CS164: Surface Reconstruction, Marching Cubes

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Computer Science Dept.
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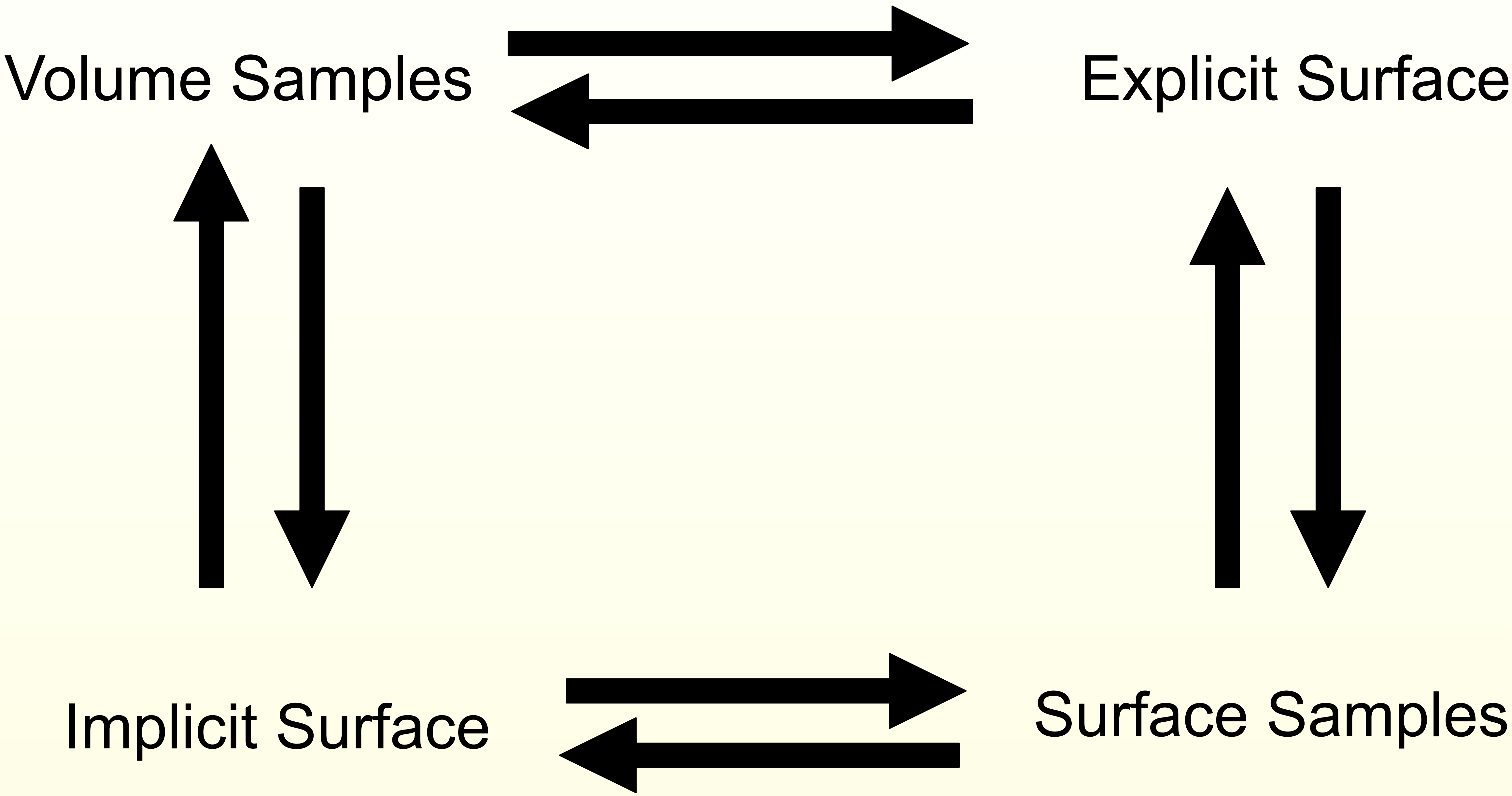
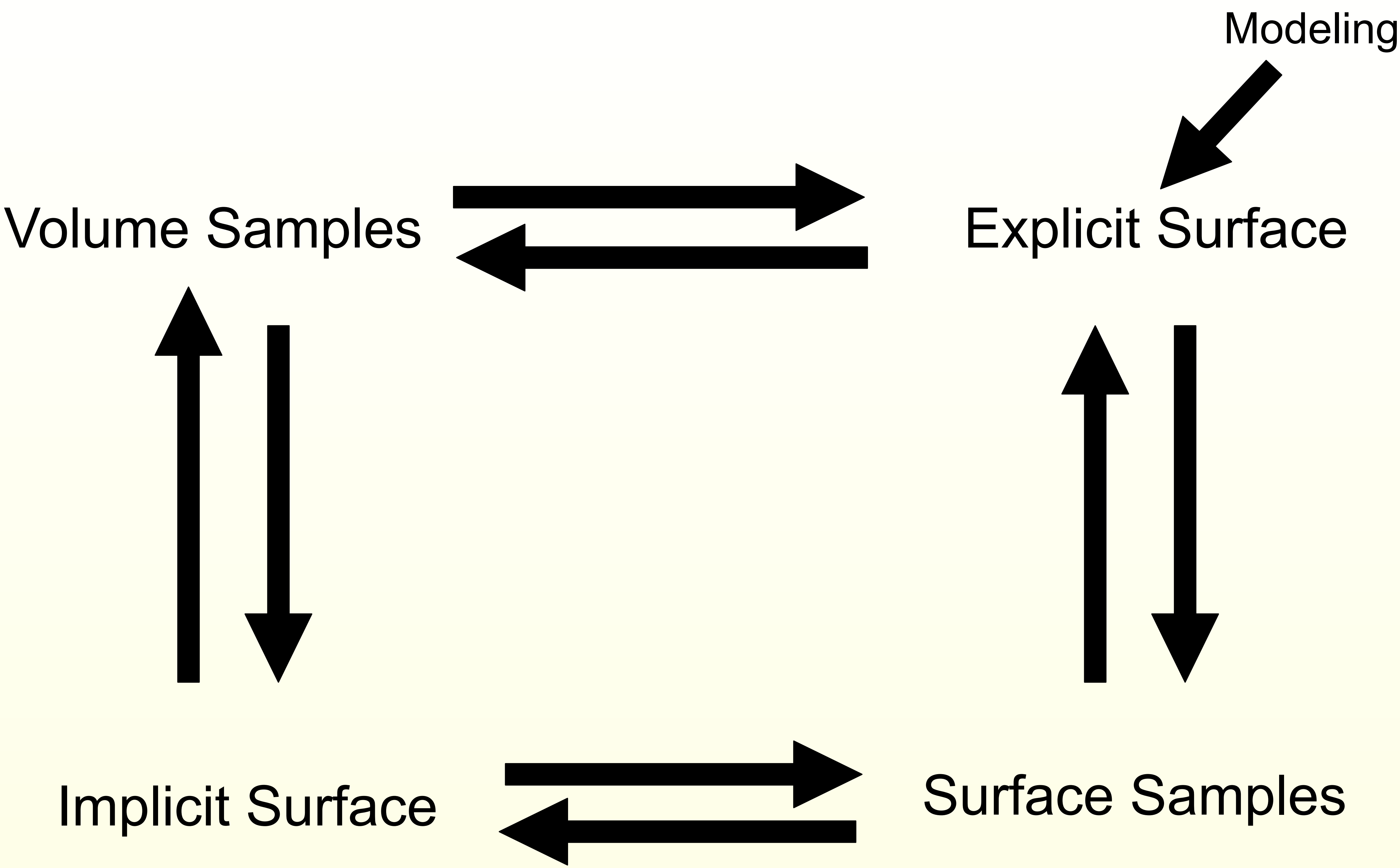
Overview

- Surface Representations
  - Explicit Surfaces
  - Implicit Surfaces
- Marching Cubes
  - Hermite Data/Extended Marching Cubes
  - Dual Contouring
  - Topological Guarantees

Surface Representations


Volume Samples
Explicit Surface
Implicit Surface
Surface Samples


Modeling
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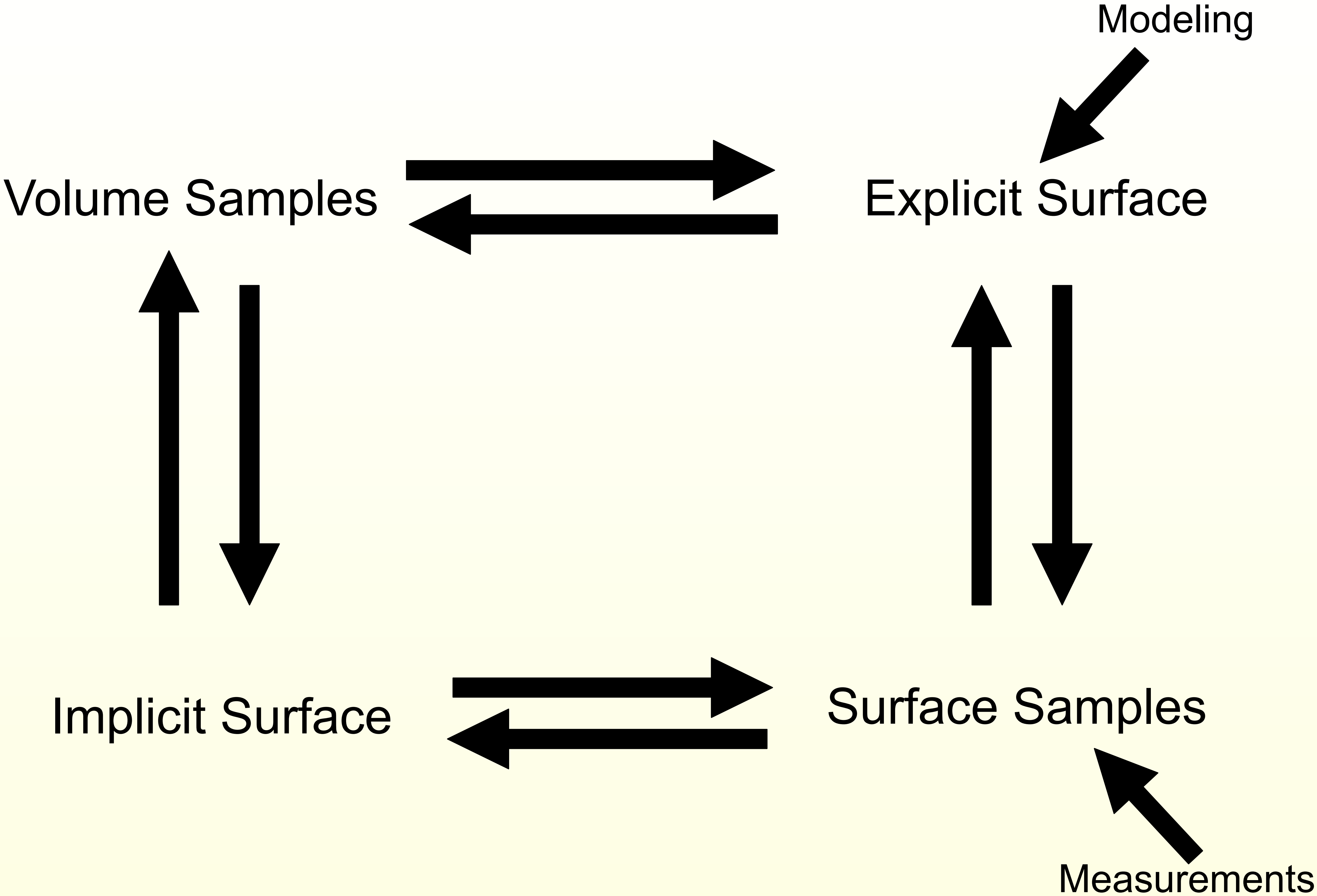
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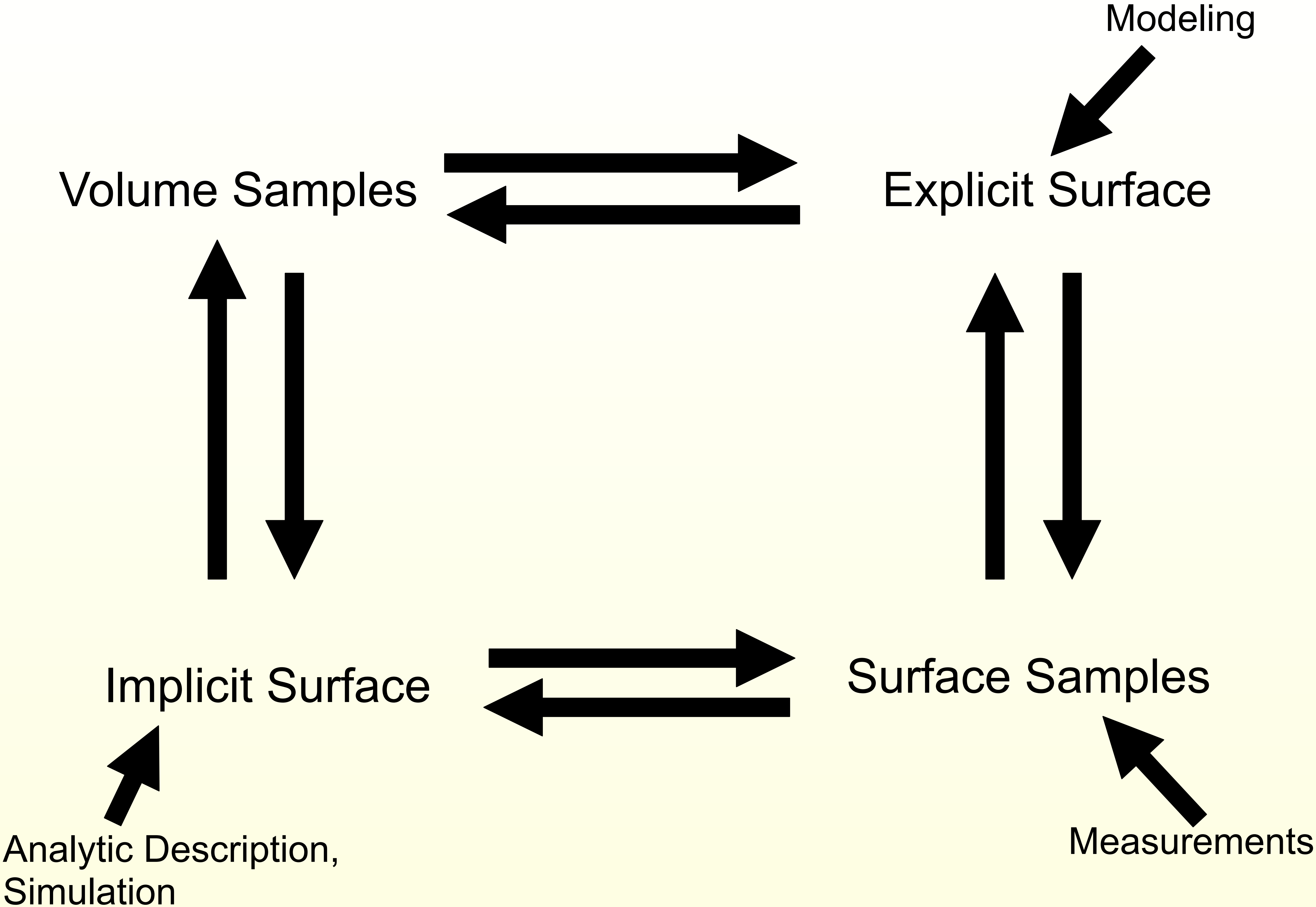
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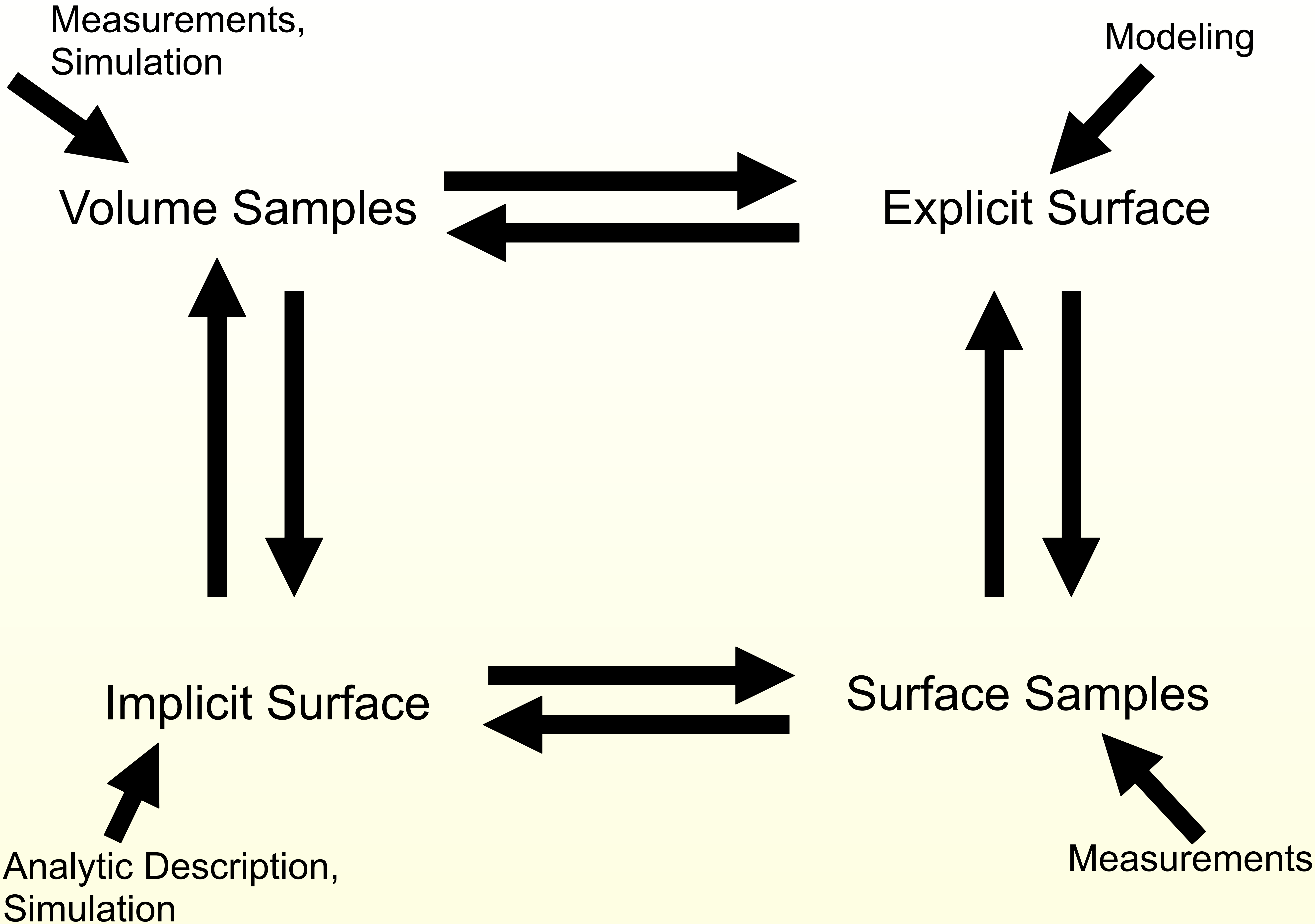
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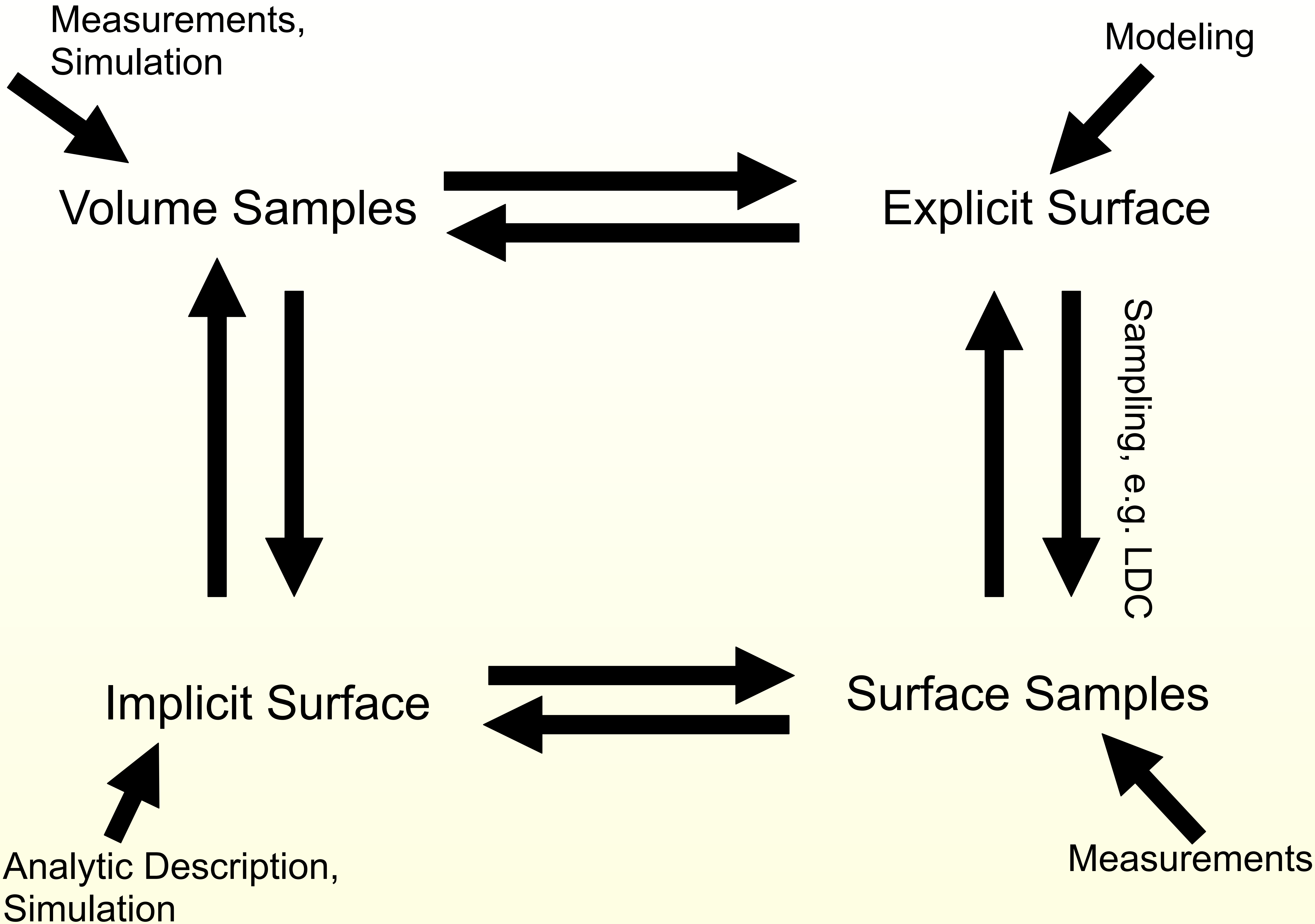

