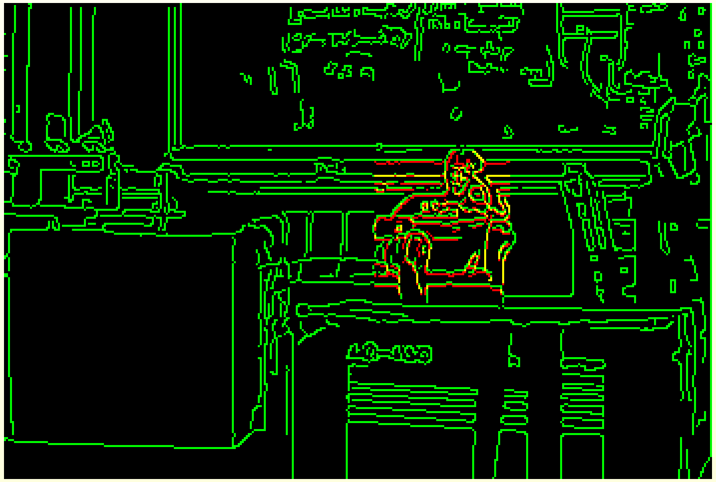
Hausdorff Distance

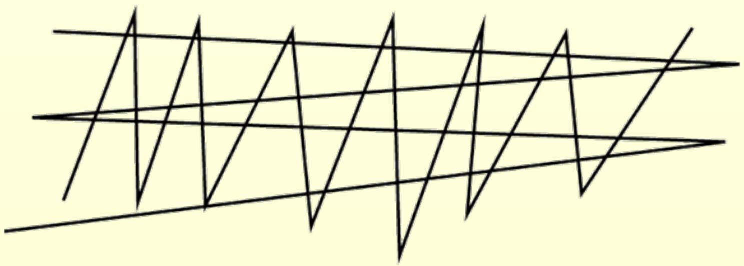
Many-to-many correspondences can be simpler ...





[Huttenlocher *et. al*., 93]





Hausdorff Definition

We are two point sets $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ in E^2. The one-sided Hausdorff distance from A to B is defined as:

$$\tilde{\delta}_H(A, B) = \max_{a \in A} \min_{b \in B} ||a - b||$$

The (bidirectional) Hausdorff distance between A and B is then defined as:

$$\delta_H(A, B) = \max\left(\tilde{\delta}_H(A, B), \tilde{\delta}_H(B, A)\right)$$

For fixed A and B, it can easily be computed in time $O((n+m) \log(n+m))$

Hausdorff Variations

- Order statistics – use percentile max (say the 90% largest distance from *A* to *B*) to avoid undue impact of outliers (fractional Hausdorff)
- Typically, one of the sets (say *B*) may be moved by a transformation group *G*

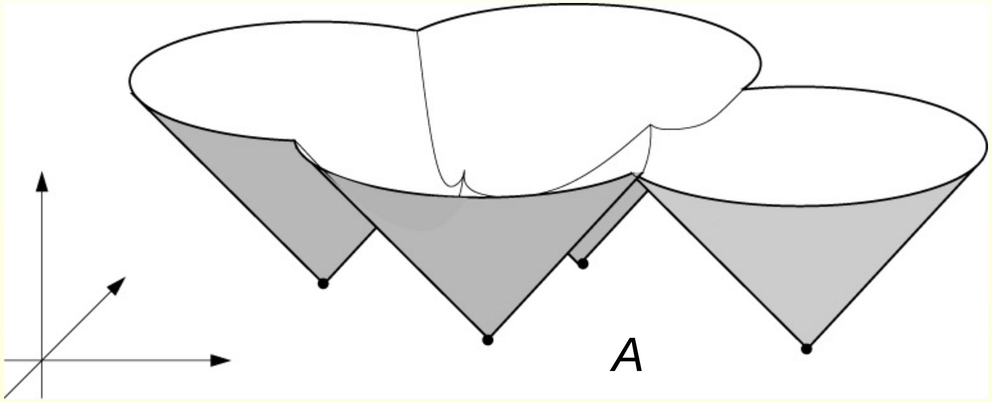
$$\tilde{\delta}_{H,\mathcal{G}}(A,B) = \min_{T \in \mathcal{G}} \max_{a \in A} \min_{b \in B} ||a - T(b)||$$

- Both “vector” and “raster” methods can be used

Computing Hausdorff

In the plane, vector form, under translation ...