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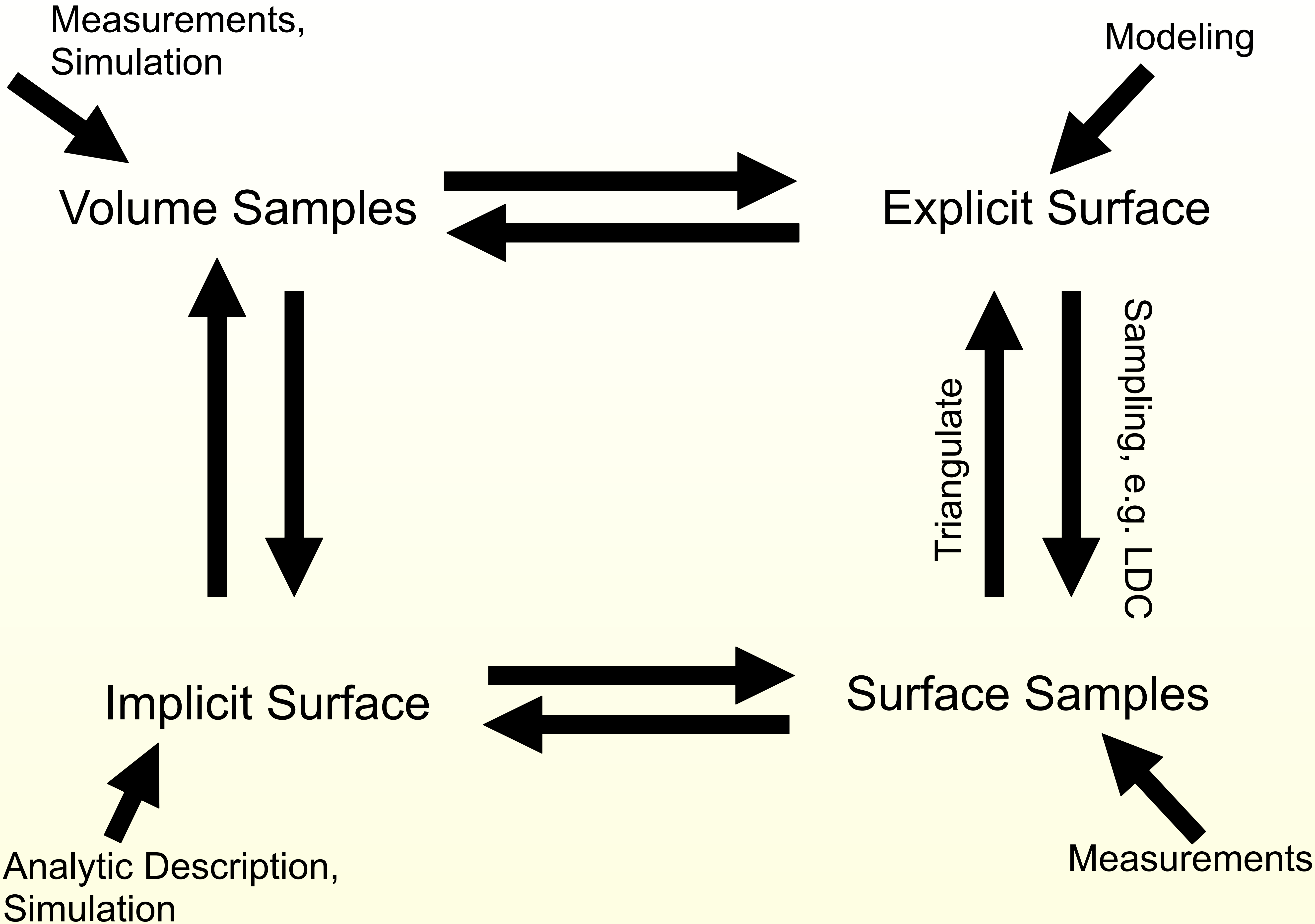
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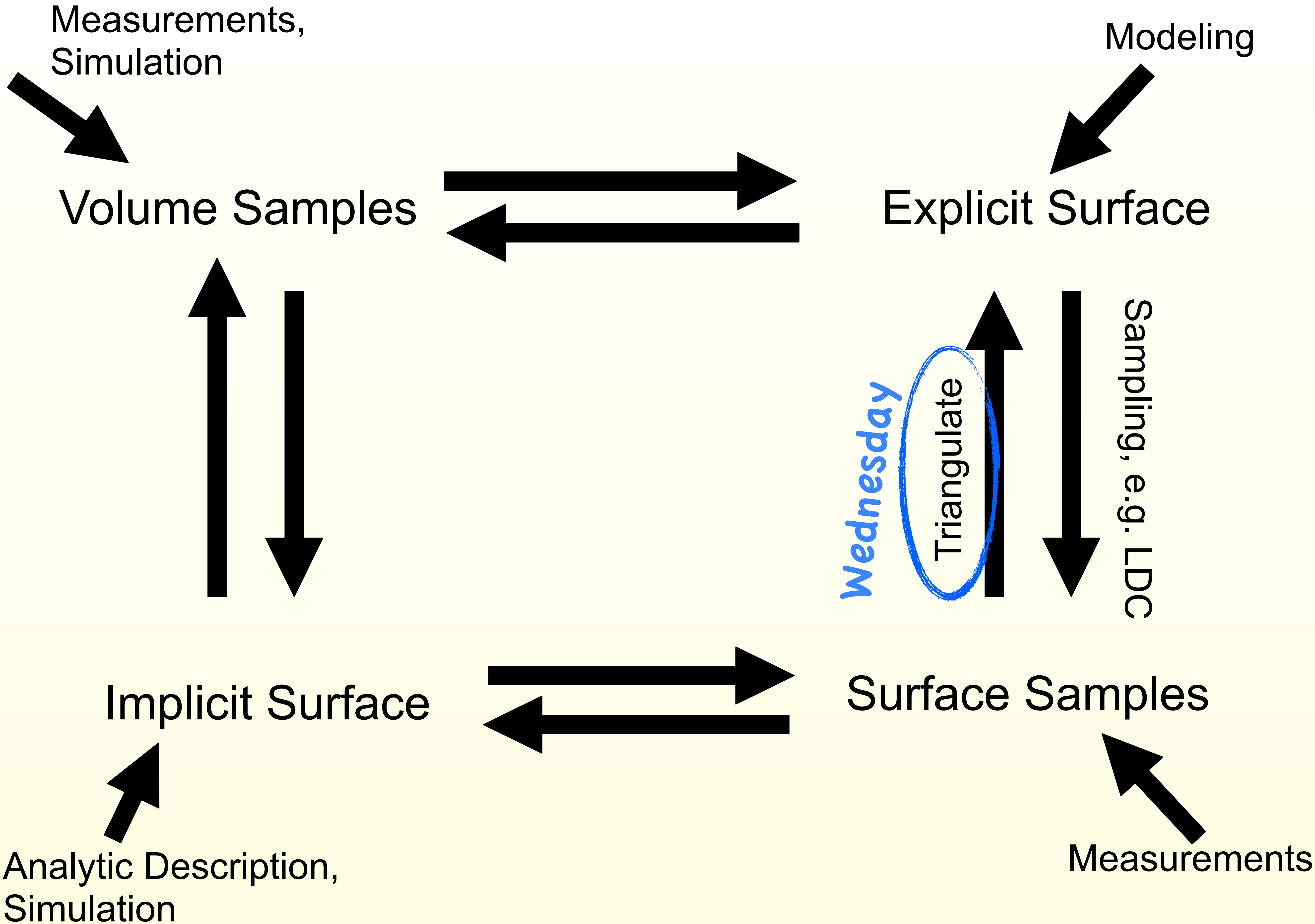
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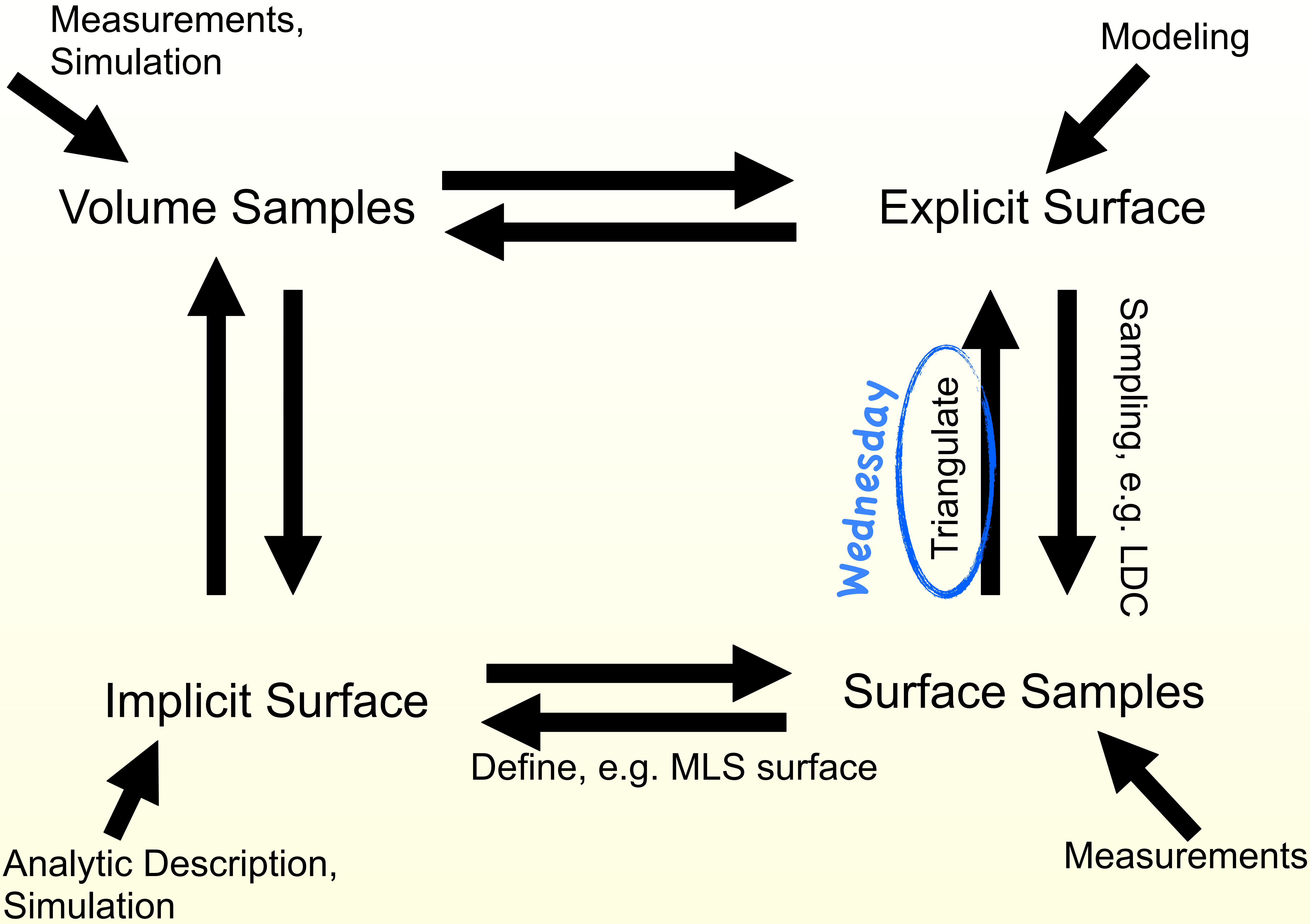
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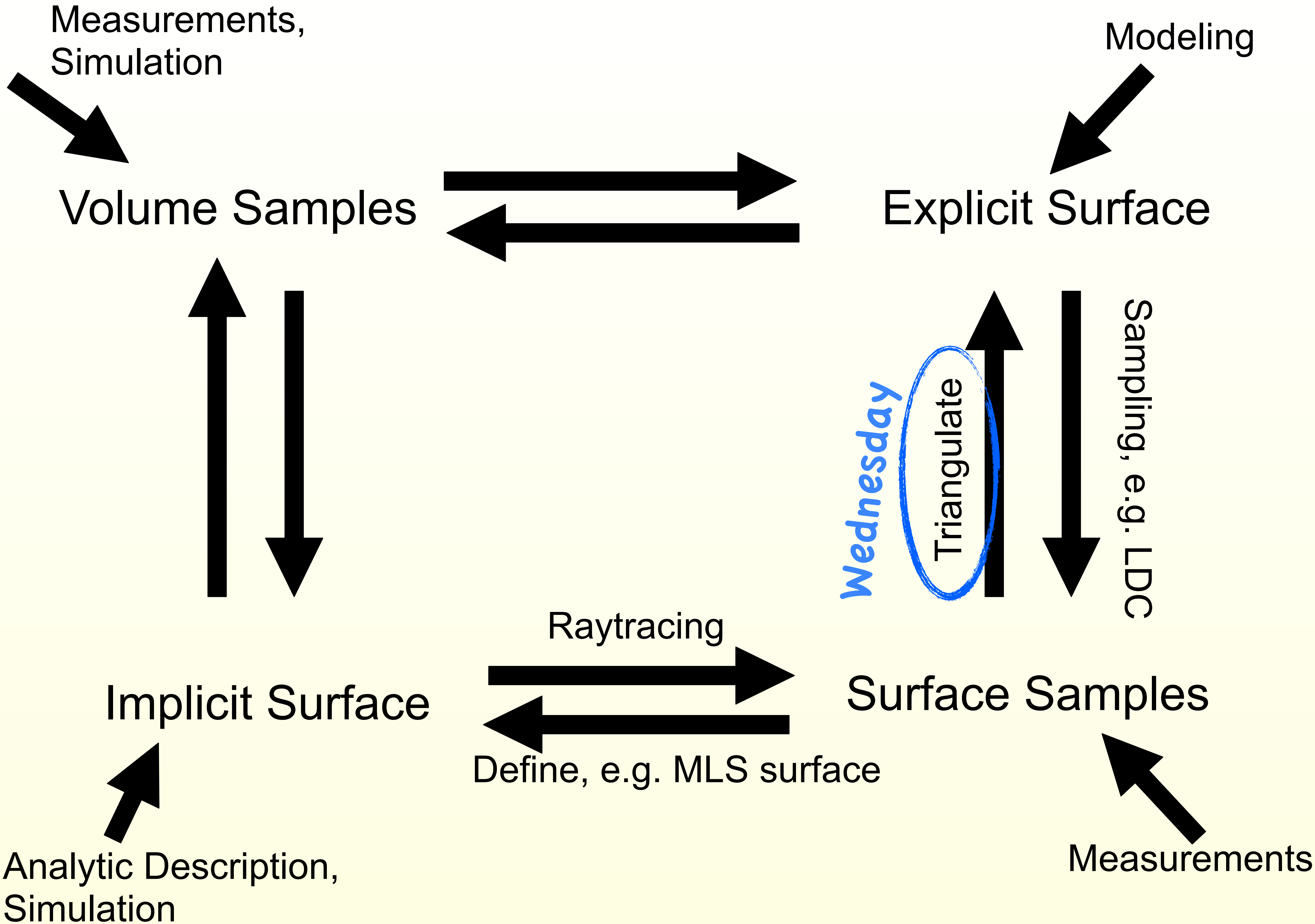
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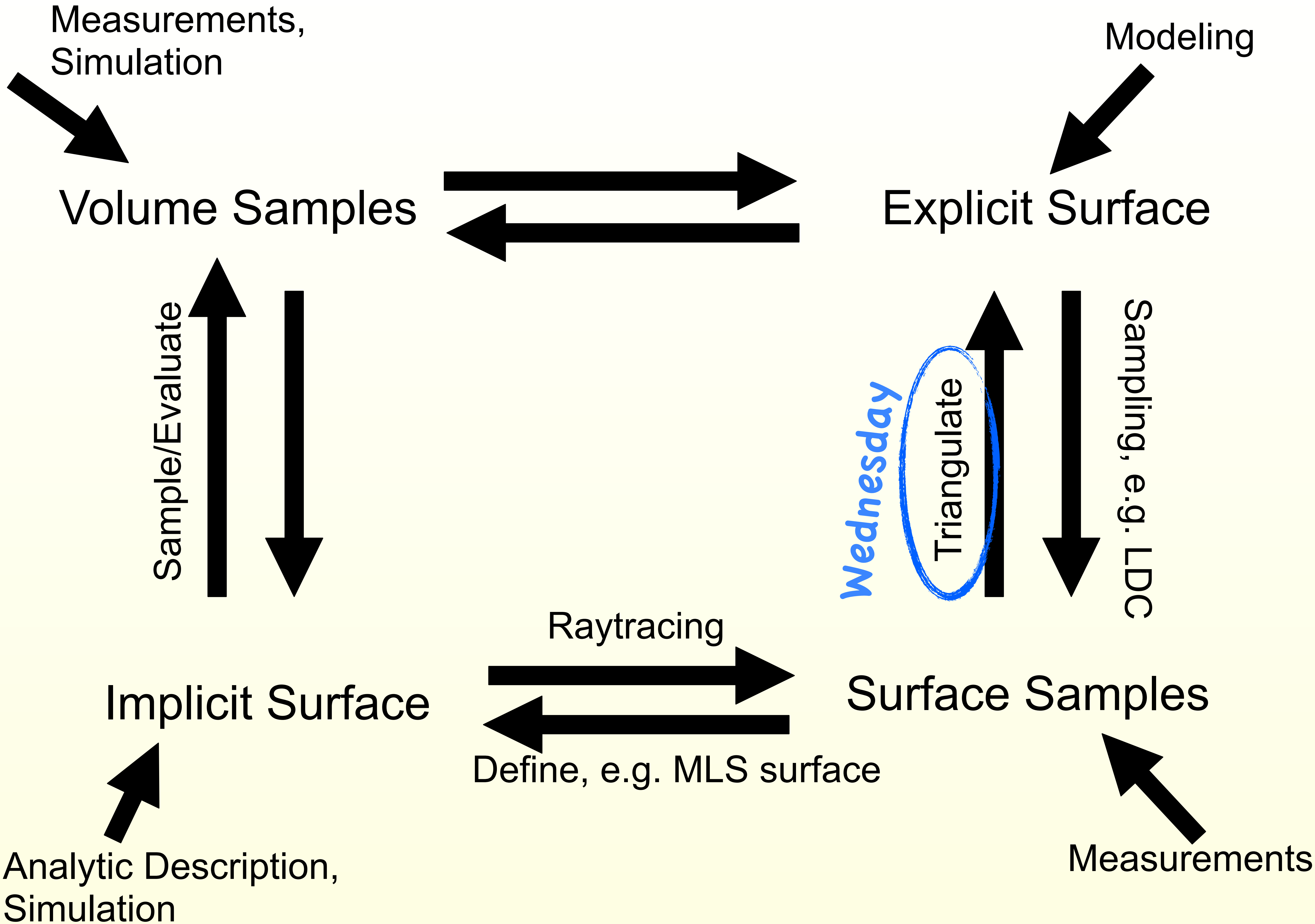
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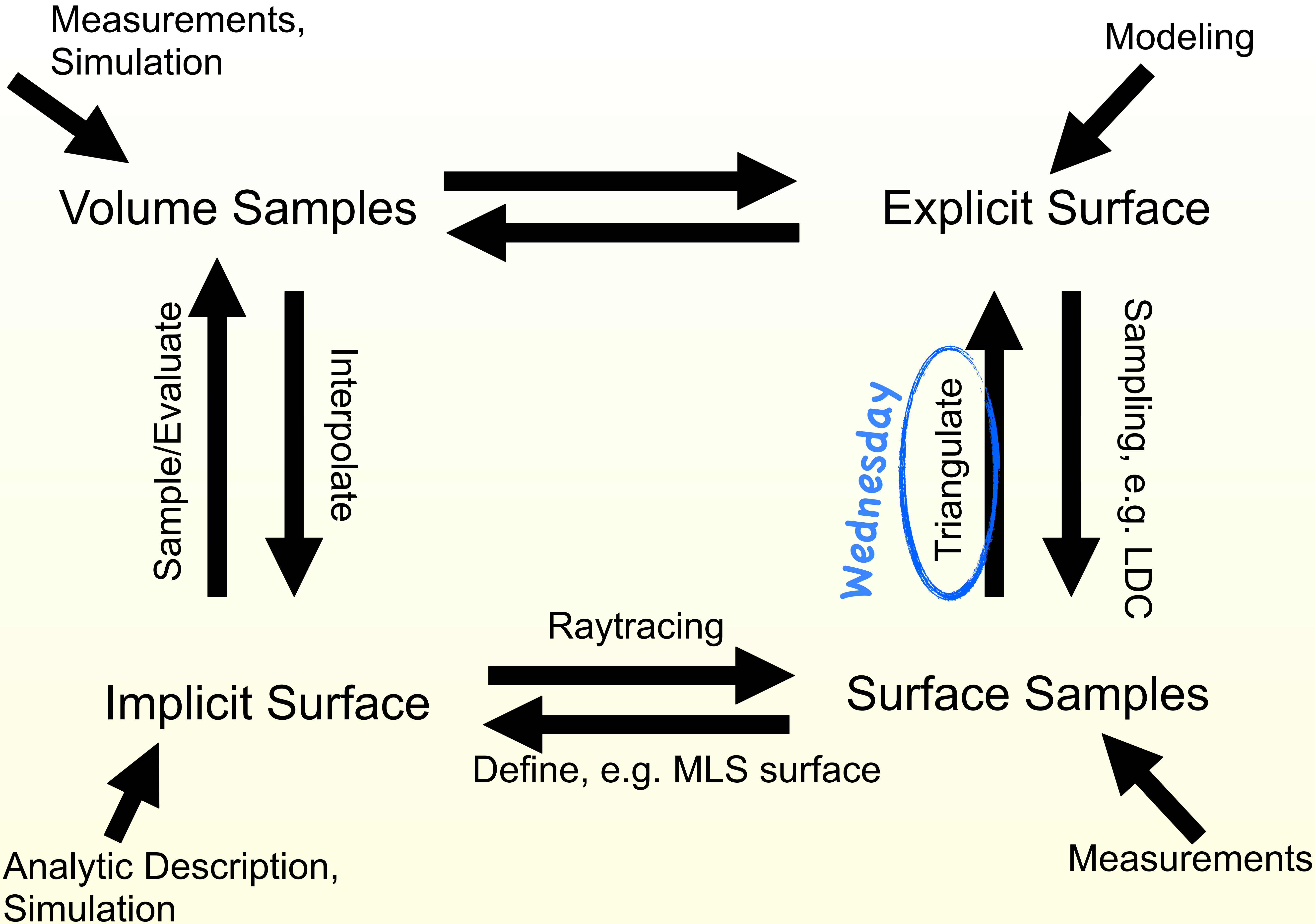
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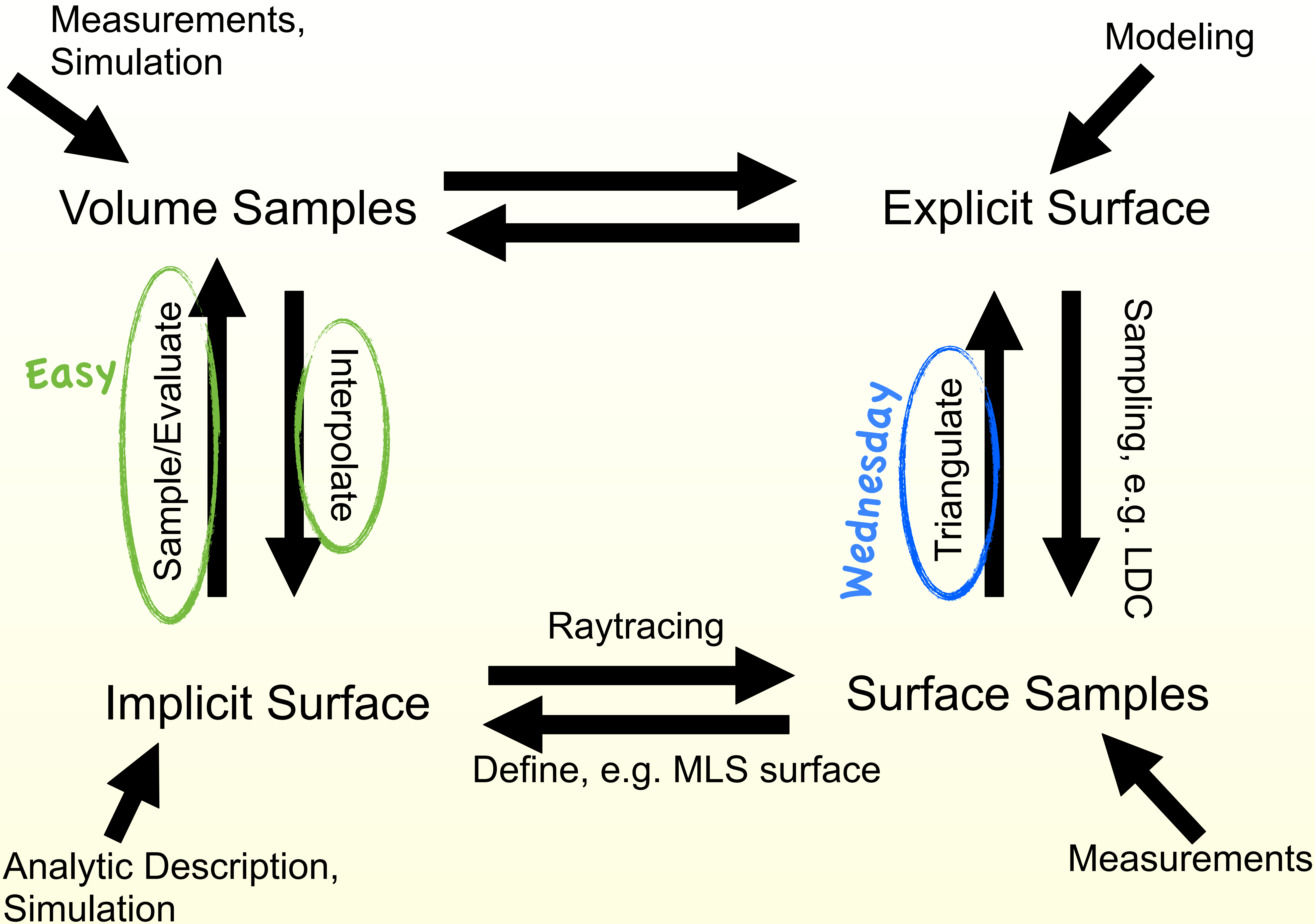
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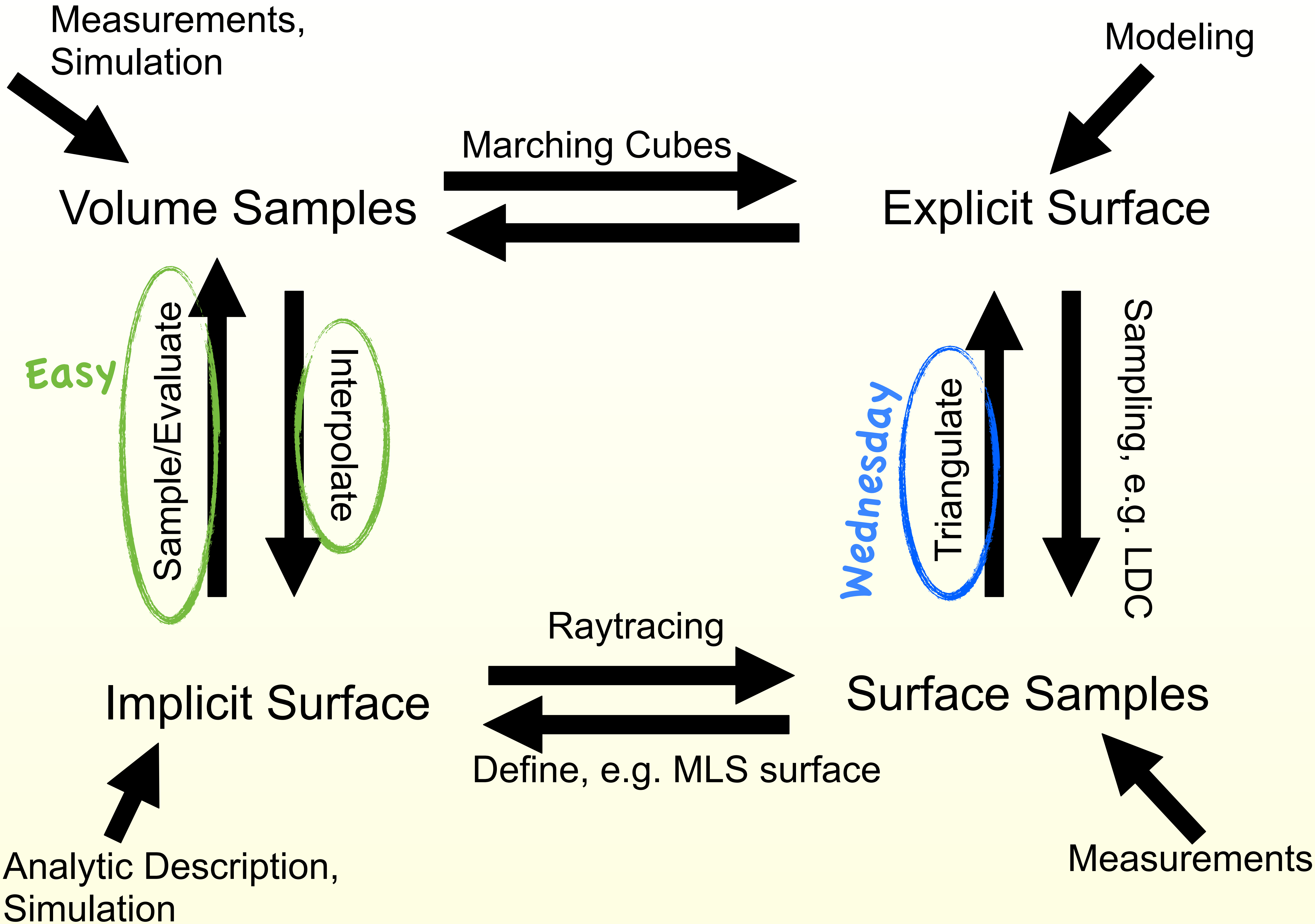
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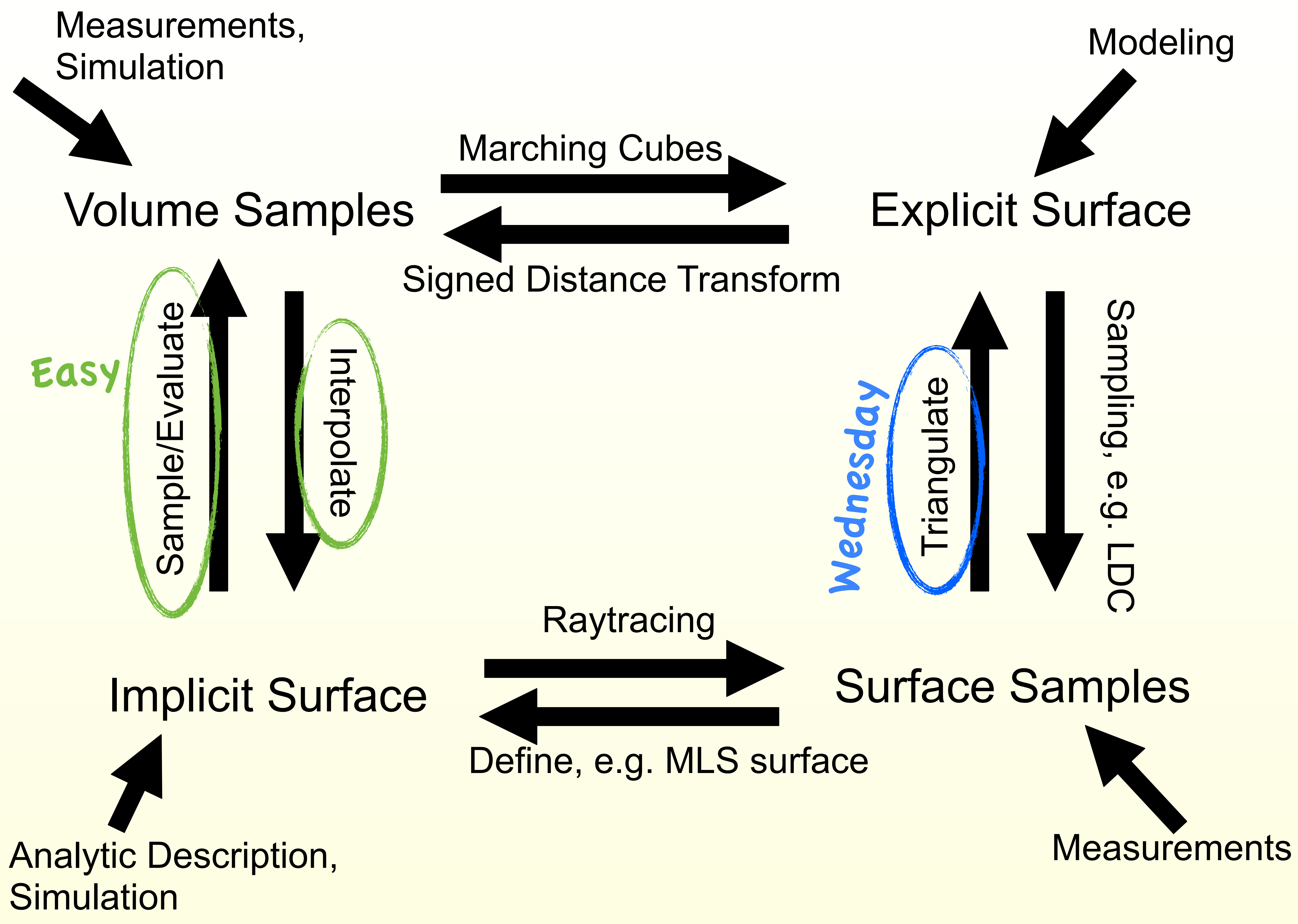