The Voronoi surface of A, a piecewise conical surface



A lower envelope surface

$$d(x) = \min_{a \in A} \|x - a\|$$

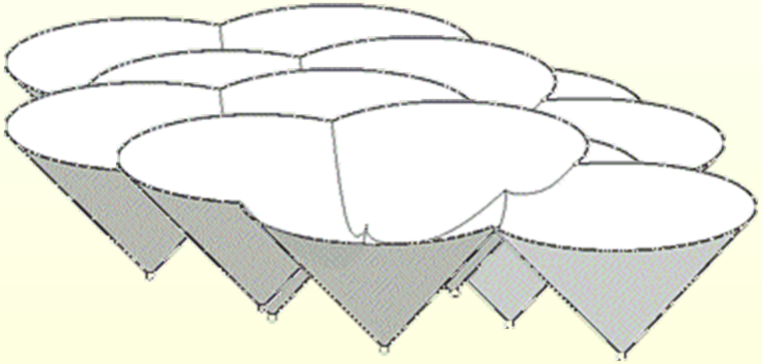
Translate *B* by *t*

$$\delta_b(t) = \min_{a \in A} \|a - (b + t)\| = \min_{a \in A} \|(a - b) - t\| = d_{-b}(t)$$

Computing Hausdorff, II

$$f(t) = \tilde{\delta}_H(B + t, A) = \max_{b \in B} \delta_b(t)$$

Upper envelope of *m* Voronoi surfaces
A-b_1, A-b_2, ..., A-b_m

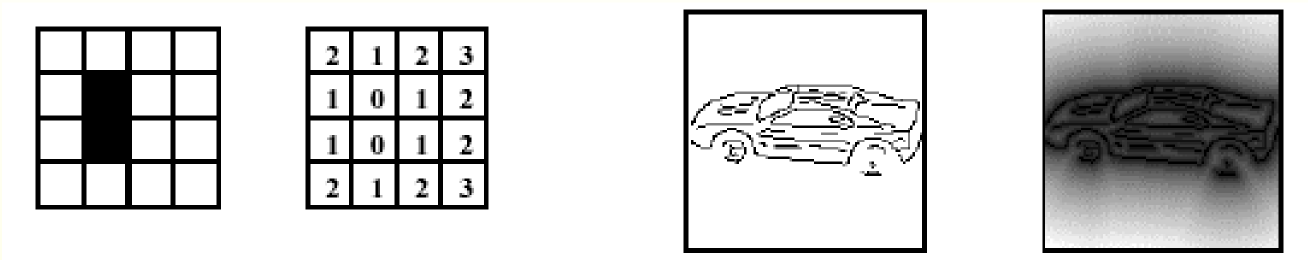


Can be done in time
$O(nm(n+m) \text{ polylog}(n+m))$

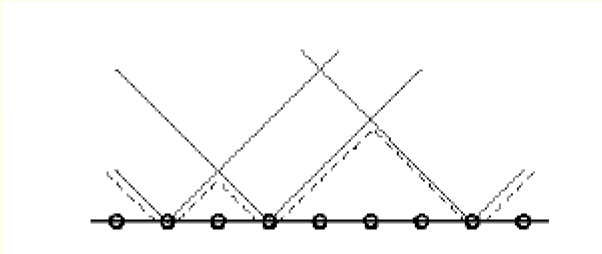
The amount of computation gets out of hand when we allow rotations and go to 3-D.

Raster Hausdorff

Distance transforms computed on a grid



(fast marching, level sets, ...)



∞	0	∞	0	∞	∞	∞	0	∞
∞	0	1	0	1	2	3	0	1
1	0	1	0	1	2	1	0	1

A 1-d example

Fast Hausdorff Search

- Branch and bound hierarchical search of transformation space
- Consider 2-D transformation space of translation in *x* and *y*
  - (Fractional) Hausdorff distance cannot change faster than linearly with translation
    - Similar constraints for other transformations
  - “Quad-tree” decomposition, compute distance for transform at center of each cell
    - If larger than cell half-width, rule out cell
    - Otherwise subdivide cell and consider children

Fast Hausdorff Search, II

- Guaranteed (or admissible) search heuristic
  - Bound on how good answer could be in unexplored region
    - Cannot miss an answer
  - In worst case won't rule anything out.
  - In practice rule out vast majority of transformations
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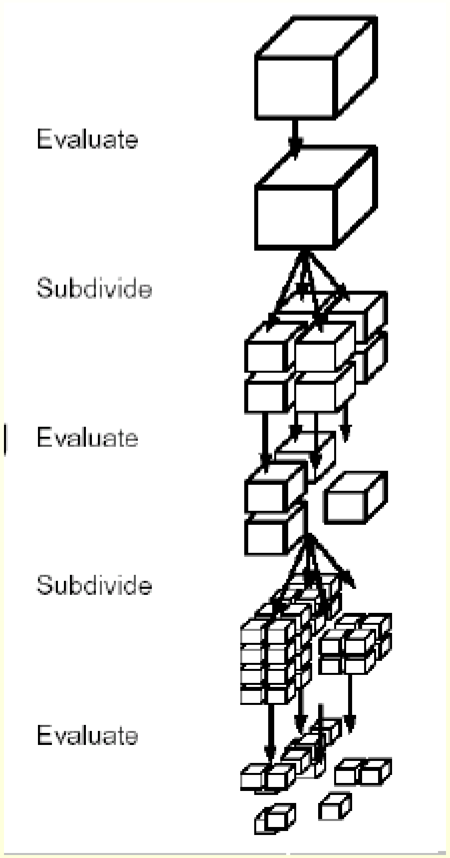
  - Can use even simpler tests than computing distance at each cell center



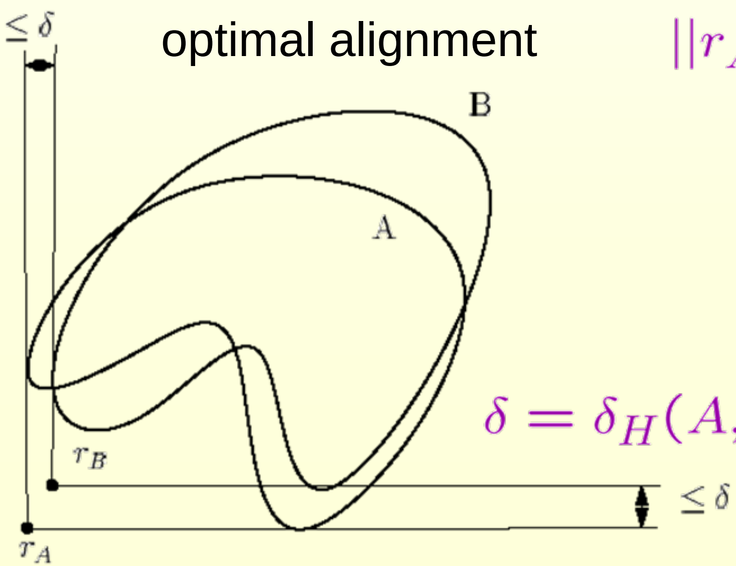
Reference Points

We match shapes by aligning certain well-chosen reference points. Such schemes can give constant-factor approximations to the Hausdorff distance [Alt *et. al.,* 91].

Example: approximate Hausdorff in 2-D under translations by matching lower left corner of bounding box