Measurements, Simulation
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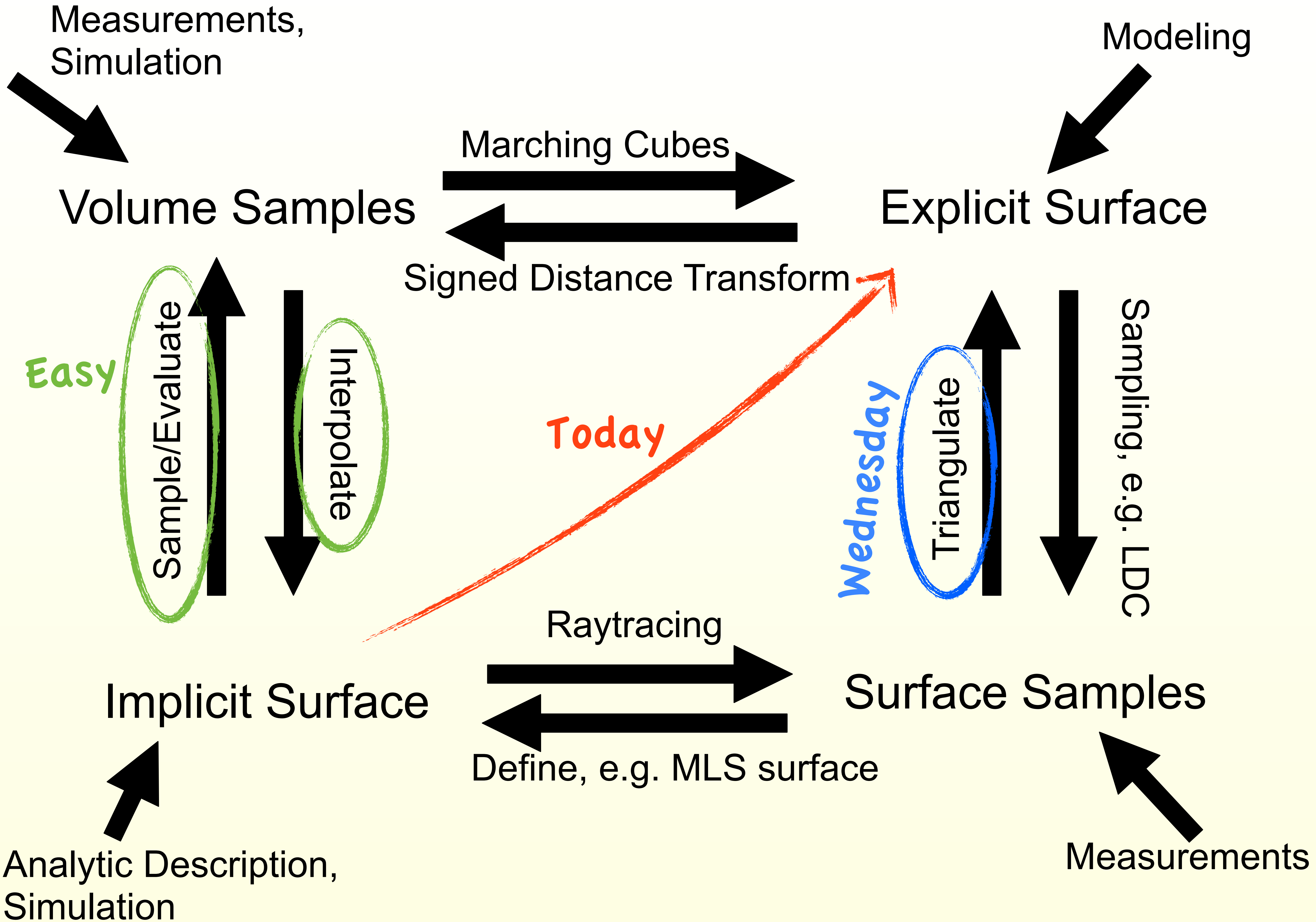

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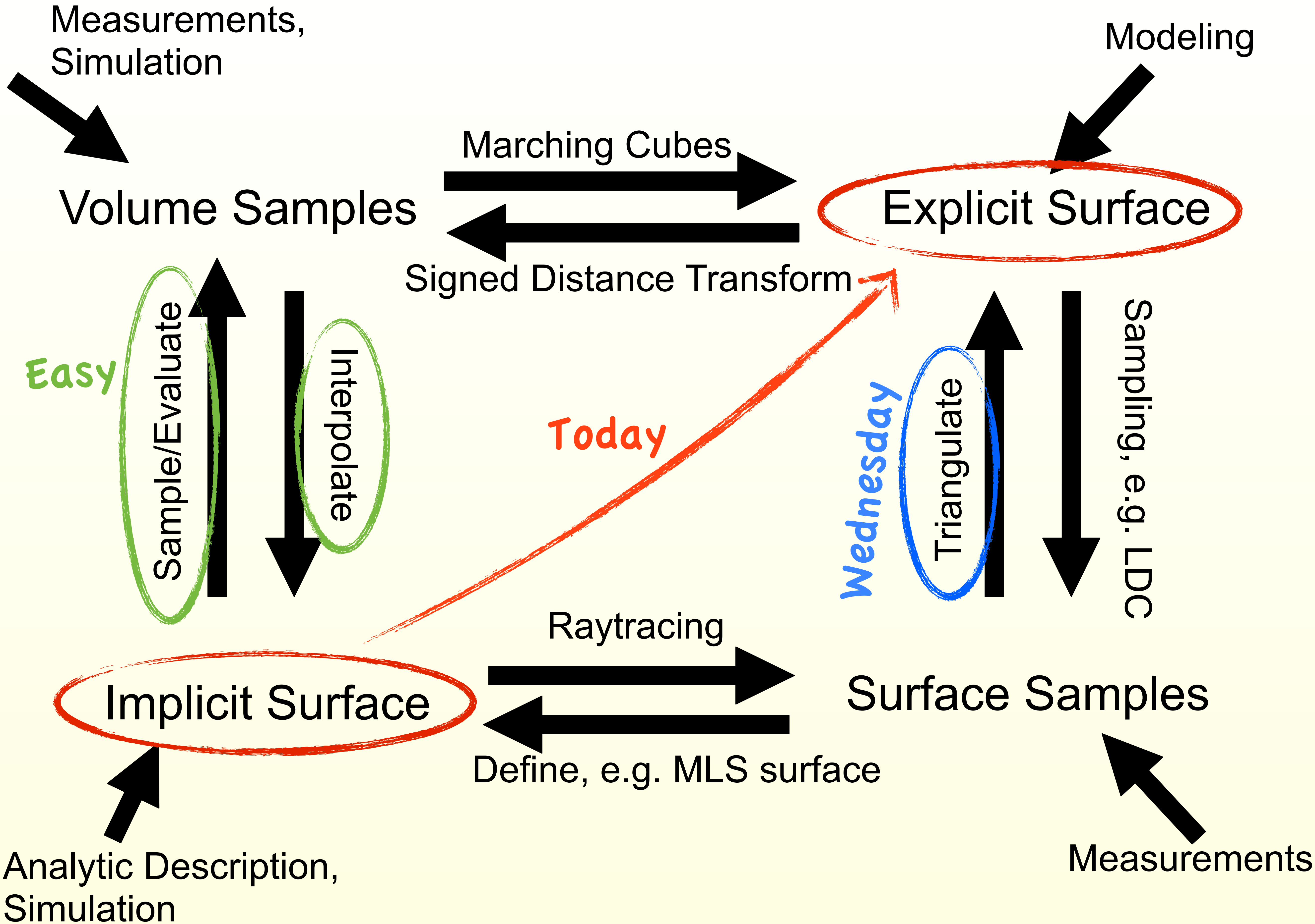
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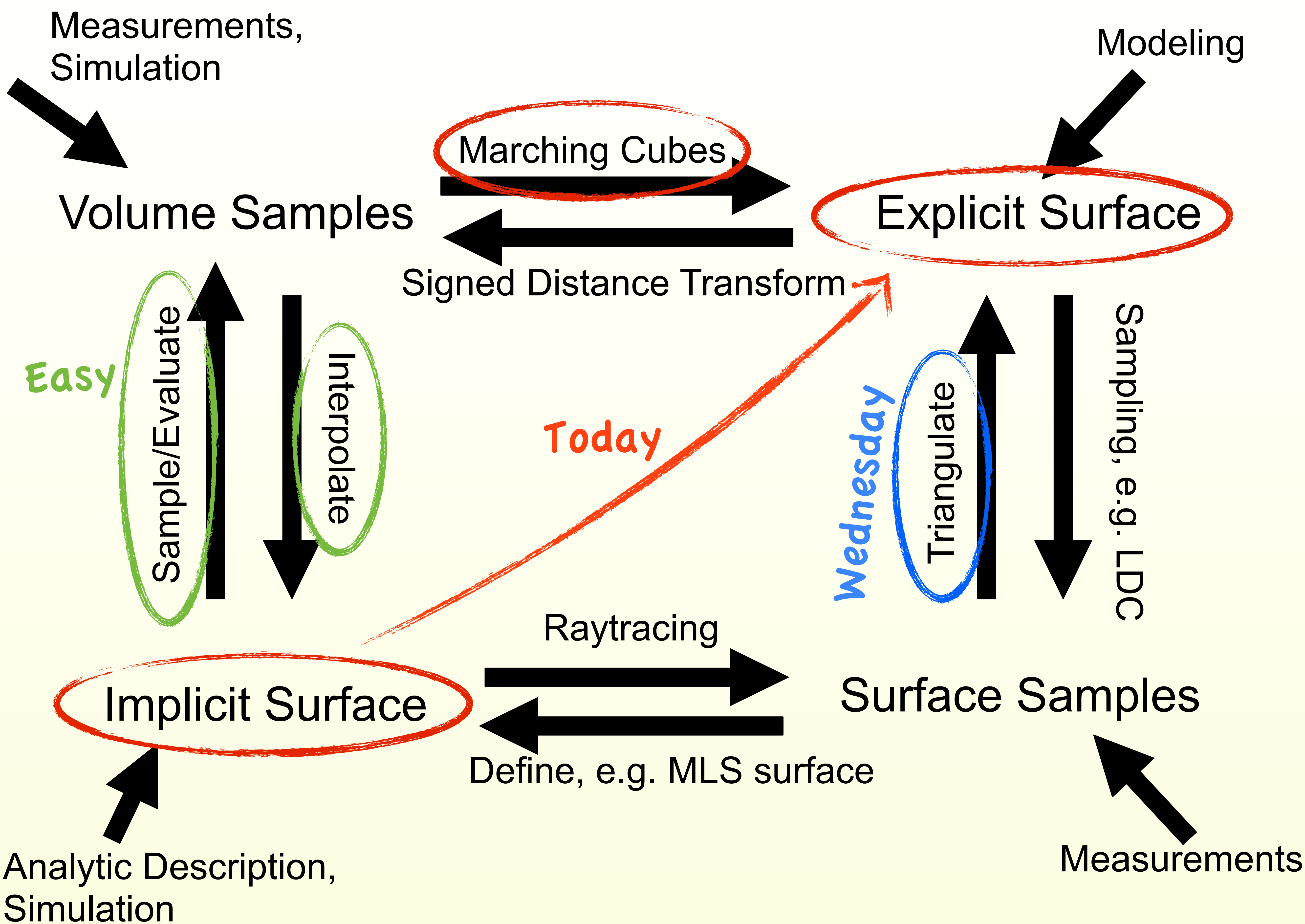

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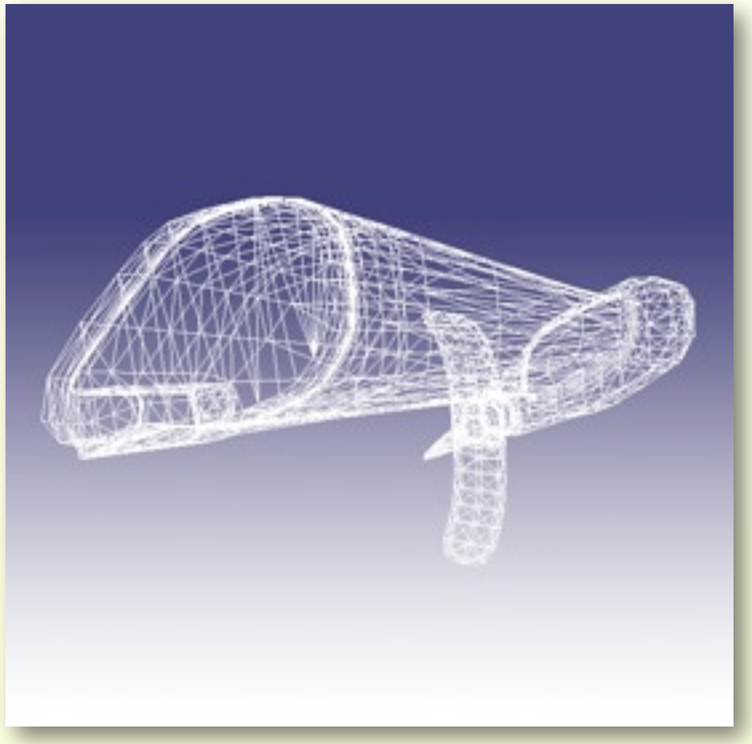
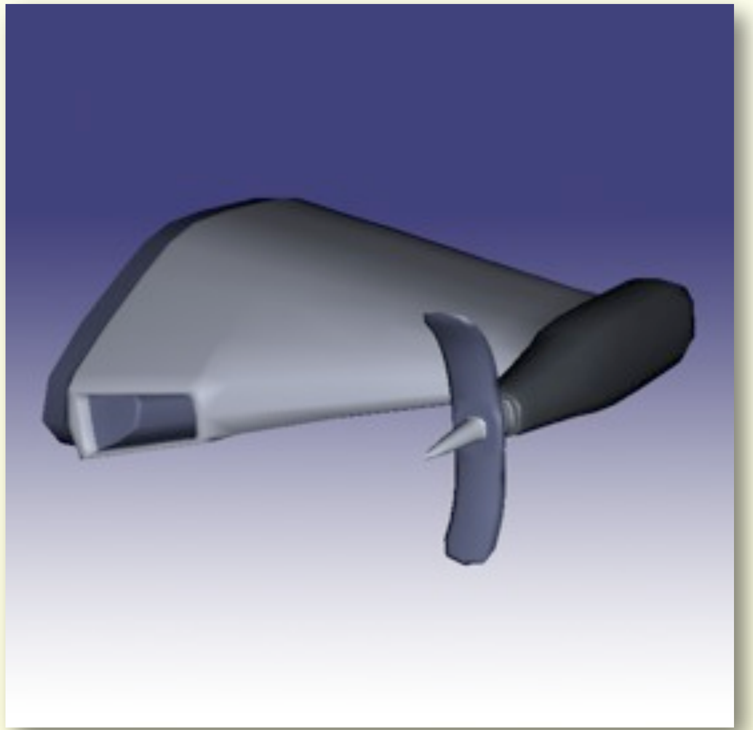
Explicit (Parametric) Surfaces

- "The surface consists of these points: ..."

$$\{f(\mathbf{u})|\mathbf{u} \in \mathbb{R}^2\}$$

- Splines (treated earlier)
- Piecewise-linear surfaces (polygonal meshes)
- Most common: triangle meshes

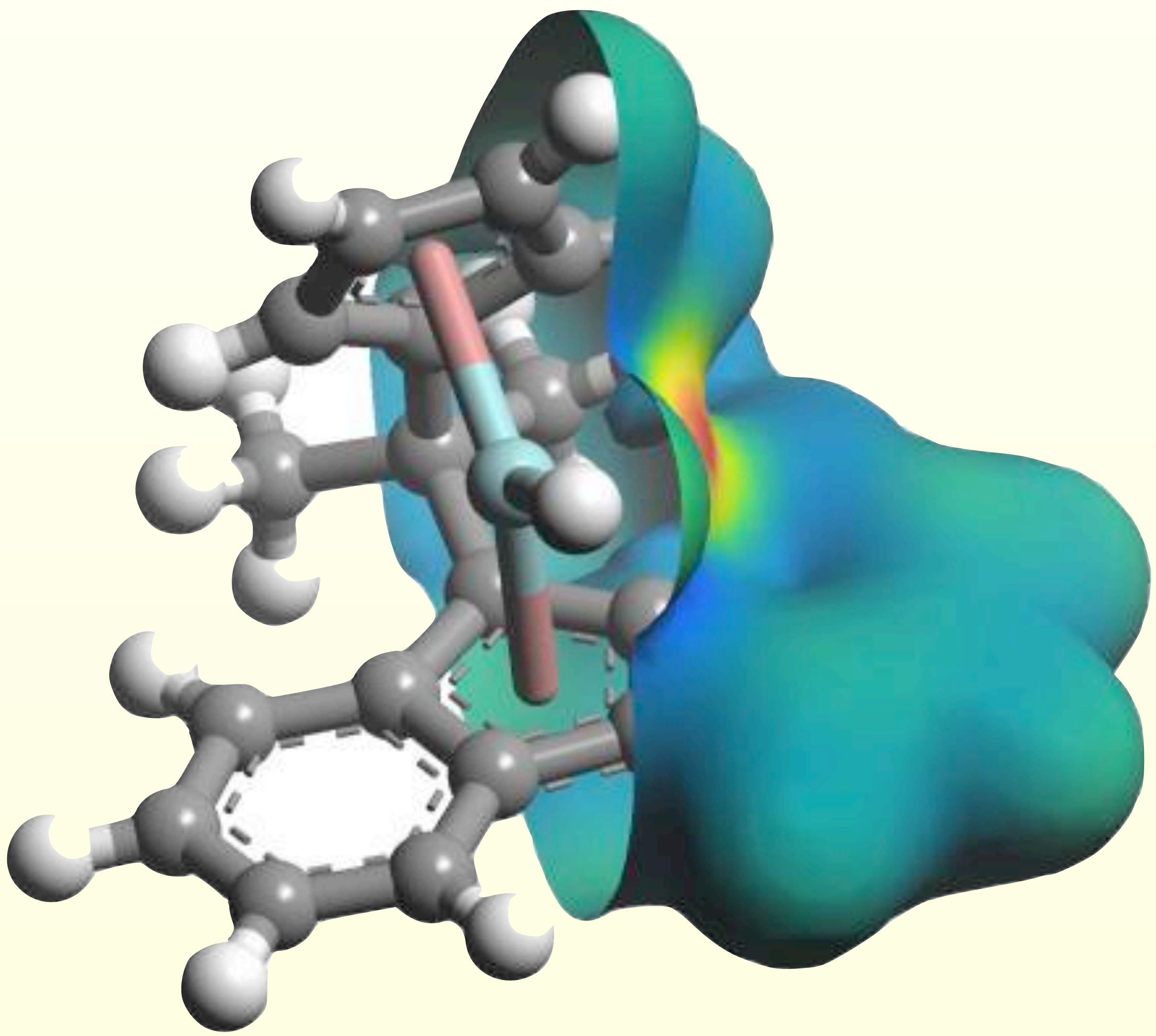




Implicit Surfaces

- "The surface consists of all points, which..."

$$\{\mathbf{x} | f(\mathbf{x}) = 0\}$$



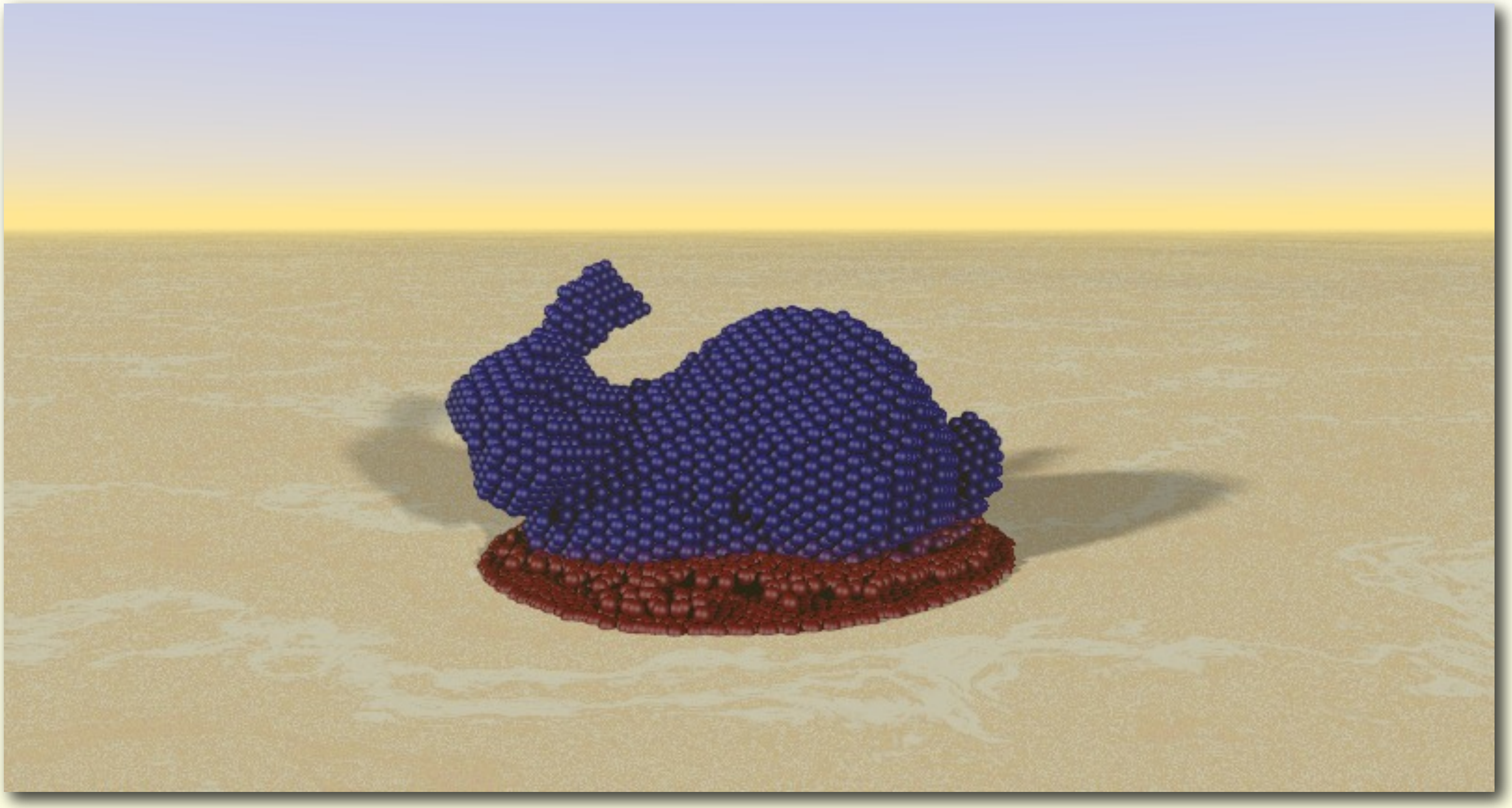
Isosurface around Zirconocene molecule [Accelrys]

Implicit vs. Explicit

- Different sources
- Explicit: $\{f(\mathbf{u})|\mathbf{u} \in \mathbb{R}^2\}$
  - Image of a function
  - Easy to enumerate points
  - Hard to check whether a given point is on the surface
- Implicit: $\{\mathbf{x}|f(\mathbf{x}) = 0\}$
  - Kernel of a function
  - Hard to enumerate points
  - Easy to check whether a given point is on the surface

Iso-Surface of a Density Field

- Example: $f(\mathbf{x}) = \sum_i w(\mathbf{x}, \mathbf{x}_i) - \rho_0$

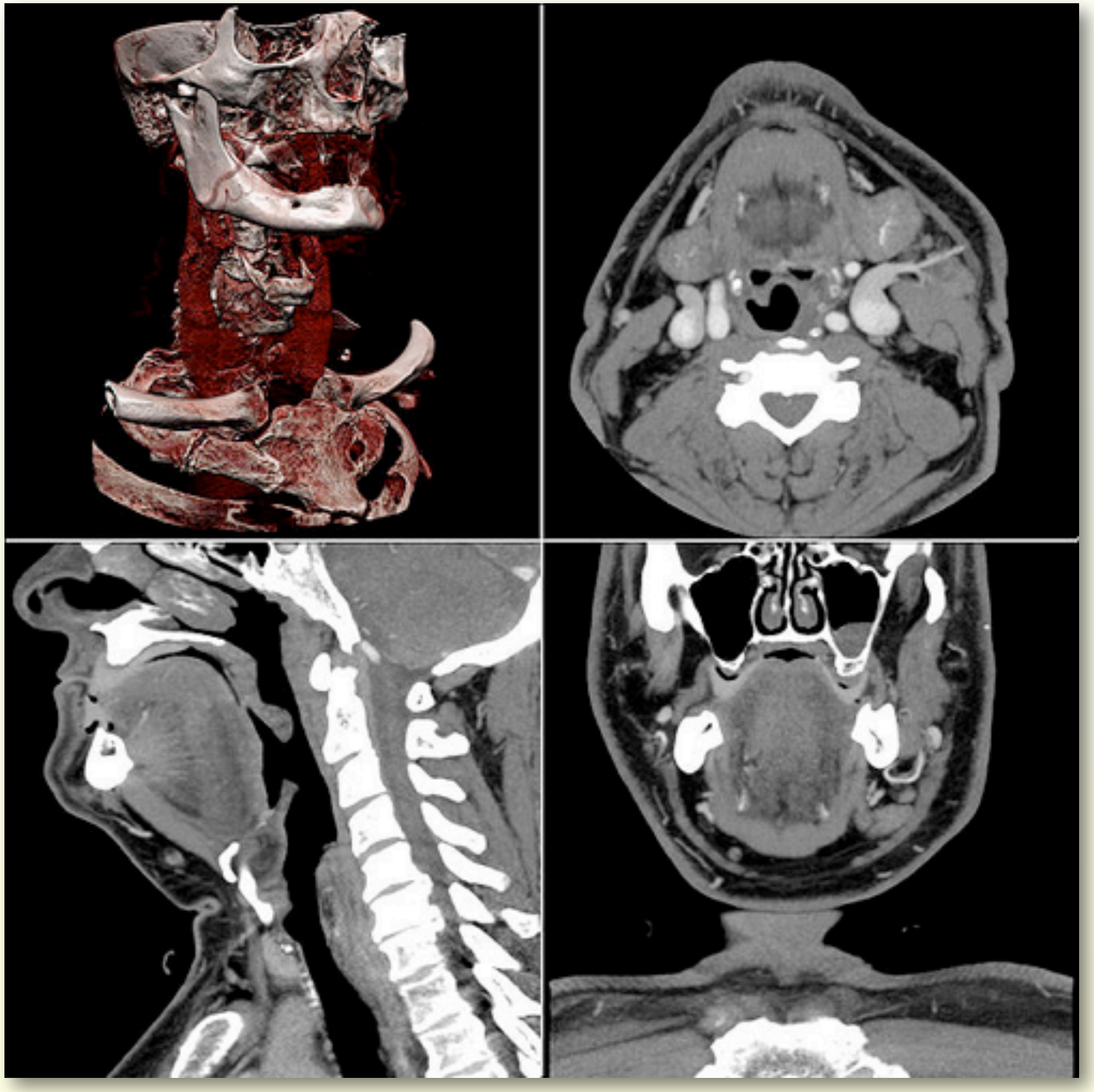


Iso-Surface of a Density Field

- Example: $f(\mathbf{x}) = \sum_i w(\mathbf{x}, \mathbf{x}_i) - \rho_0$

Iso-Surface in a CAT Scan

- $f(\mathbf{x}) = I(\mathbf{x}) - q$
- Samples in a regular grid
- Trilinear within voxels
- How can we make an isosurface explicit?