optimal alignment

$$||r_A - r_B|| \leq \sqrt{2}\delta$$



$$\delta = \delta_H(A, B)$$

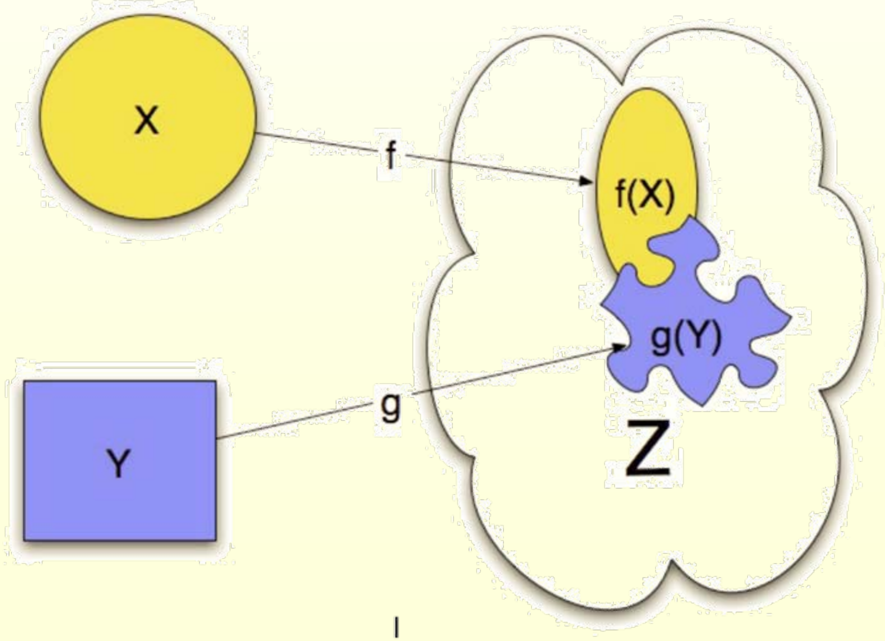
$$\begin{aligned} \delta_H(A, B') &\leq \delta_H(A, B) + \delta_H(B, B') \\ &\leq (\sqrt{2} + 1)\delta \end{aligned}$$

Can be improved by local resampling

Possible for rigid motions, etc.

Gromov-Hausdoff Distance

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



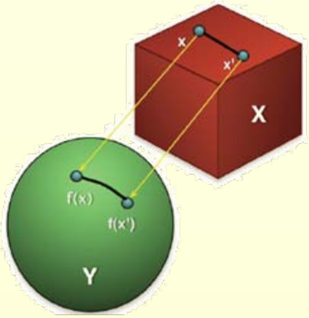
Gromov-Hausdorff Alternate Form

For compact spaces (X, d_X) and (Y, d_Y) let

$$d_{\mathcal{GH}}^{(2)}(X,Y) = \frac{1}{2} \inf_R \max_{(x,y),(x',y') \in R} |d_X(x,x') - d_Y(y,y')|$$

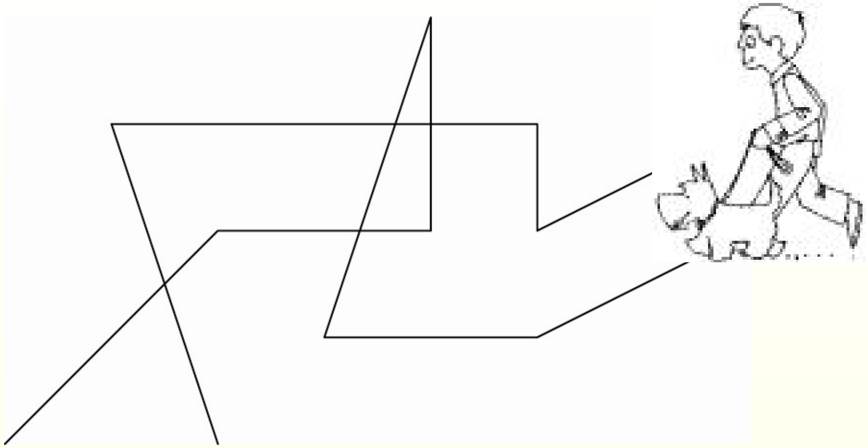
We write, compactly,

$$d_{\mathcal{GH}}^{(2)}(X,Y) = \frac{1}{2} \inf_R \|d_X - d_Y\|_{L^\infty(R \times R)}$$