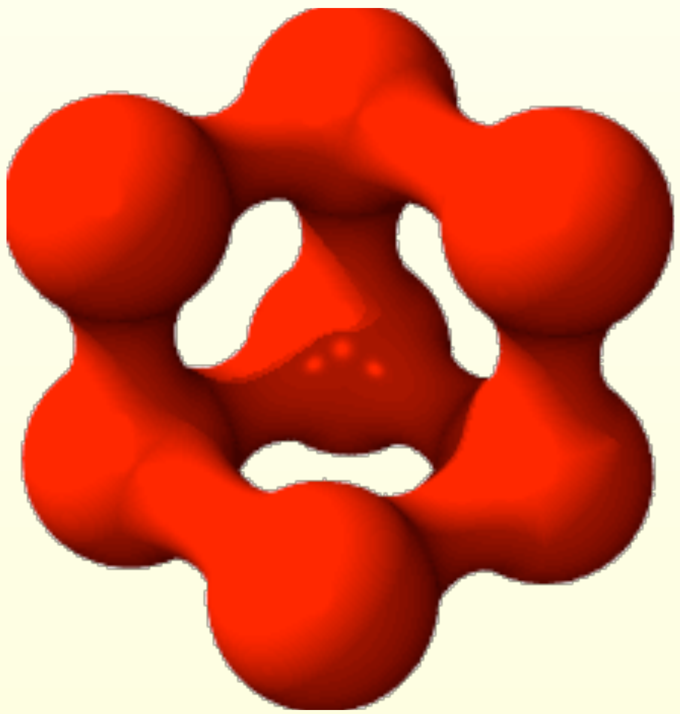
Marching Cubes
(and variants)

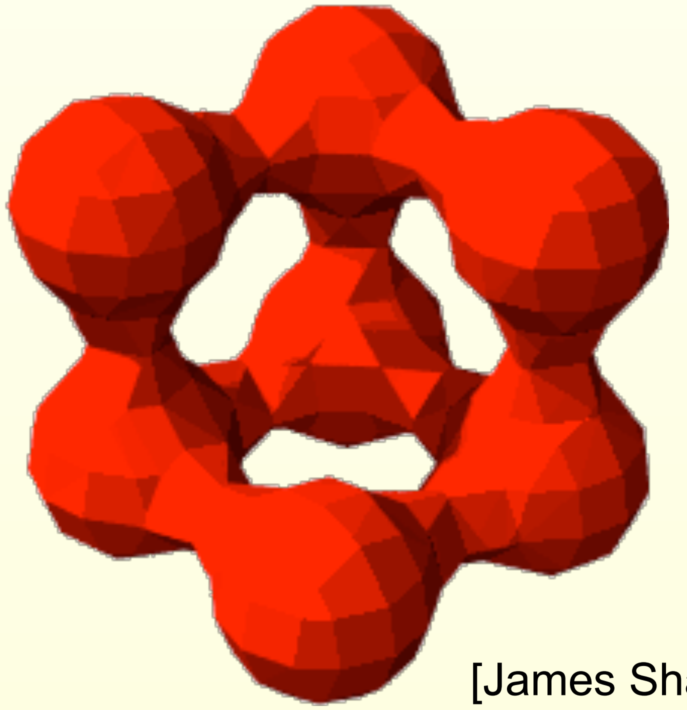
Problem Statement

Given a function $I(\mathbf{x})$ defining an implicit surface

$$\mathcal{S} = \{\mathbf{x} | f(\mathbf{x}) = I(\mathbf{x}) - q = 0\},$$

create a triangle mesh that approximates the surface S.





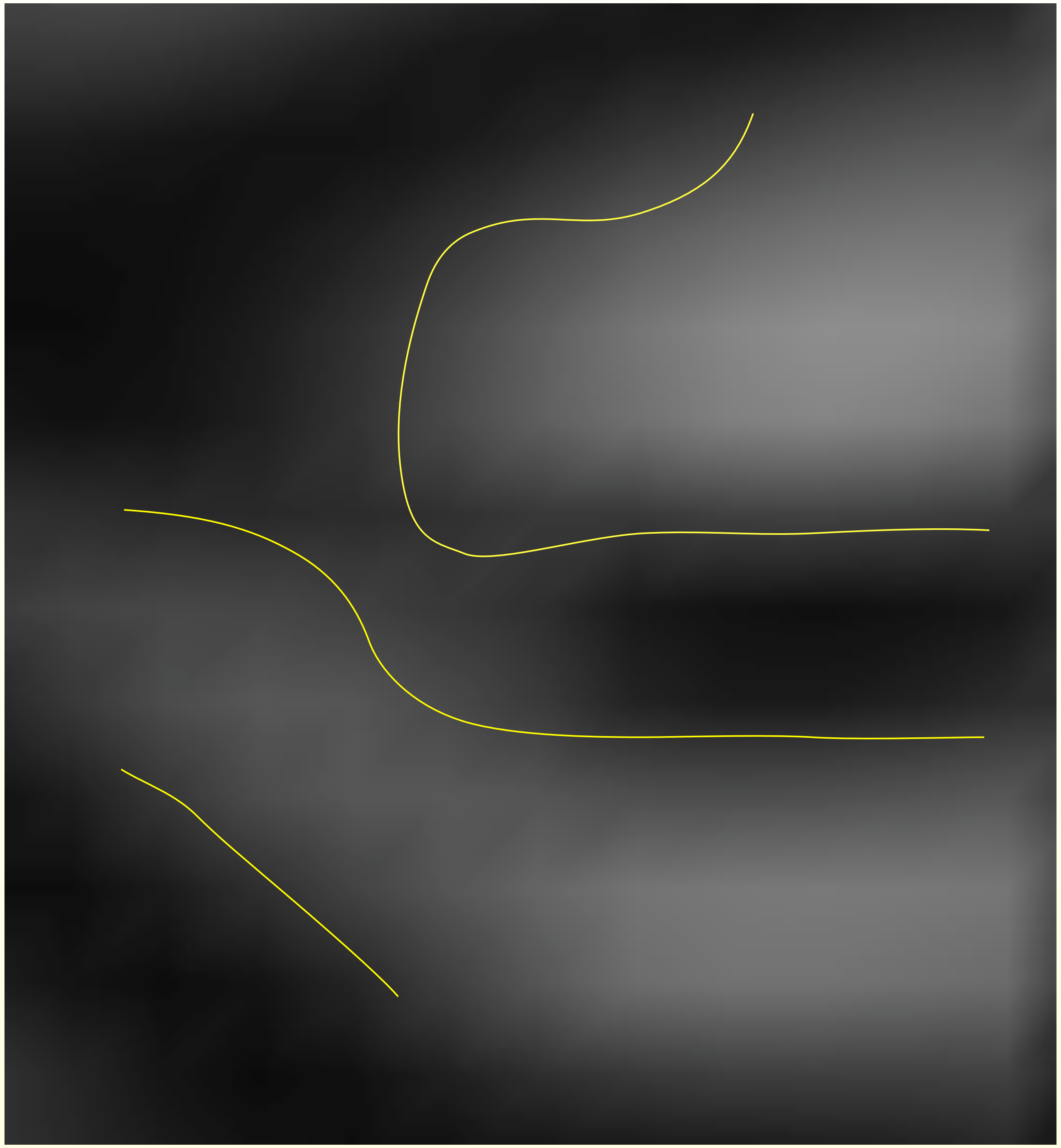
[James Sharman]

Overview

- Marching Cubes
  - 2D case: Marching Squares
  - 3D case: Marching Cubes
  - Marching Tetrahedra
- Extended Marching Cubes
- Dual Contouring

Marching Squares

- Given $I(\mathbf{x})$
  - $I(\mathbf{x}) < q$: $\mathbf{x}$ outside
  - $I(\mathbf{x}) > q$: $\mathbf{x}$ inside
- Discretize space
- Evaluate on the grid



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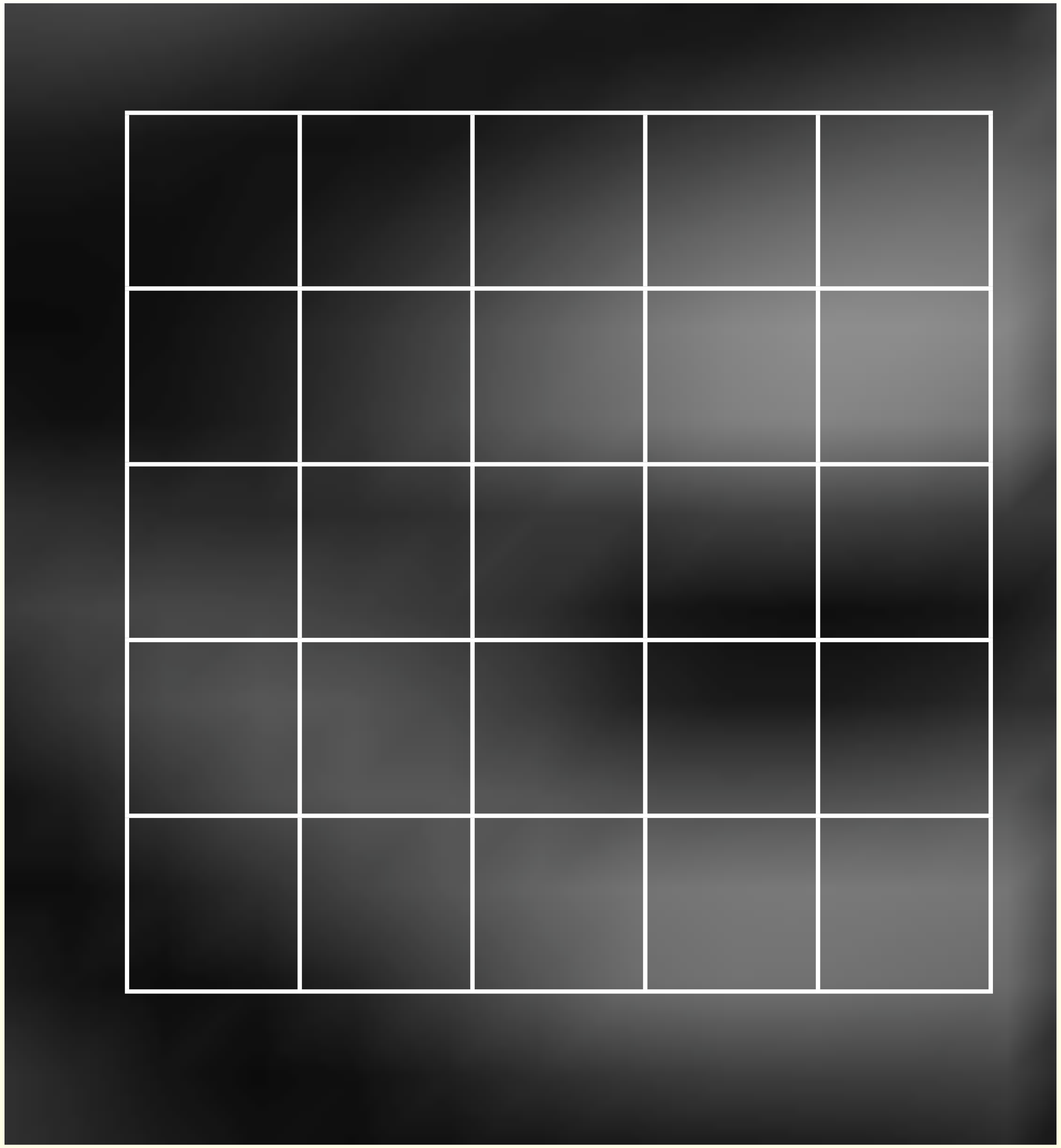

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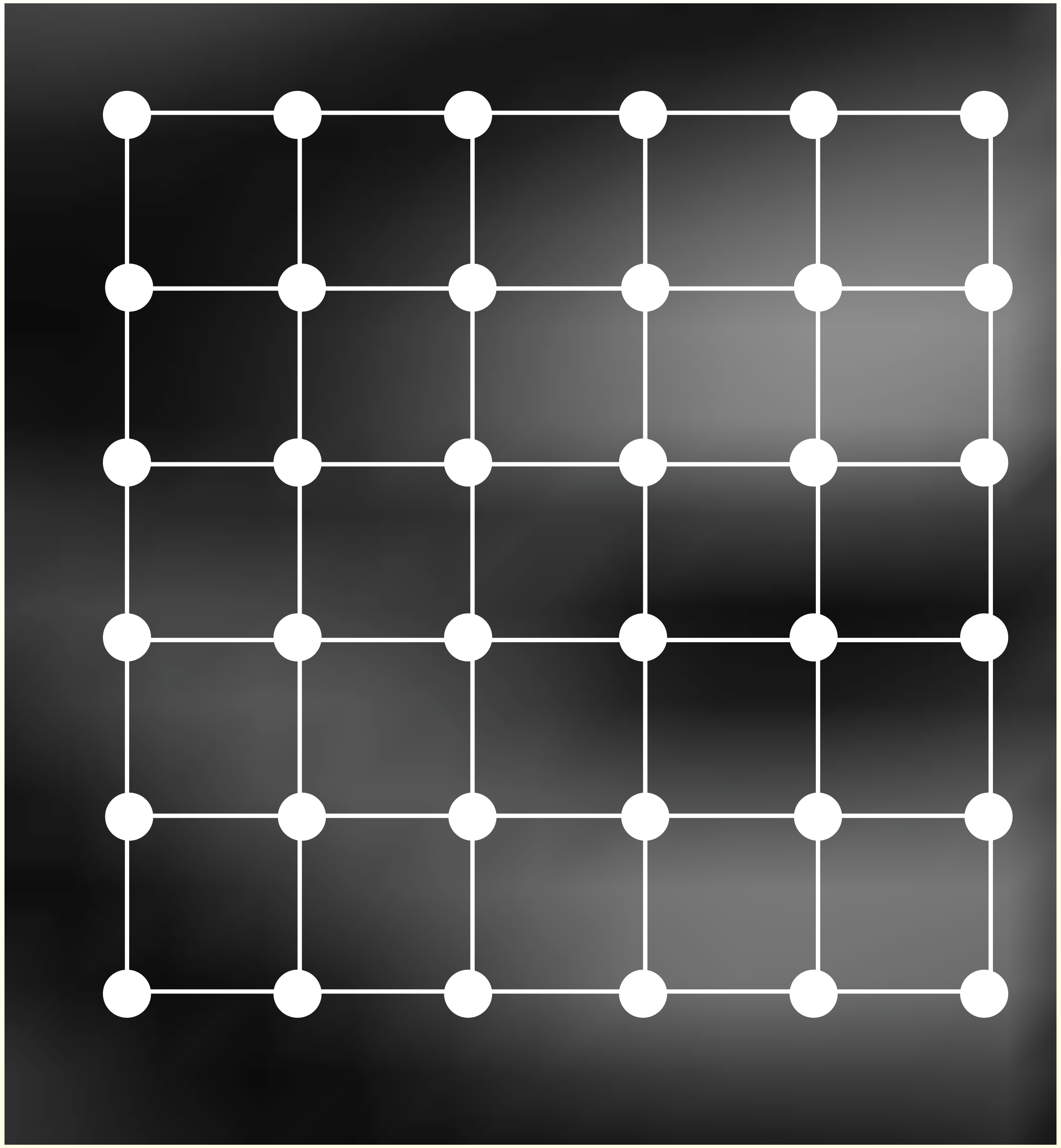

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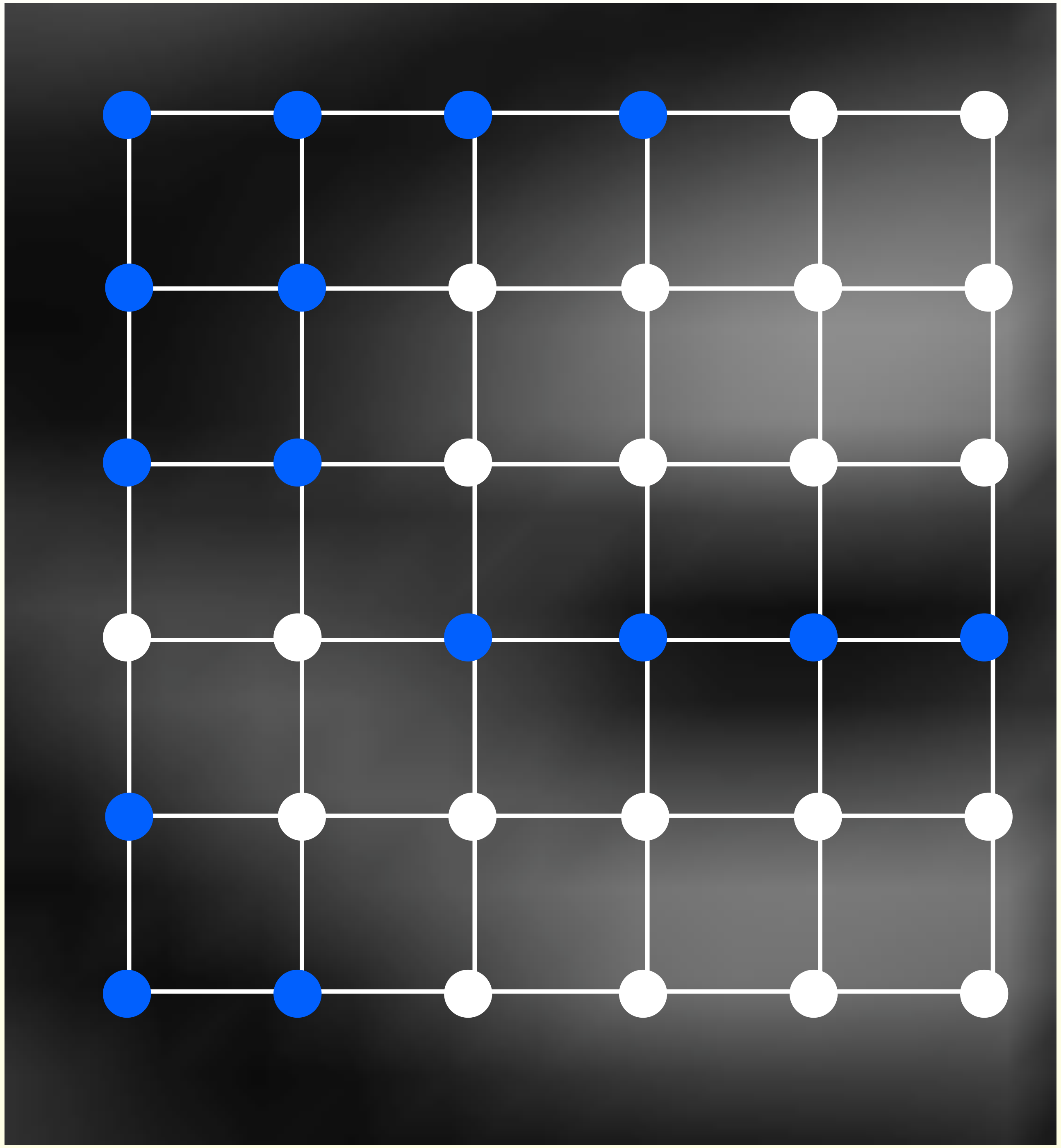
- Evaluate $f(\mathbf{x})$ on the grid
- Classify grid points



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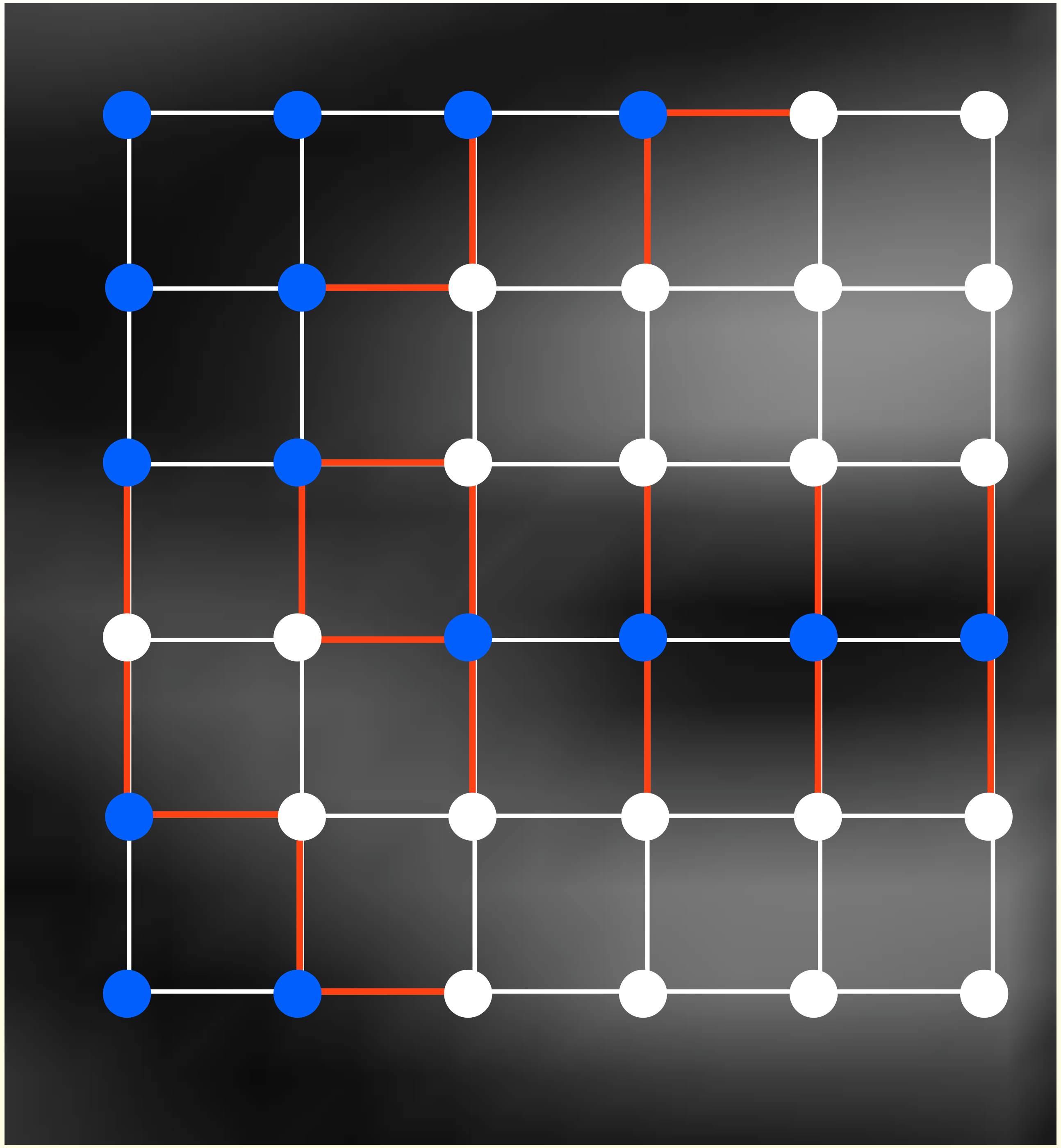
- Classify grid edges



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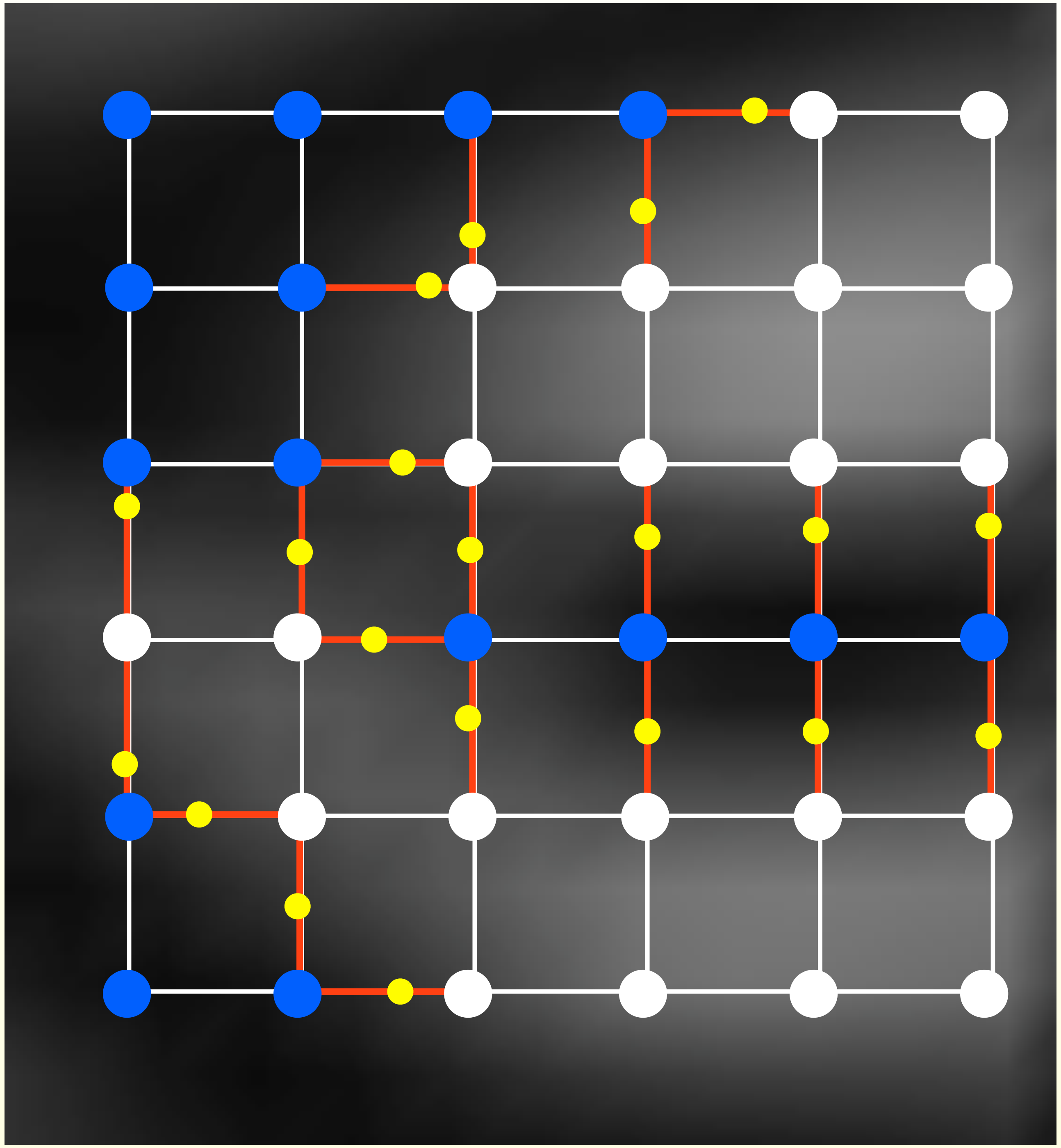
- Evaluate $f(\mathbf{x})$ on the grid
- Classify grid points
- Classify grid edges
- Compute intersections