Hard to compute …

Fréchet Distance

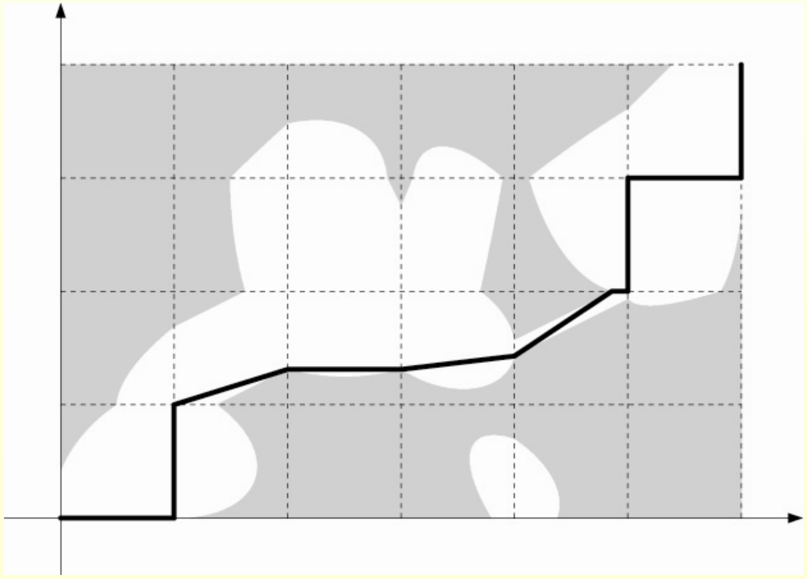


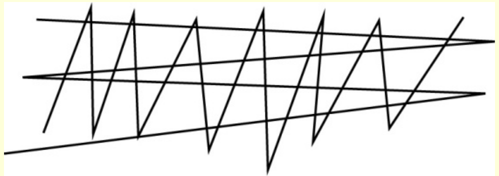
$$\delta_F(f,g) = \inf_{\alpha,\beta} \max_{t\in[0,1]} ||f(\alpha(t)) - g(\beta(t))||$$

From a decision problem to
an optimization problem.

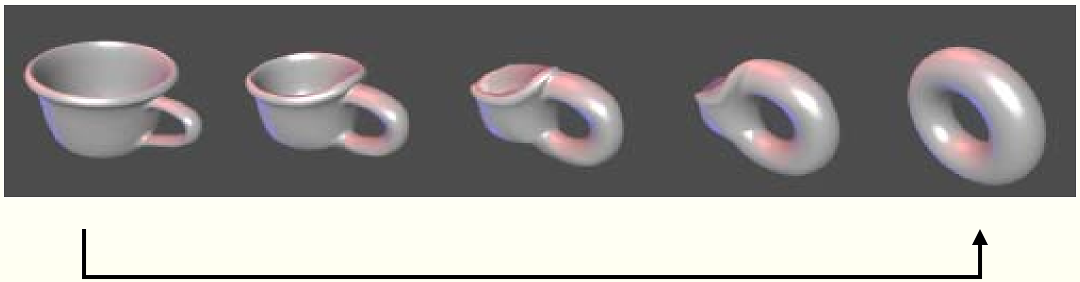
Guess and verify ...

O(*mn* log(*mn*))