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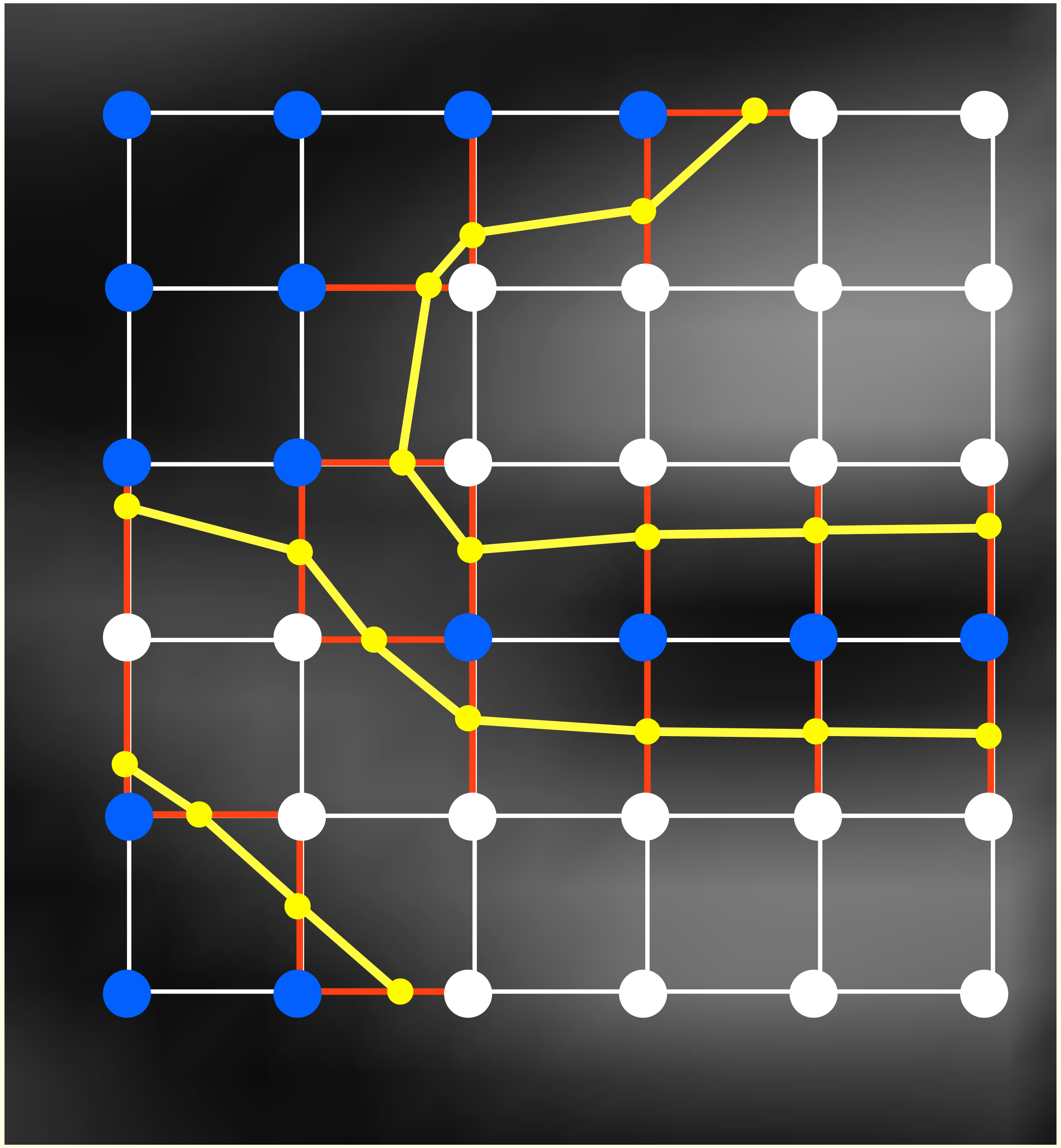
- Connect intersections



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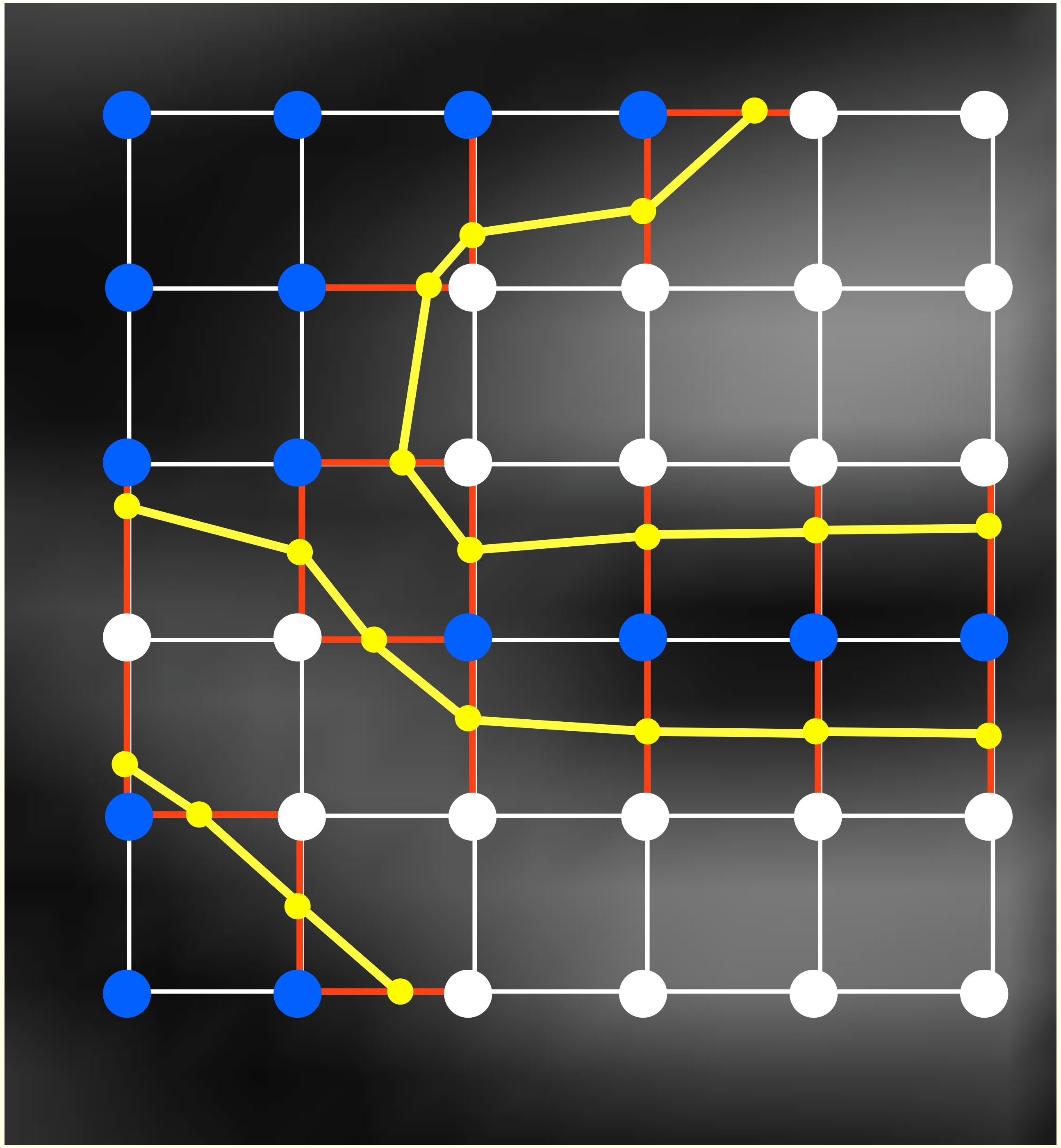

Computing Intersections

- Edges with a sign switch contain intersections

$$
\begin{aligned}
& f(\mathbf{x}_1) < 0 \text{ and } f(\mathbf{x}_2) \geq 0 \\
\Rightarrow \quad & f(\mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)) = 0 \text{ for some } 0 < t \leq 1
\end{aligned}
$$

- Nonlinear equation, use raycasting to find root

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- Nonlinear equation, use raycasting to find root
- Sampled data
  - f is trilinear
  - f is linear along $\mathbf{x}_2 - \mathbf{x}_1$

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$$t = -f(\mathbf{x}_1)/(f(\mathbf{x}_2) - f(\mathbf{x}_1))$$

Connecting Intersections

- Treat each cell separately

Connecting Intersections

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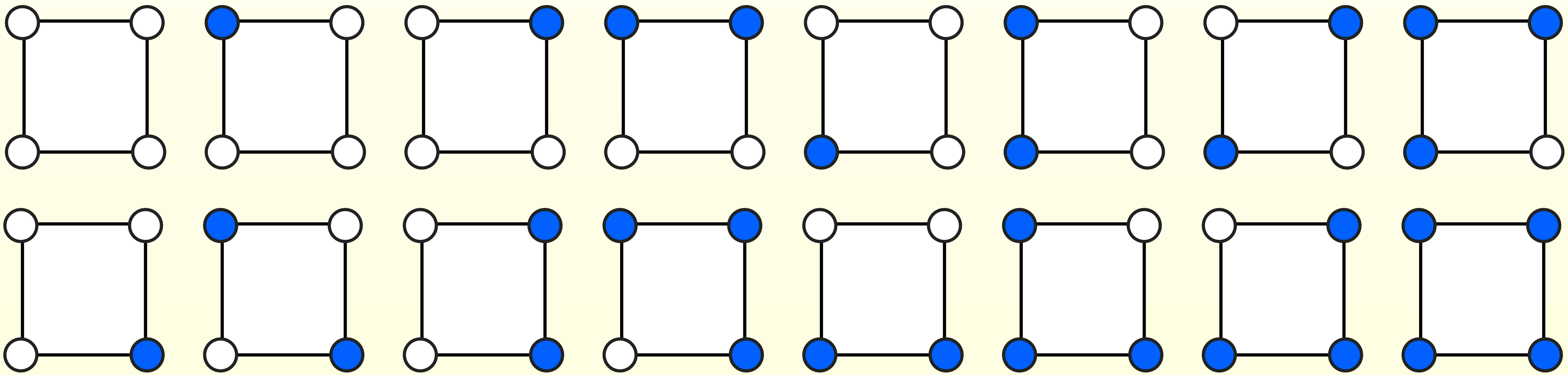
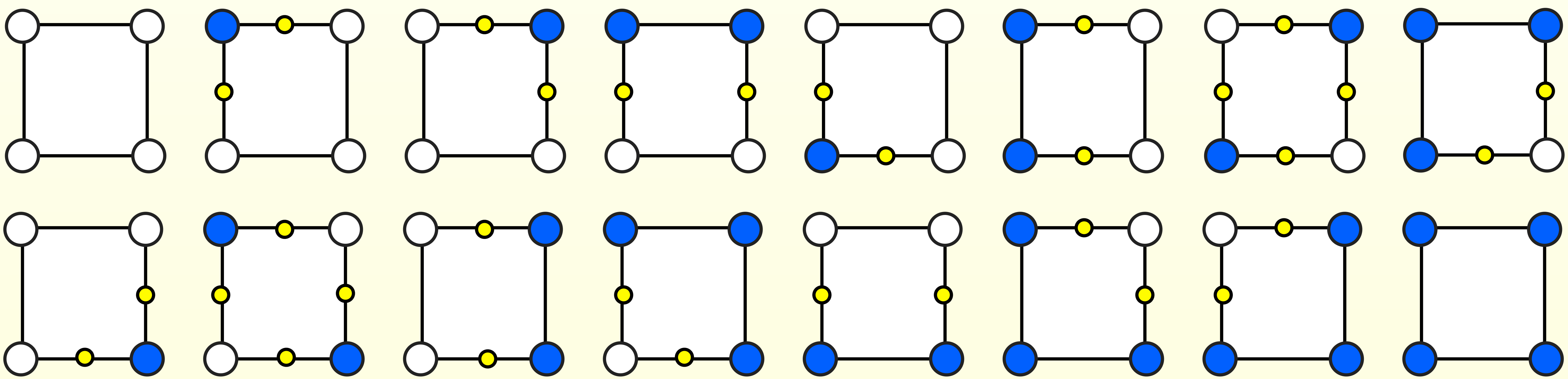
- Enumerate all possible inside/outside combinations



Connecting Intersections

- Treat each cell separately
- Enumerate all possible inside/outside combinations
- Group those leading to the same intersections



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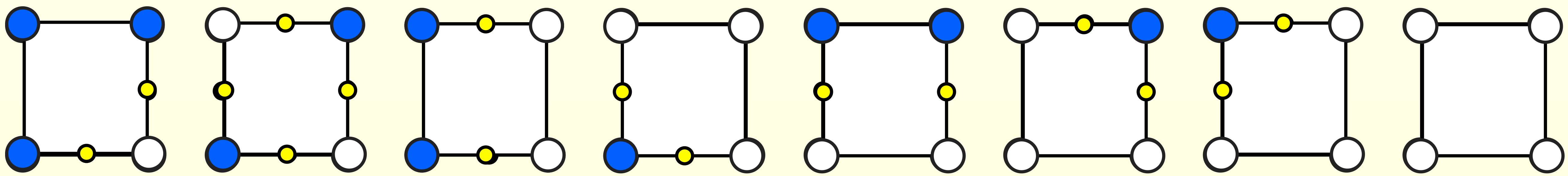

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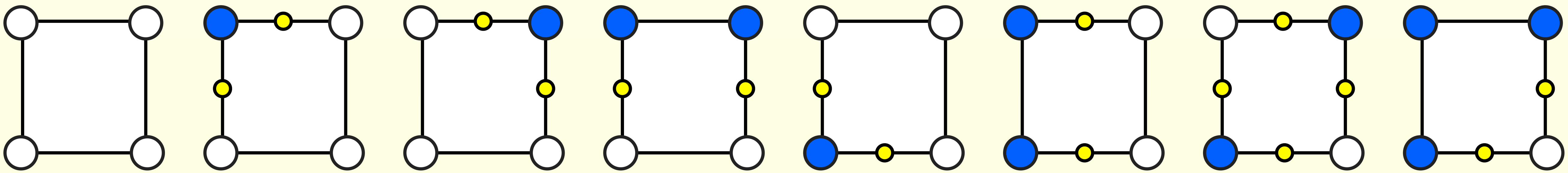
- Group those equivalent after rotations



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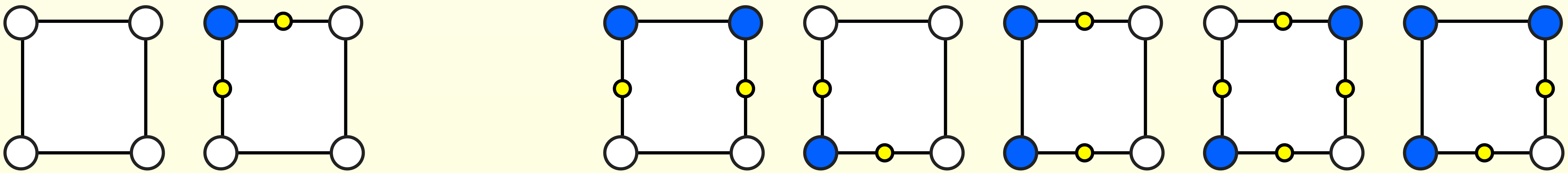

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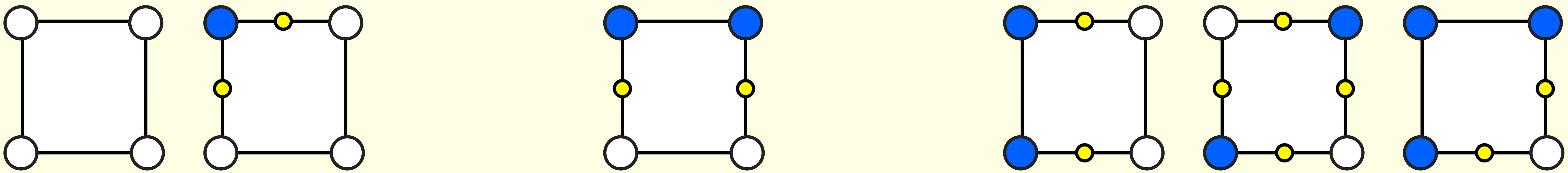

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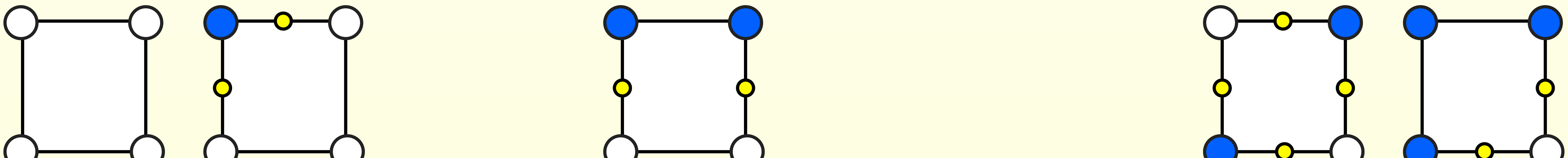

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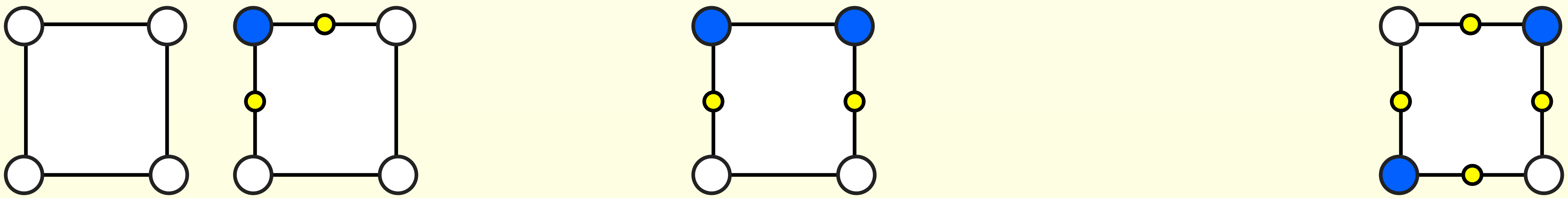

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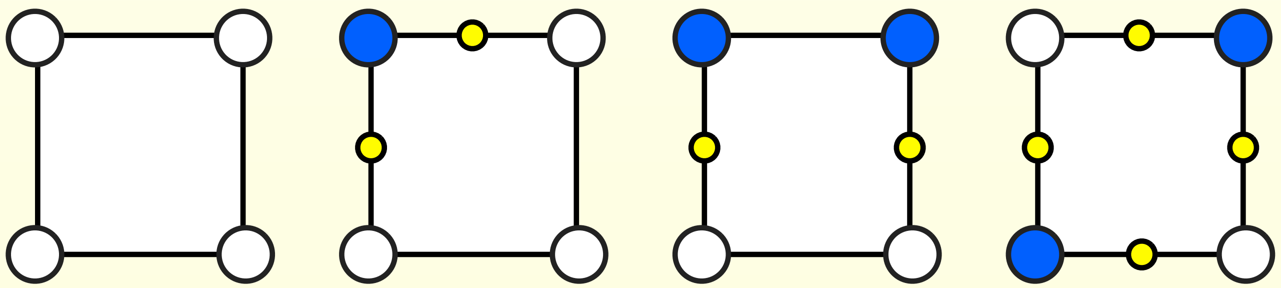

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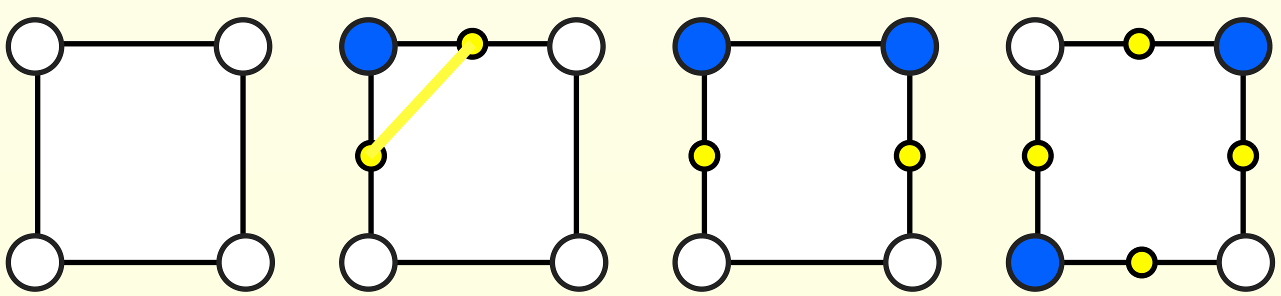
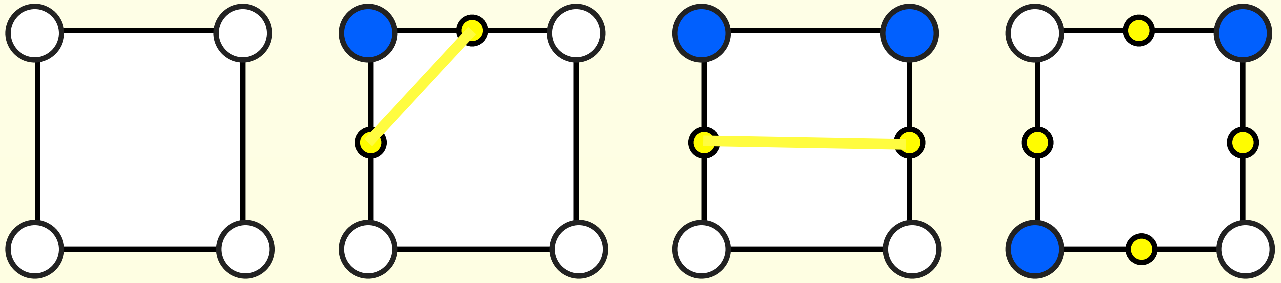


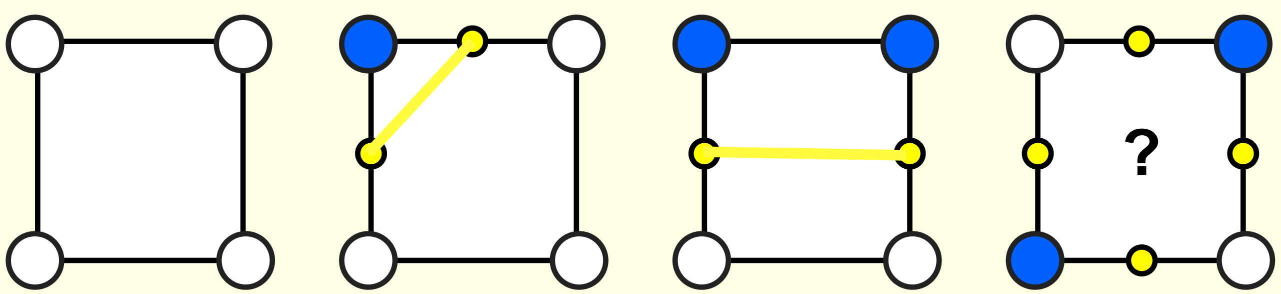
Connecting Intersections

- Treat each cell separately
- Enumerate all possible inside/outside combinations
- Group those leading to the same intersections
- Group those equivalent after rotations
- Connect intersections

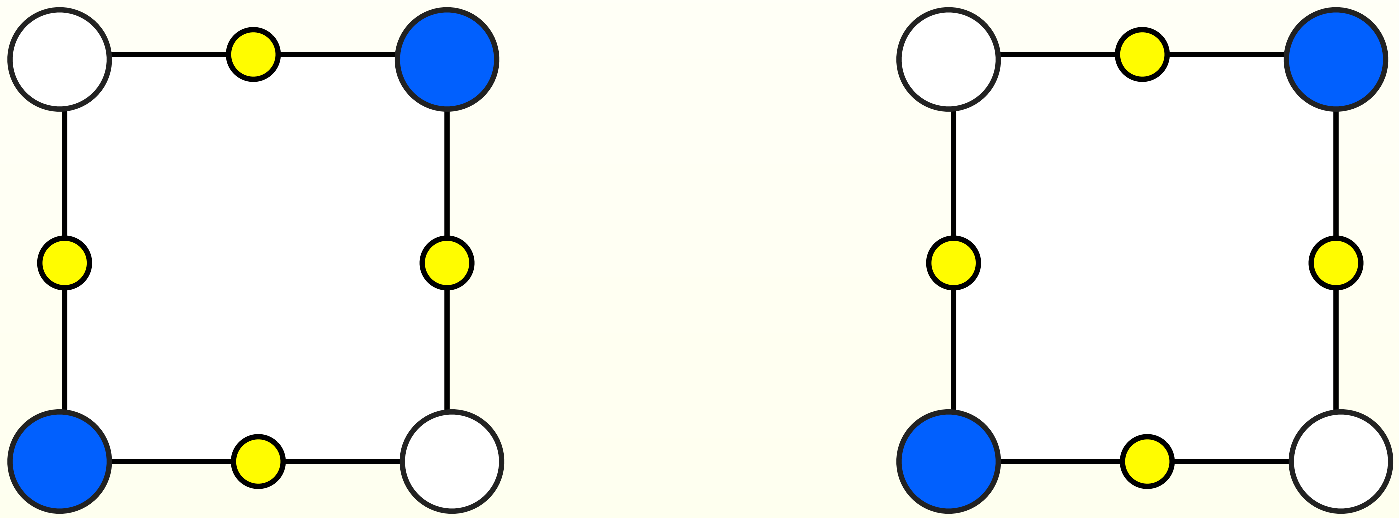


Connecting Intersections

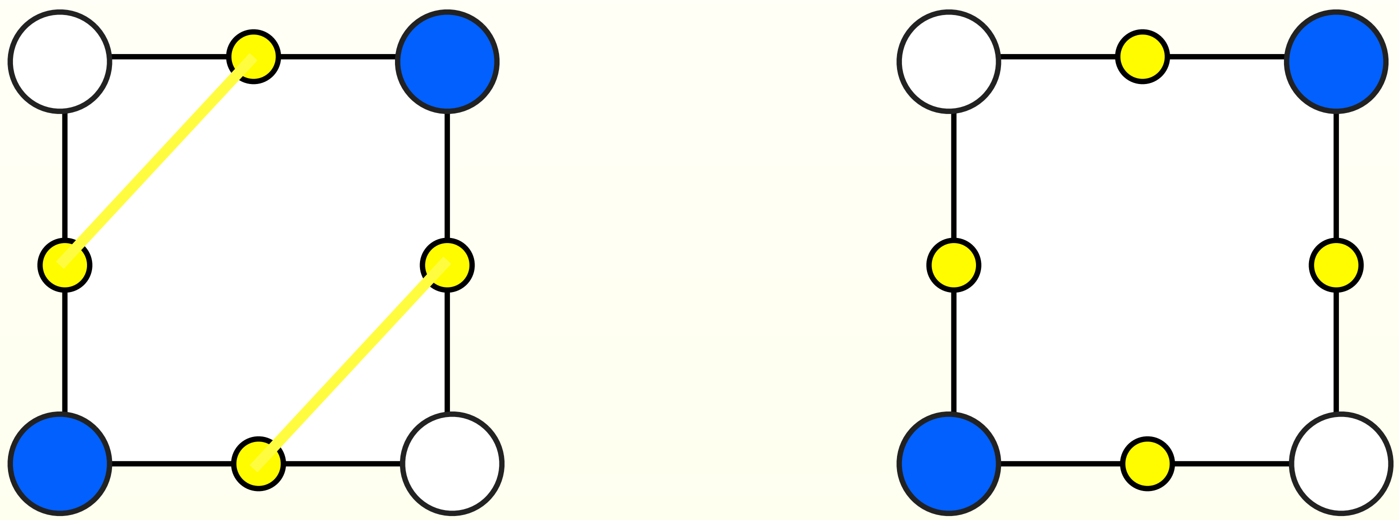
- Treat each cell separately
- Enumerate all possible inside/outside combinations
- Group those leading to the same intersections
- Group those equivalent after rotations
- Connect intersections



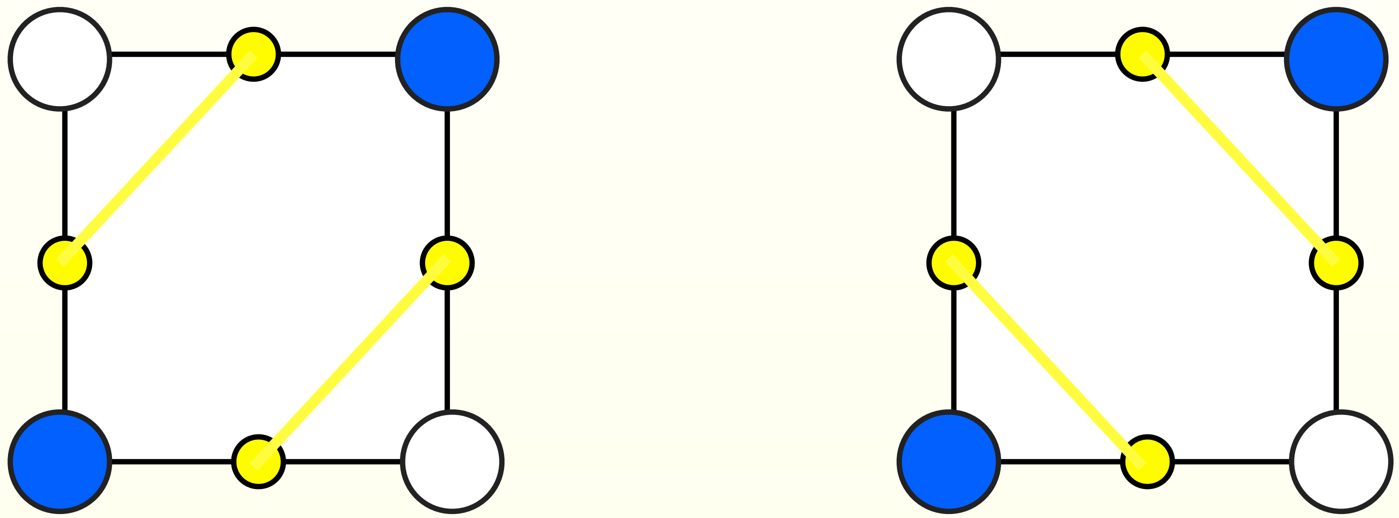
Ambiguous case



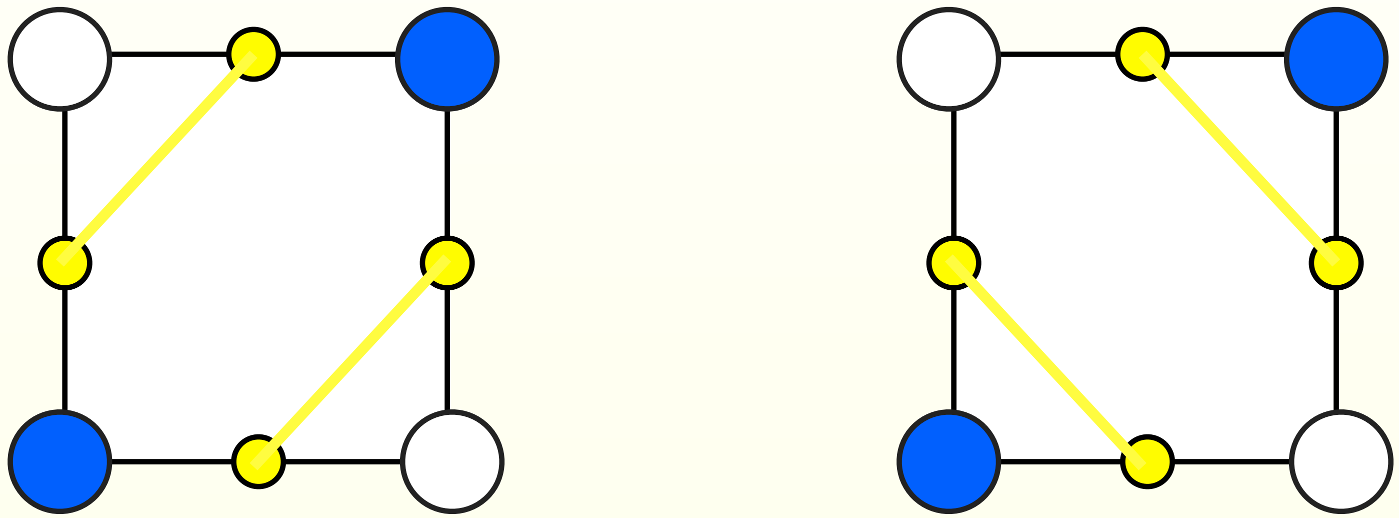
Ambiguous case



Ambiguous case

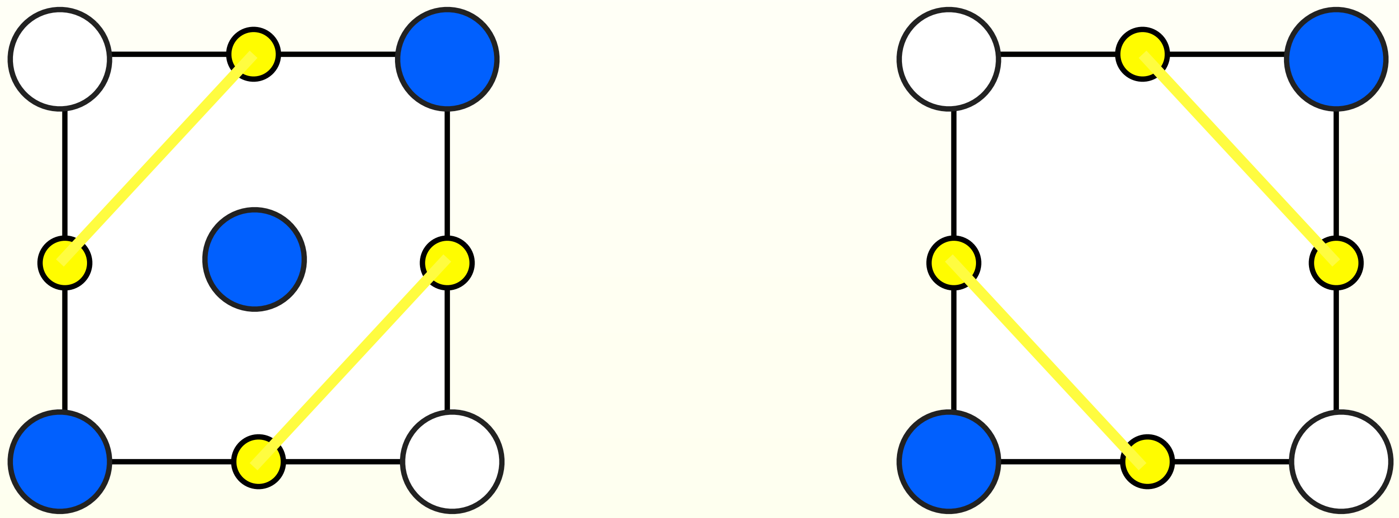


Ambiguous case



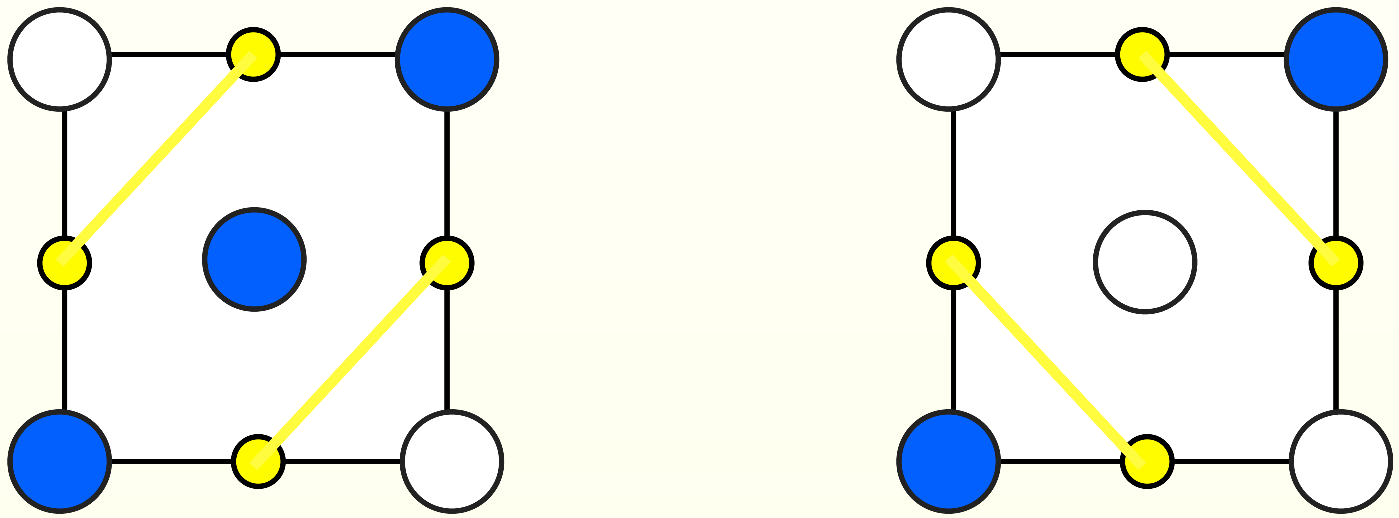
- No way to decide without further samples

Ambiguous case



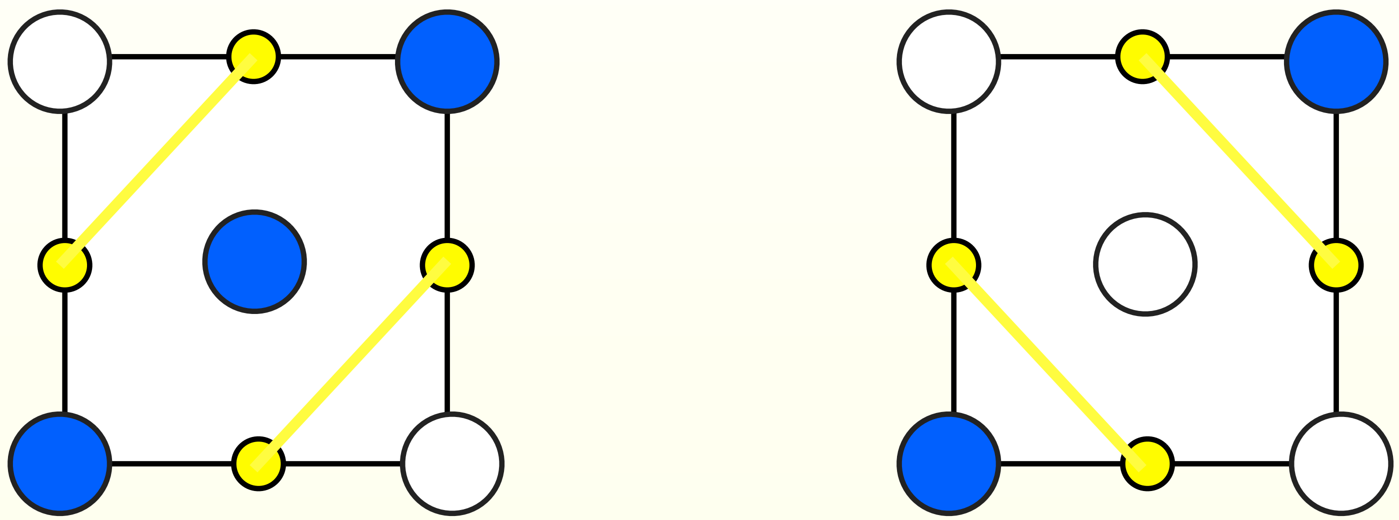
- No way to decide without further samples

Ambiguous case



- No way to decide without further samples

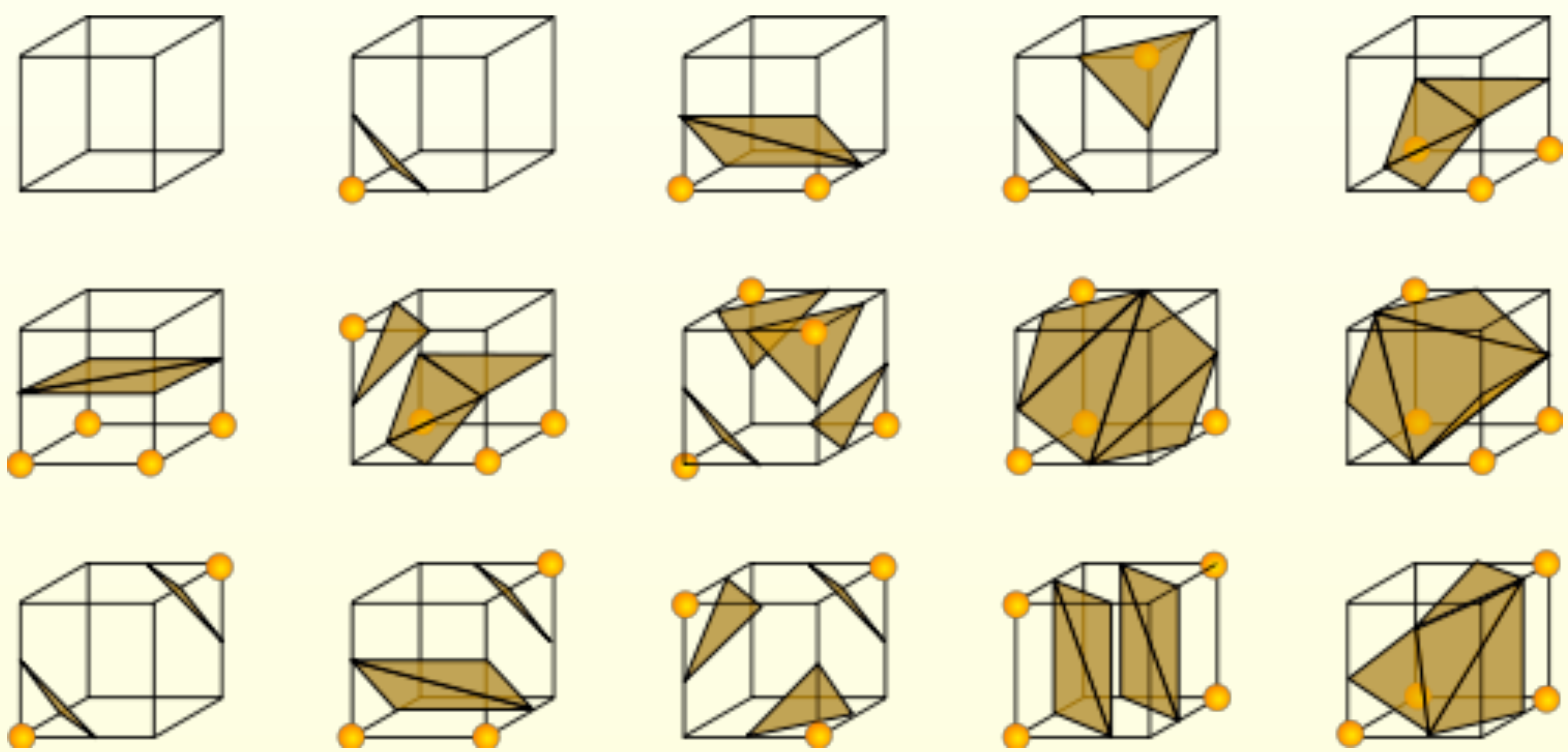
Ambiguous case



- No way to decide without further samples
- No samples available: Just choose one

Marching Cubes

- Same basic principle in 3D
- Lines become surface patches
  - Up to 4 triangles per voxel
- 256 different cases, 15 after symmetries



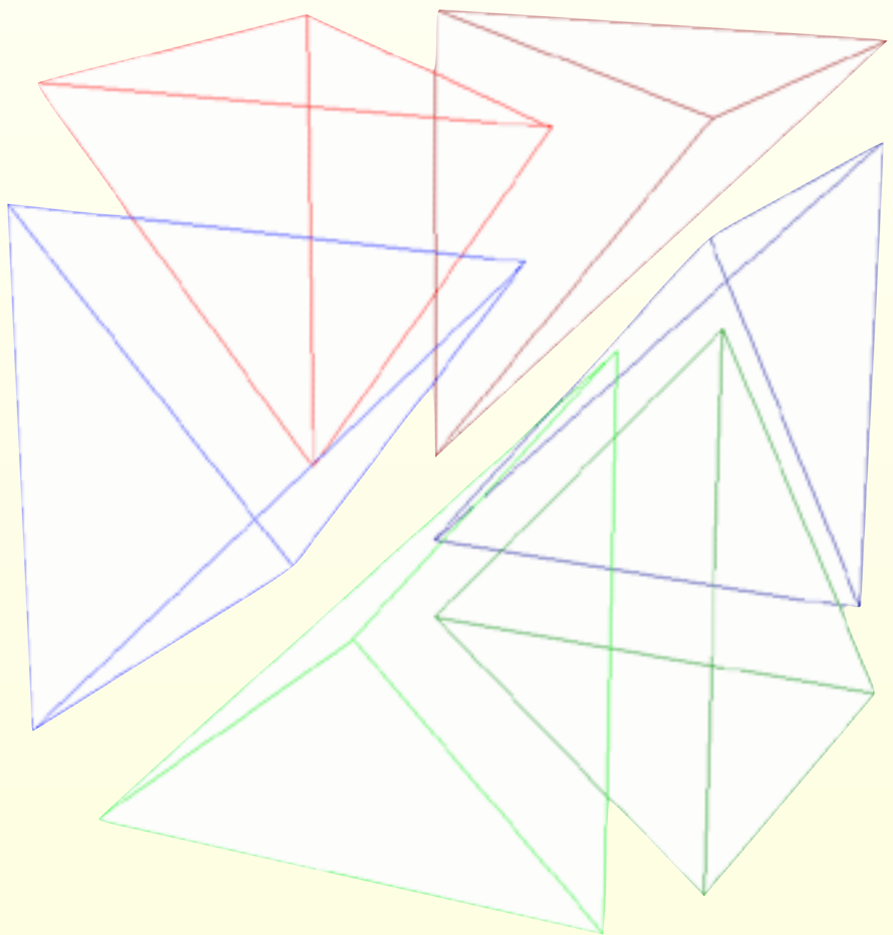
Marching Tetrahedra

- Different discretization: Tetrahedra

[Paul Bourke]

Marching Tetrahedra

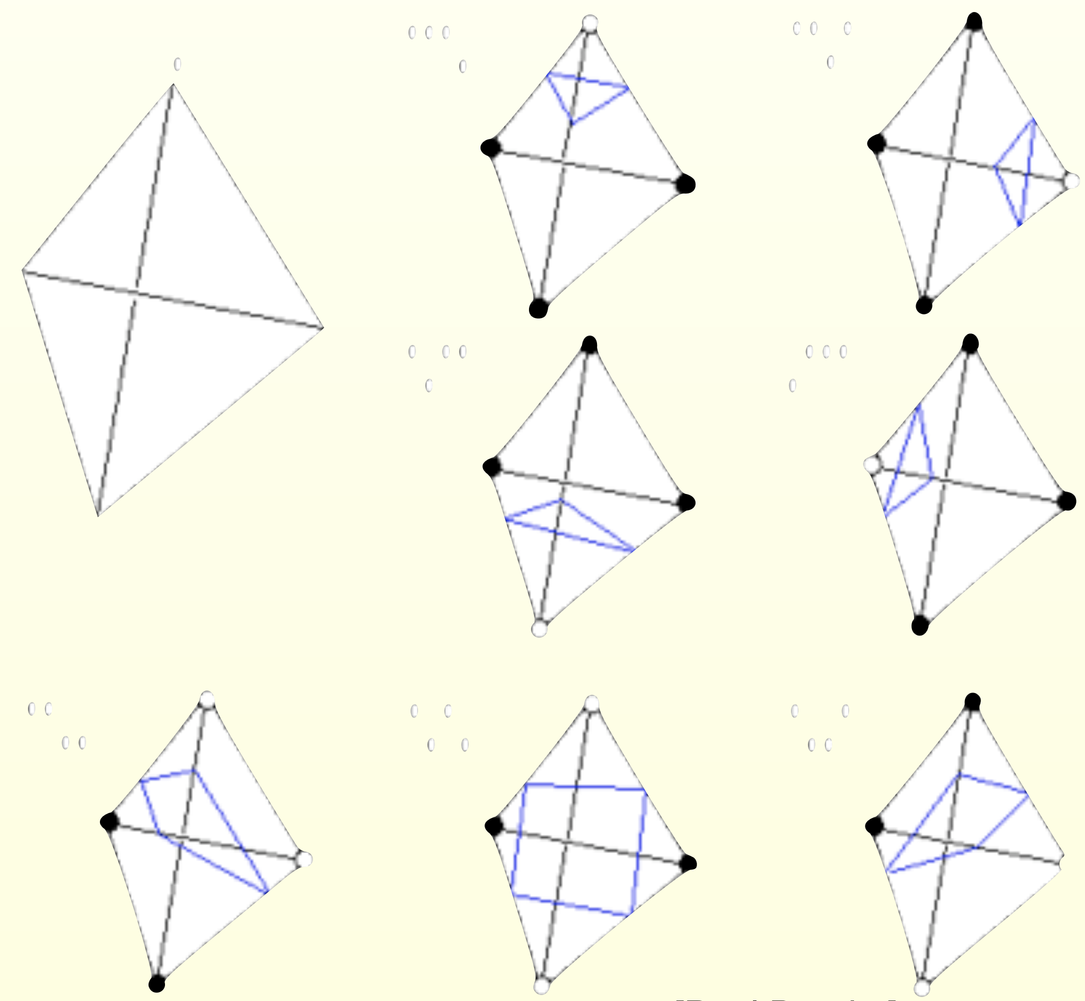
- Different discretization: Tetrahedra
  - 6 tetrahedra per voxel (if we start from cubes)



[Paul Bourke]

Marching Tetrahedra

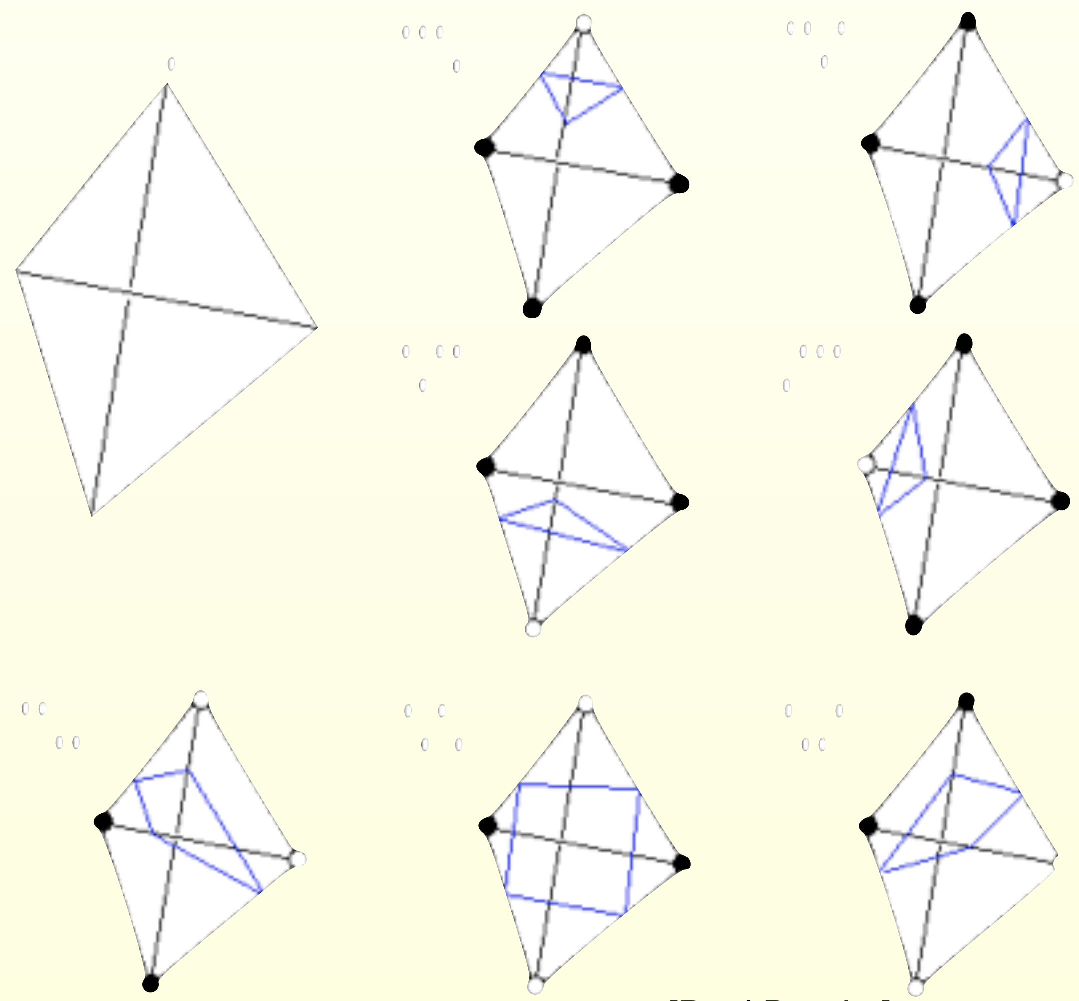
- Different discretization: Tetrahedra
  - 6 tetrahedra per voxel (if we start from cubes)
  - 16 cases, 8 after symmetry



[Paul Bourke]

Marching Tetrahedra

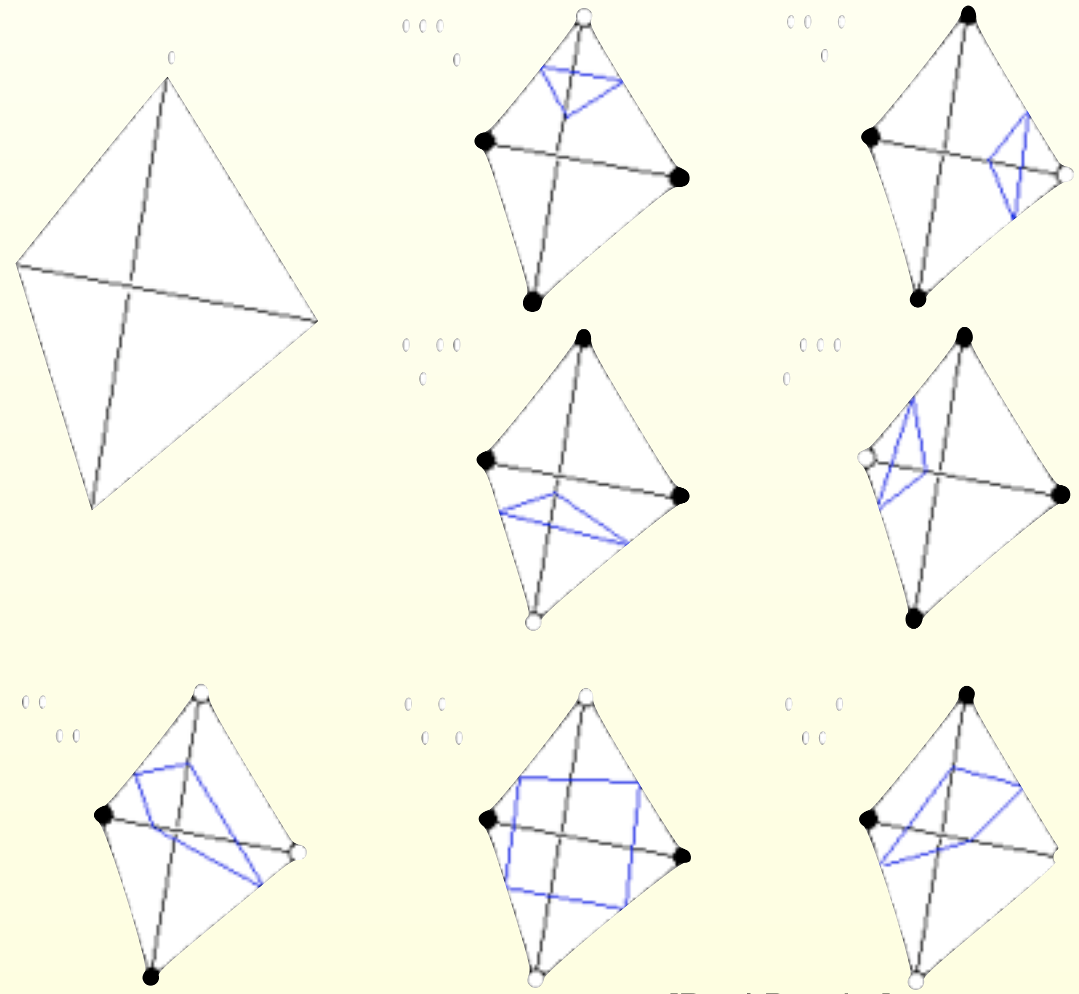
- Different discretization: Tetrahedra
  - 6 tetrahedra per voxel (if we start from cubes)
  - 16 cases, 8 after symmetry
  - Up to 2 triangles per tet