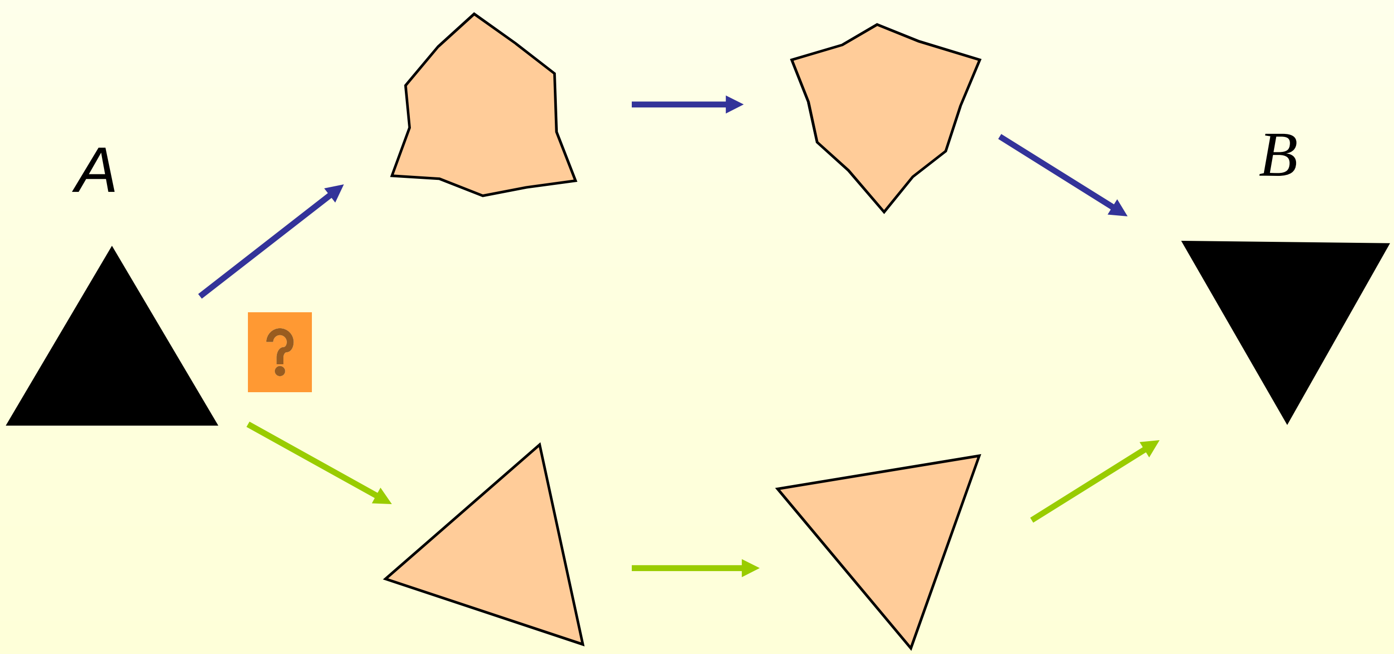




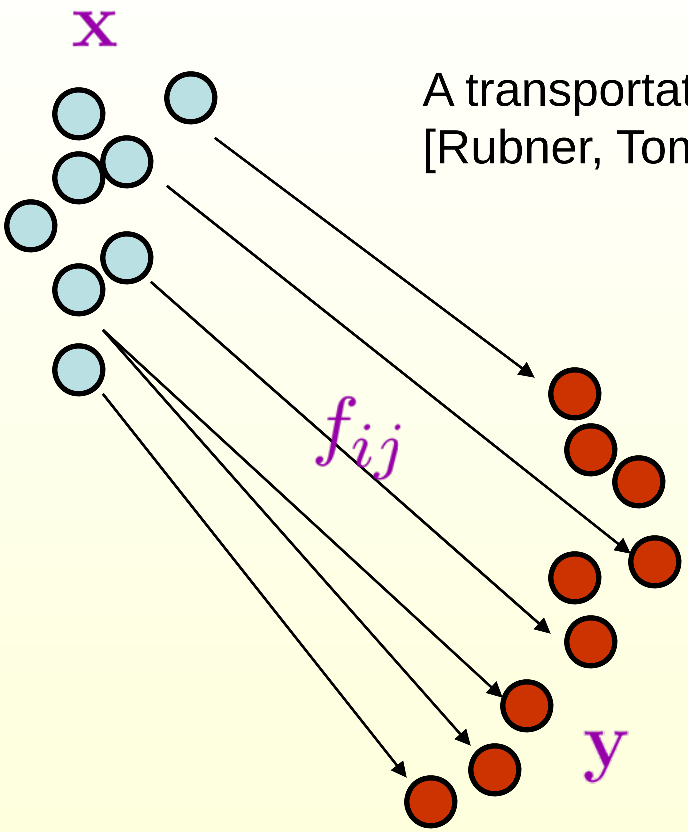
Morphing Distance



minimum cost transformation
from *A* to *B*



The Earth Mover's Distance



A transportation metric via linear programming
[Rubner, Tomasi, G., 98]

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} f_{ij}$$

$$\begin{aligned} f_{ij} &\geq 0 & i \in \mathcal{I}, j \in \mathcal{J} \\ \sum_{i \in \mathcal{I}} f_{ij} &= y_j & j \in \mathcal{J} \\ \sum_{j \in \mathcal{J}} f_{ij} &\leq x_i & i \in \mathcal{I}, \end{aligned}$$

$$\mathrm{EMD}(\mathbf{x}, \mathbf{y}) = \frac{\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} f_{ij}}{\sum_{j \in \mathcal{J}} y_j}$$

Also Wesserstein distance,
Kantorovich-Rubinstein distance …