[Paul Bourke]

Marching Tetrahedra

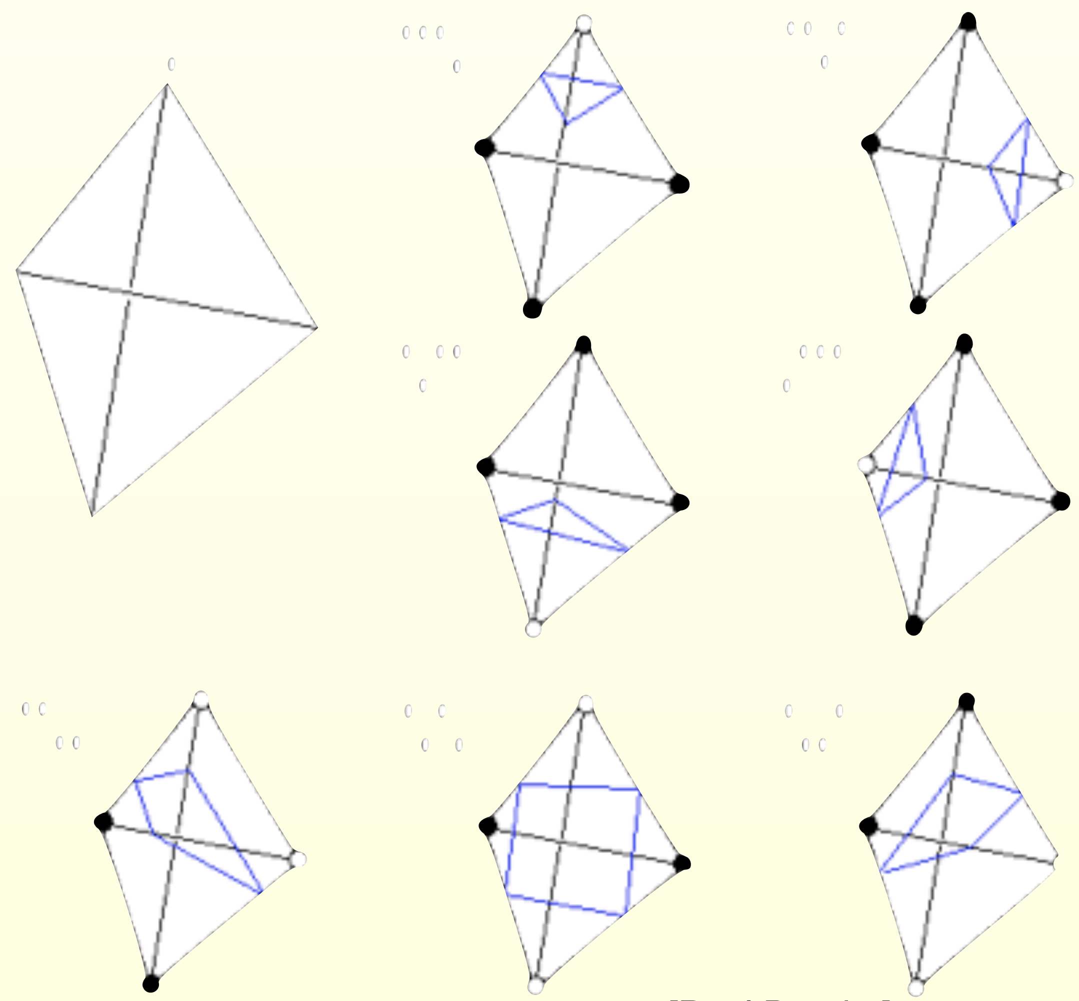
- Different discretization: Tetrahedra
  - 6 tetrahedra per voxel (if we start from cubes)
  - 16 cases, 8 after symmetry
  - Up to 2 triangles per tet
  - No ambiguities



[Paul Bourke]

Marching Tetrahedra

- Different discretization: Tetrahedra
  - 6 tetrahedra per voxel (if we start from cubes)
  - 16 cases, 8 after symmetry
  - Up to 2 triangles per tet
  - No ambiguities
- Used when input data discretized as tetrahedra



[Paul Bourke]

Implementation

- Big lookup tables

```
int edgeTable[256]={
0x0  , 0x109, 0x203, 0x30a, 0x406, 0x50f, 0x605, 0x70c,
0x80c, 0x905, 0xa0f, 0xb06, 0xc0a, 0xd03, 0xe09, 0xf00,
0x190, 0x99 , 0x393, 0x29a, 0x596, 0x49f, 0x795, 0x69c,
0x99c, 0x895, 0xb9f, 0xa96, 0xd9a, 0xc93, 0xf99, 0xe90,
0x230, 0x339, 0x33 , 0x13a, 0x636, 0x73f, 0x435, 0x53c,
0xa3c, 0xb35, 0x83f, 0x936, 0xe3a, 0xf33, 0xc39, 0xd30,
0x3a0, 0x2a9, 0x1a3, 0xaa , 0x7a6, 0x6af, 0x5a5, 0x4ac,
0xbac, 0xaa5, 0x9af, 0x8a6, 0xfaa, 0xea3, 0xda9, 0xca0,
0x460, 0x569, 0x663, 0x76a, 0x66 , 0x16f, 0x265, 0x36c,
0xc6c, 0xd65, 0xe6f, 0xf66, 0x86a, 0x963, 0xa69, 0xb60,
0x5f0, 0x4f9, 0x7f3, 0x6fa, 0x1f6, 0xff , 0x3f5, 0x2fc,
0xdfc, 0xcf5, 0xfff, 0xef6, 0x9fa, 0x8f3, 0xbf9, 0xaf0,
0x650, 0x759, 0x453, 0x55a, 0x256, 0x35f, 0x55 , 0x15c,
0xe5c, 0xf55, 0xc5f, 0xd56, 0xa5a, 0xb53, 0x859, 0x950,
0x7c0, 0x6c9, 0x5c3, 0x4ca, 0x3c6, 0x2cf, 0x1c5, 0xcc ,
0xfcc, 0xec5, 0xdcf, 0xcc6, 0xbca, 0xac3, 0x9c9, 0x8c0,
0x8c0, 0x9c9, 0xac3, 0xbca, 0xcc6, 0xdcf, 0xec5, 0xfcc,
0xcc , 0x1c5, 0x2cf, 0x3c6, 0x4ca, 0x5c3, 0x6c9, 0x7c0,
0x950, 0x859, 0xb53, 0xa5a, 0xd56, 0xc5f, 0xf55, 0xe5c,
0x15c, 0x55 , 0x35f, 0x256, 0x55a, 0x453, 0x759, 0x650,
0xaf0, 0xbf9, 0x8f3, 0x9fa, 0xef6, 0xfff, 0xcf5, 0xdfc,
0x2fc, 0x3f5, 0xff , 0x1f6, 0x6fa, 0x7f3, 0x4f9, 0x5f0,
0xb60, 0xa69, 0x963, 0x86a, 0xf66, 0xe6f, 0xd65, 0xc6c,
0x36c, 0x265, 0x16f, 0x66 , 0x76a, 0x663, 0x569, 0x460,
0xca0, 0xda9, 0xea3, 0xfaa, 0x8a6, 0x9af, 0xaa5, 0xbac,
0x4ac, 0x5a5, 0x6af, 0x7a6, 0xaa , 0x1a3, 0x2a9, 0x3a0,
0xd30, 0xc39, 0xf33, 0xe3a, 0x936, 0x83f, 0xb35, 0xa3c,
0x53c, 0x435, 0x73f, 0x636, 0x13a, 0x33 , 0x339, 0x230,
0xe90, 0xf99, 0xc93, 0xd9a, 0xa96, 0xb9f, 0x895, 0x99c,
0x69c, 0x795, 0x49f, 0x596, 0x29a, 0x393, 0x99 , 0x190,
0xf00, 0xe09, 0xd03, 0xc0a, 0xb06, 0xa0f, 0x905, 0x80c,
0x70c, 0x605, 0x50f, 0x406, 0x30a, 0x203, 0x109, 0x0   };
```

```
int triTable[256][16] =
{{-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 8, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 1, 9, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{1, 8, 3, 9, 8, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{1, 2, 10, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 8, 3, 1, 2, 10, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{9, 2, 10, 0, 2, 9, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{2, 8, 3, 2, 10, 8, 10, 9, 8, -1, -1, -1, -1, -1, -1, -1},
{3, 11, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 11, 2, 8, 11, 0, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{1, 9, 0, 2, 3, 11, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{1, 11, 2, 1, 9, 11, 9, 8, 11, -1, -1, -1, -1, -1, -1, -1},
{3, 10, 1, 11, 10, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 10, 1, 0, 8, 10, 8, 11, 10, -1, -1, -1, -1, -1, -1, -1},
{3, 9, 0, 3, 11, 9, 11, 10, 9, -1, -1, -1, -1, -1, -1, -1},
{9, 8, 10, 10, 8, 11, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{4, 7, 8, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{4, 3, 0, 7, 3, 4, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 1, 9, 8, 4, 7, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{4, 1, 9, 4, 7, 1, 7, 3, 1, -1, -1, -1, -1, -1, -1, -1},
{1, 2, 10, 8, 4, 7, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{3, 4, 7, 3, 0, 4, 1, 2, 10, -1, -1, -1, -1, -1, -1, -1},
{9, 2, 10, 9, 0, 2, 8, 4, 7, -1, -1, -1, -1, -1, -1, -1},
{2, 10, 9, 2, 9, 7, 2, 7, 3, 7, 9, 4, -1, -1, -1, -1},
{8, 4, 7, 3, 11, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{11, 4, 7, 11, 2, 4, 2, 0, 4, -1, -1, -1, -1, -1, -1, -1},
{9, 0, 1, 8, 4, 7, 2, 3, 11, -1, -1, -1, -1, -1, -1, -1},
{4, 7, 11, 9, 4, 11, 9, 11, 2, 9, 2, 1, -1, -1, -1, -1},
{3, 10, 1, 3, 11, 10, 7, 8, 4, -1, -1, -1, -1, -1, -1, -1},
{1, 11, 10, 1, 4, 11, 1, 0, 4, 7, 11, 4, -1, -1, -1, -1},
{4, 7, 8, 9, 0, 11, 9, 11, 10, 11, 0, 3, -1, -1, -1, -1},
{4, 7, 11, 4, 11, 9, 9, 11, 10, -1, -1, -1, -1, -1, -1, -1},
{9, 5, 4, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{9, 5, 4, 0, 8, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 5, 4, 1, 5, 0, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{8, 5, 4, 8, 3, 5, 3, 1, 5, -1, -1, -1, -1, -1, -1, -1},
{1, 2, 10, 9, 5, 4, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
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{9, 5, 4, 2, 3, 11, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1},
{0, 11, 2, 0, 8, 11, 4, 9, 5, -1, -1, -1, -1, -1, -1, -1},
```
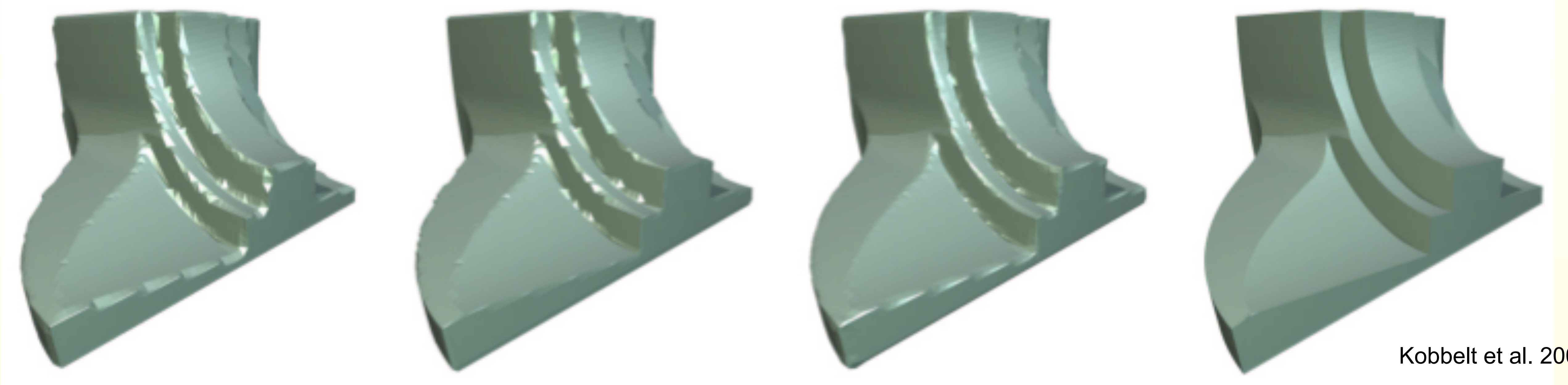

[Paul Bourke]

Problems & Solutions

No Sharp Features

- Increasing grid resolution does not help
- Normals do not converge



Kobbelt et al. 2001

No Sharp Features

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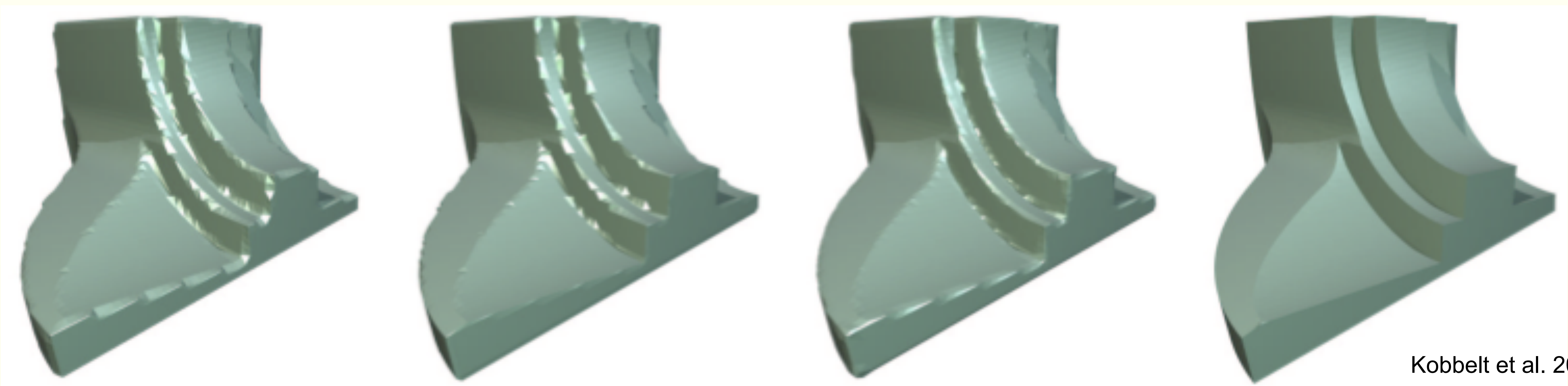
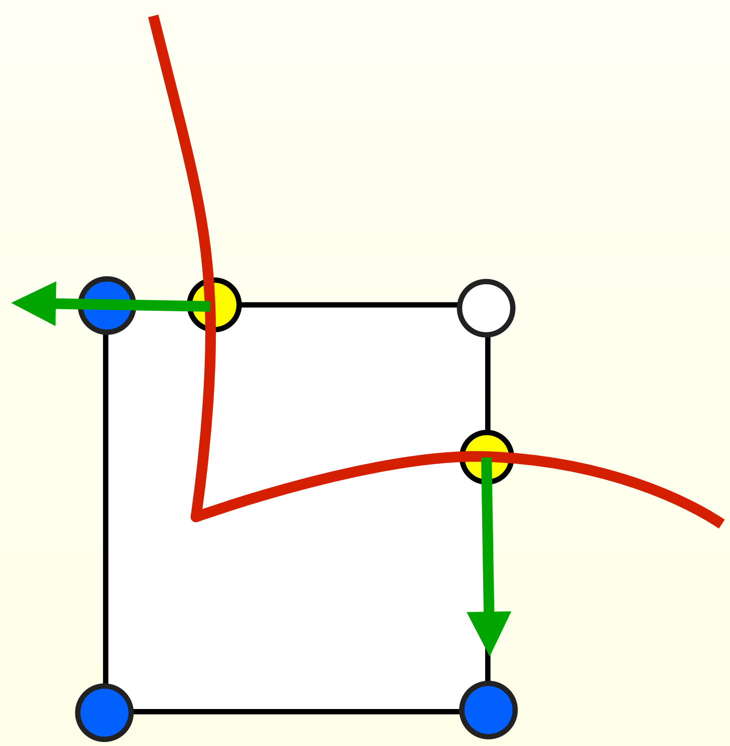


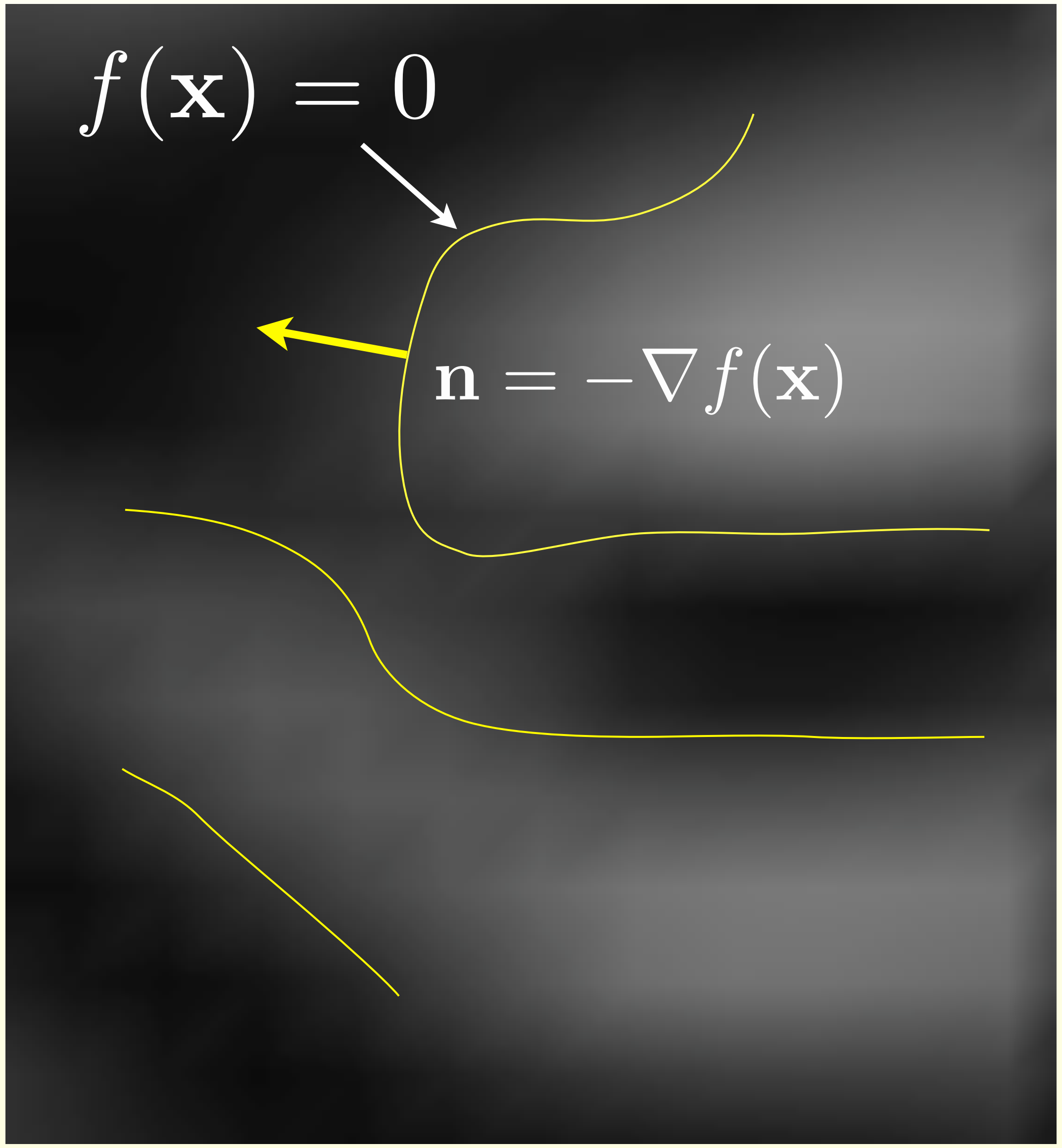
- Use normal information to find edges and corners

Using Hermite Data



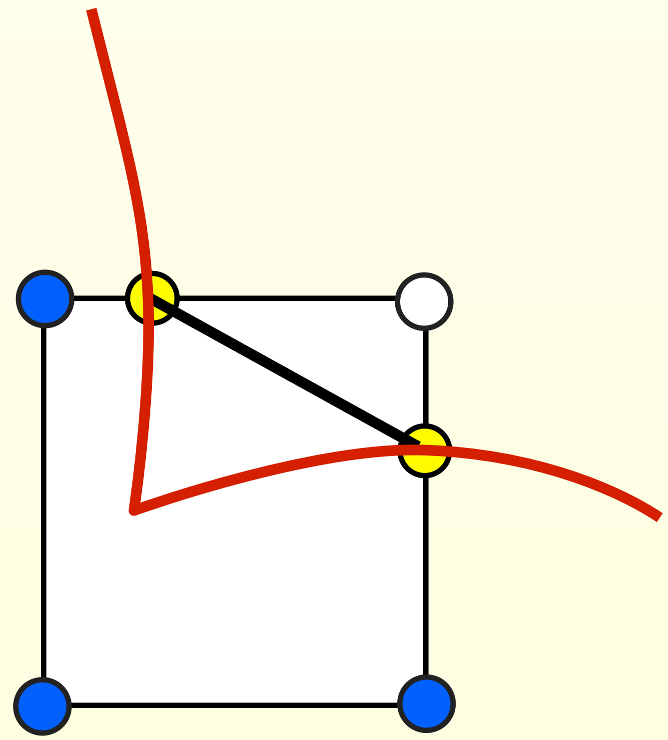
Where Do Normals Come From?

- Gradient of the function $f(\mathbf{x})$ defining the surface



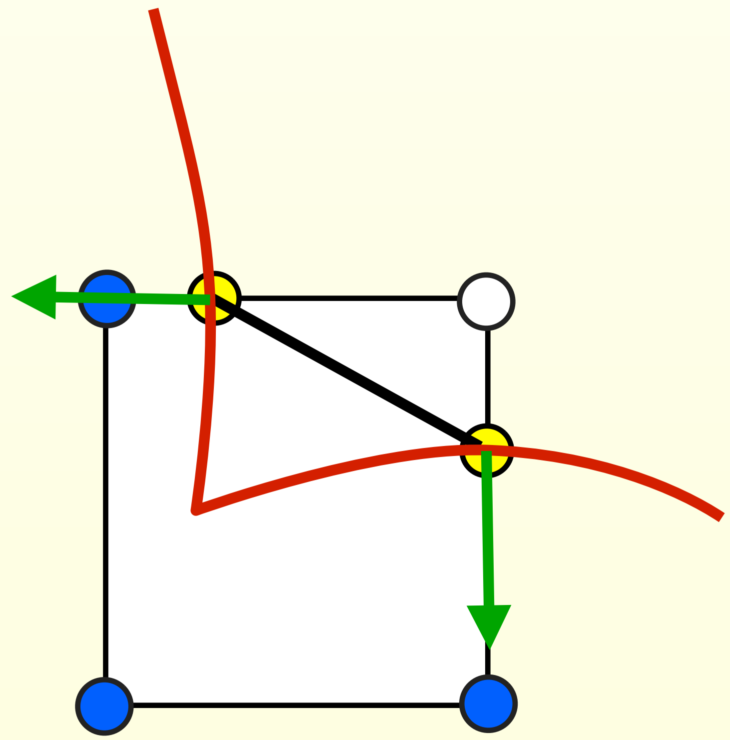
Extended Marching Cubes

- Sharp features are not well approximated



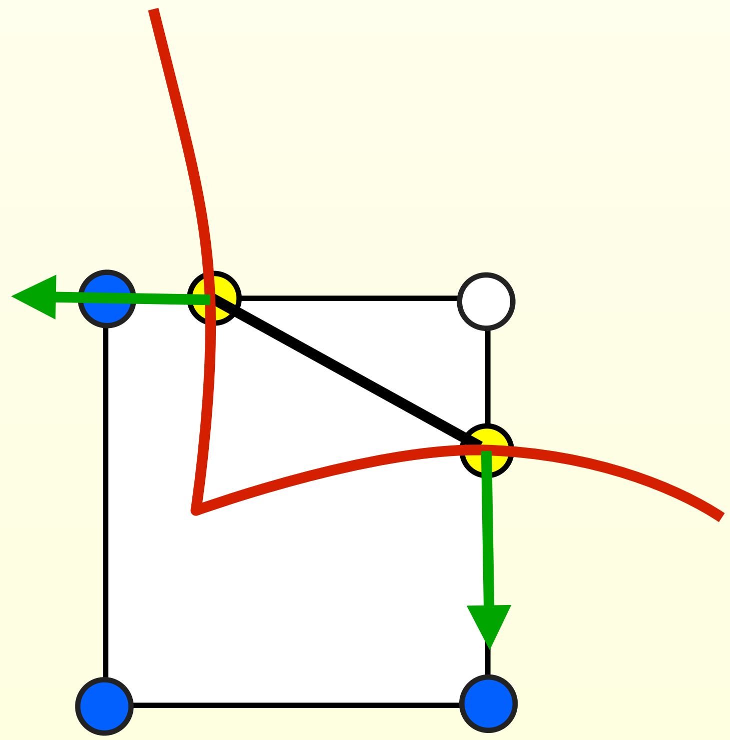
Extended Marching Cubes

- Sharp features are not well approximated
- Use normal information



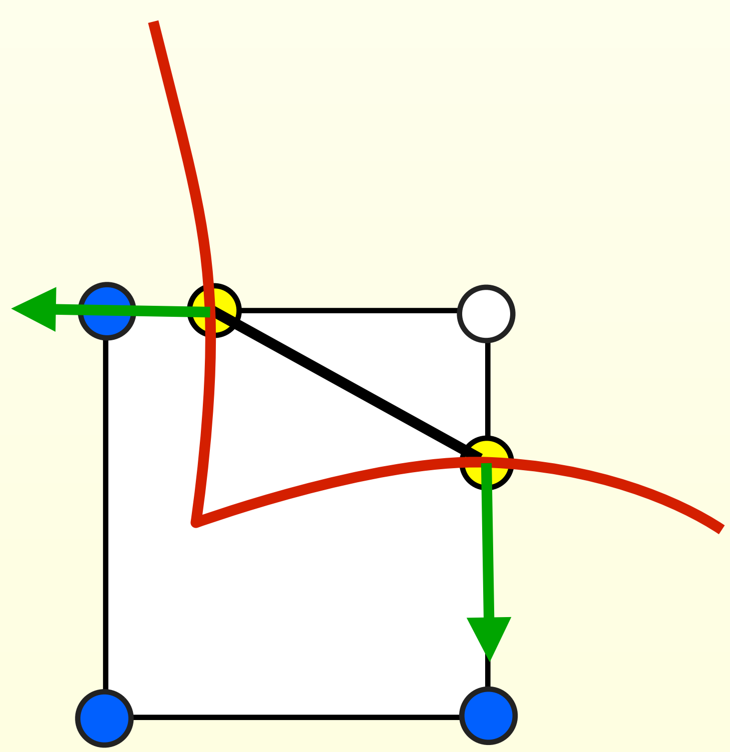
Extended Marching Cubes

- Sharp features are not well approximated
- Use normal information
- Special treatment for corners and edges



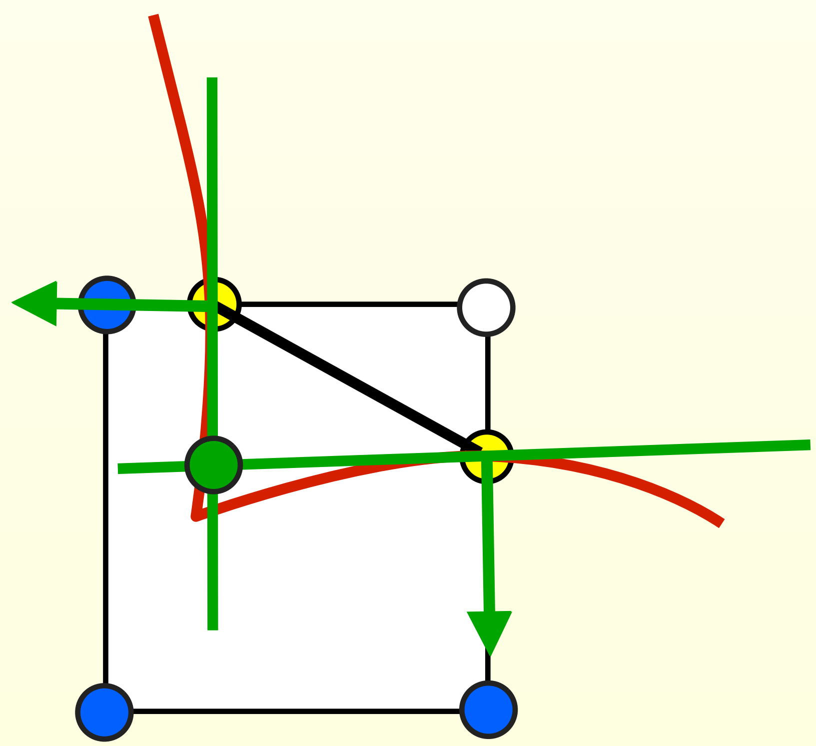
Extended Marching Cubes

- Sharp features are not well approximated
- Use normal information
- Special treatment for corners and edges
  - Corner/edge if large angle between normals



Extended Marching Cubes

- Sharp features are not well approximated
- Use normal information
- Special treatment for corners and edges
  - Corner/edge if large angle between normals
  - Add a vertex at the intersection of tangent planes



Extended Marching Cubes

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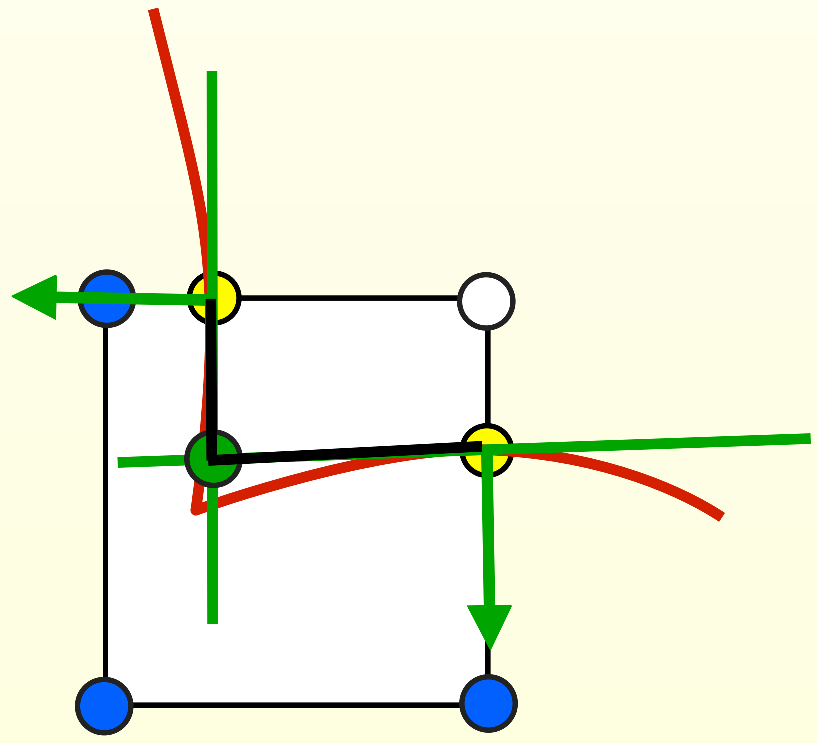
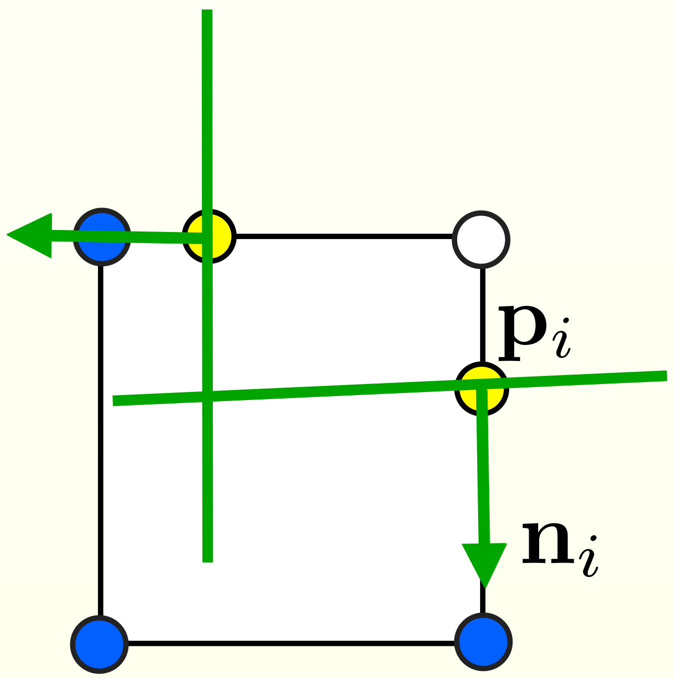


Extended Marching Cubes

- Computing the intersection

$$\mathbf{n}_i \mathbf{x} = \mathbf{n}_i \mathbf{p}_i$$

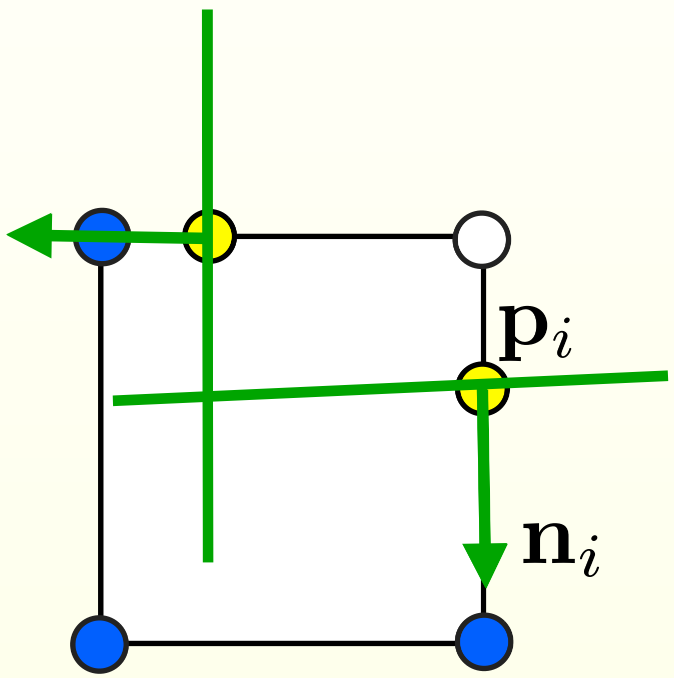


- Sometimes underdetermined
  - When planes are (almost) parallel
- Sometimes overdetermined
  - When too many planes

Extended Marching Cubes

- Computing the intersection

$$\begin{bmatrix} \mathbf{n}_1^T \\ \vdots \\ \mathbf{n}_k^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{n}_1^T \mathbf{p}_1 \\ \vdots \\ \mathbf{n}_k^T \mathbf{p}_k \end{bmatrix}$$



- Sometimes underdetermined
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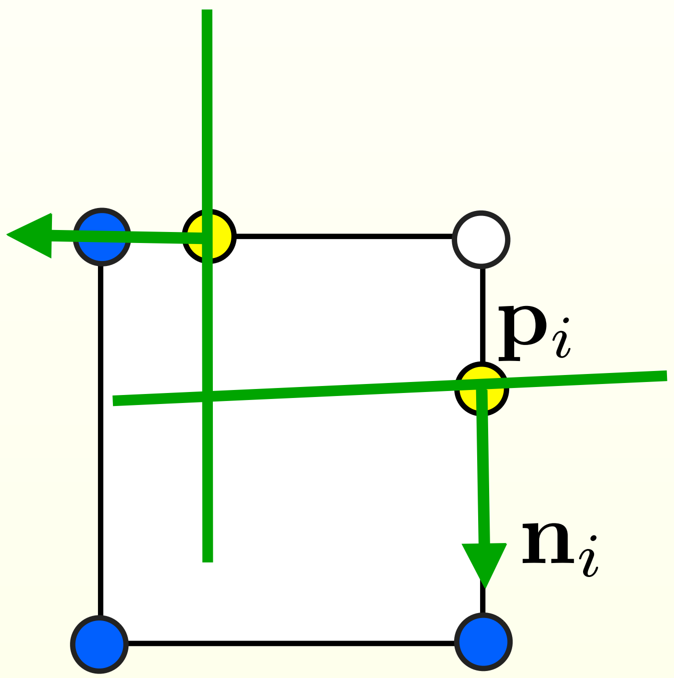
Extended Marching Cubes

- Computing the intersection

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{n}_1^T \\ \vdots \\ \mathbf{n}_k^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{n}_1^T \mathbf{p}_1 \\ \vdots \\ \mathbf{n}_k^T \mathbf{p}_k \end{bmatrix} = \mathbf{b}$$



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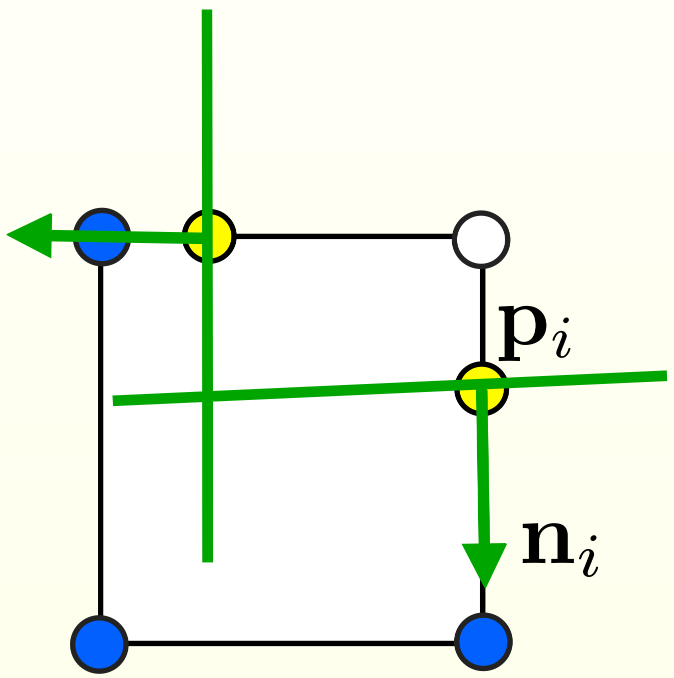
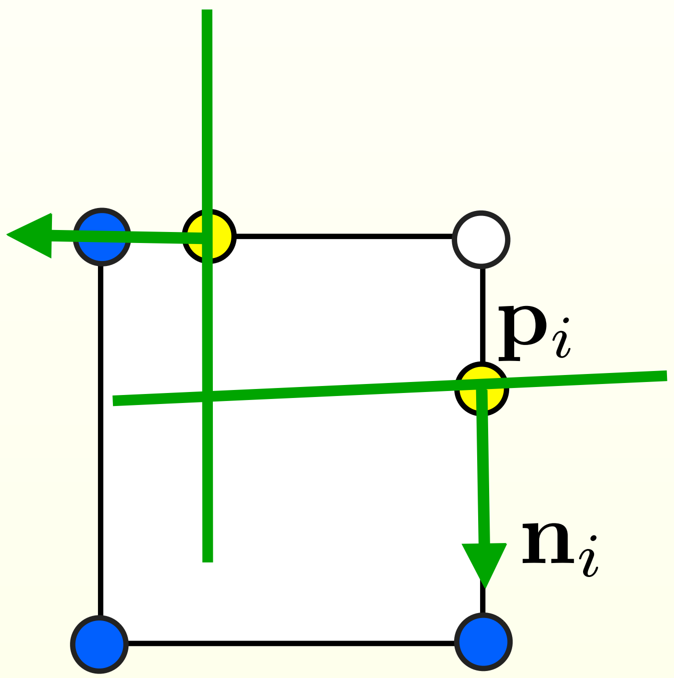


- Sometimes underdetermined
  - When planes are (almost) parallel

Extended Marching Cubes

- Computing the intersection

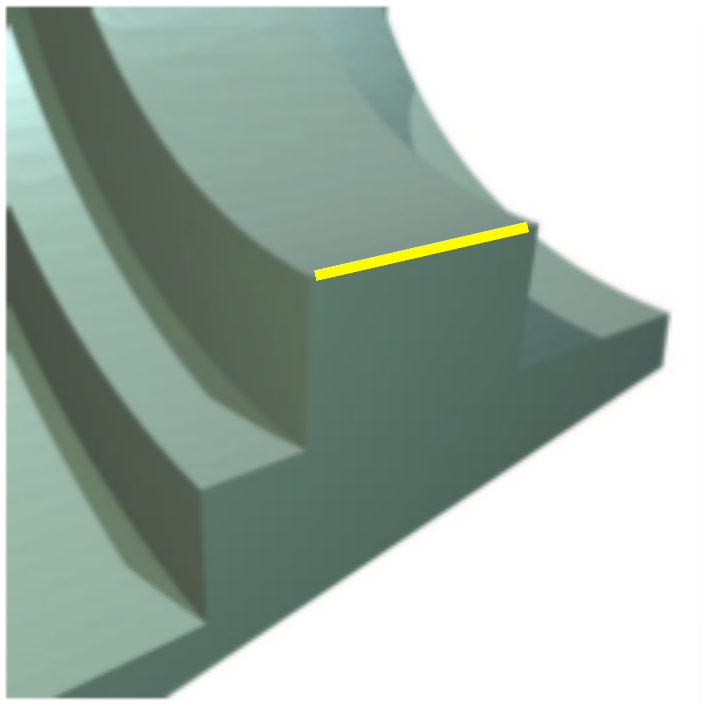
$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{n}_1^T \\ \vdots \\ \mathbf{n}_k^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{n}_1^T \mathbf{p}_1 \\ \vdots \\ \mathbf{n}_k^T \mathbf{p}_k \end{bmatrix} = \mathbf{b}$$



- Sometimes underdetermined
  - When planes are (almost) parallel
- Sometimes overdetermined
  - When too many planes

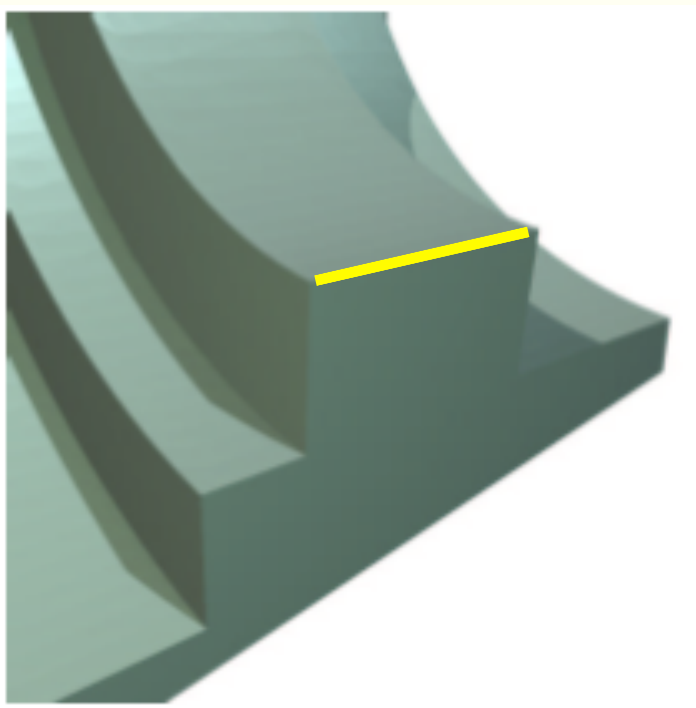
Underdetermined

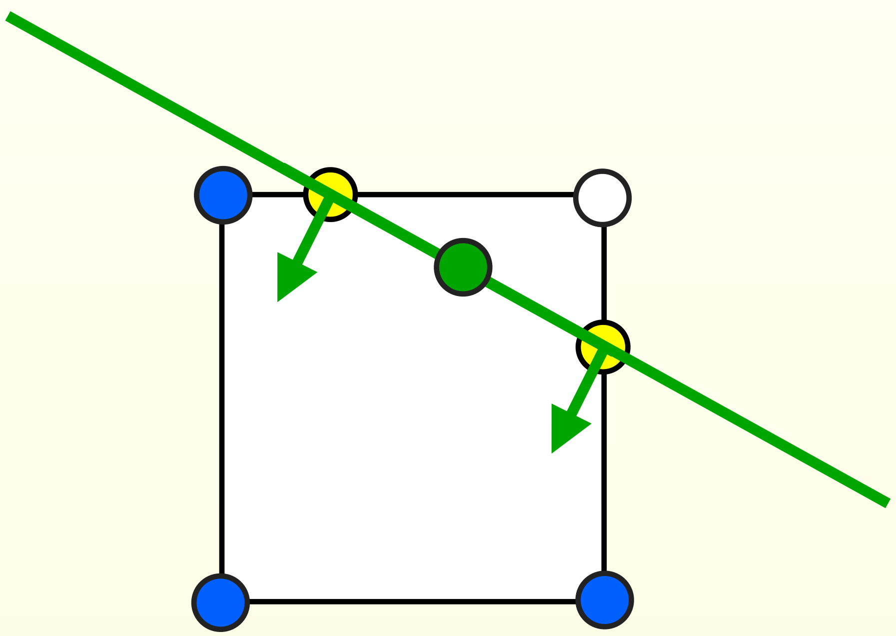
- Many solutions to $\mathbf{Ax} = \mathbf{b}$
- Standard edge case in 3D



Underdetermined

- Many solutions to $\mathbf{Ax} = \mathbf{b}$
- Standard edge case in 3D





- Choose the solution closest to the center of gravity of all intersections

Overdetermined

- Find least-squares solution: minimize $\|\mathbf{Ax} - \mathbf{b}\|^2$

Overdetermined

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- Compute the pseudo-inverse $\mathbf{A}^+$ using SVD

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} = \mathbf{A}^+[\mathbf{n}_1^T\mathbf{p}_1 \ldots \mathbf{n}_k^T\mathbf{p}_k]^T$$

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- Solution for under- and over-determined system

$$\mathbf{A}\mathbf{y} = [\ldots \mathbf{n}_i^T(\mathbf{p}_i - \sum_i \mathbf{p}_i/k) \ldots]$$

$$\mathbf{x} = \mathbf{A}^+([\ldots \mathbf{n}_i(\mathbf{p}_i - \sum_i \mathbf{p}_i/k) \ldots]) + \sum_i \mathbf{p}_i/k$$

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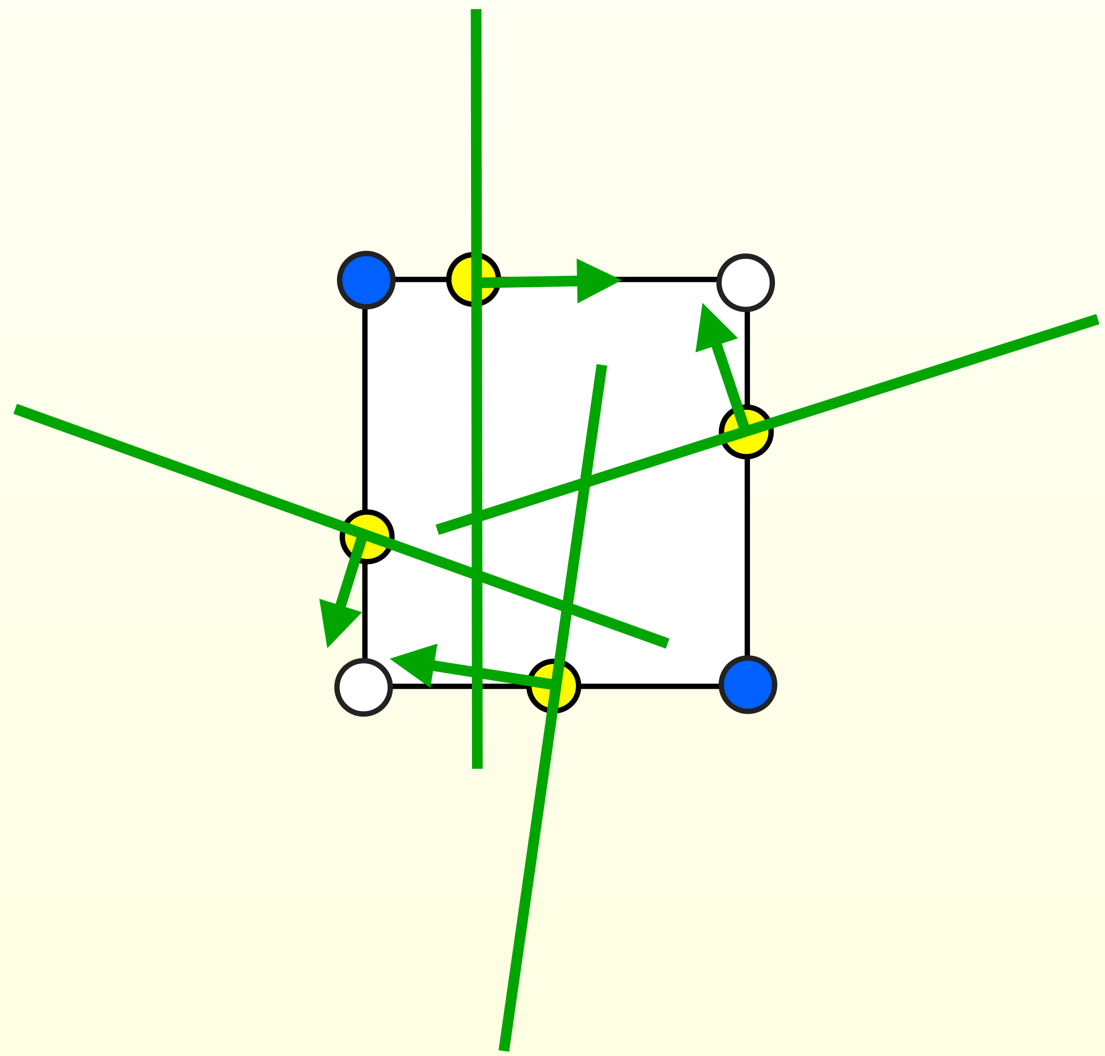
Extended Marching Cubes

- No ambiguous cases



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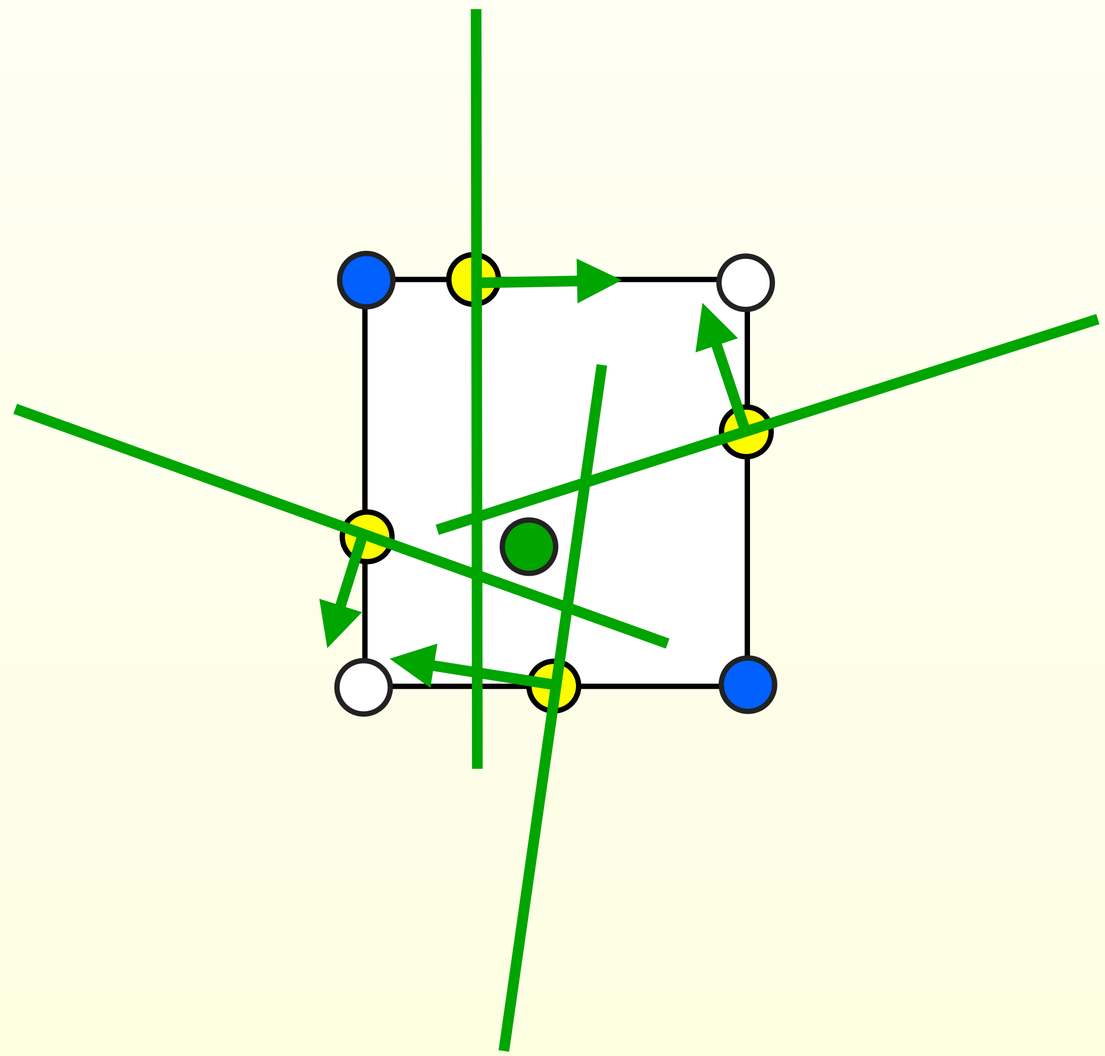

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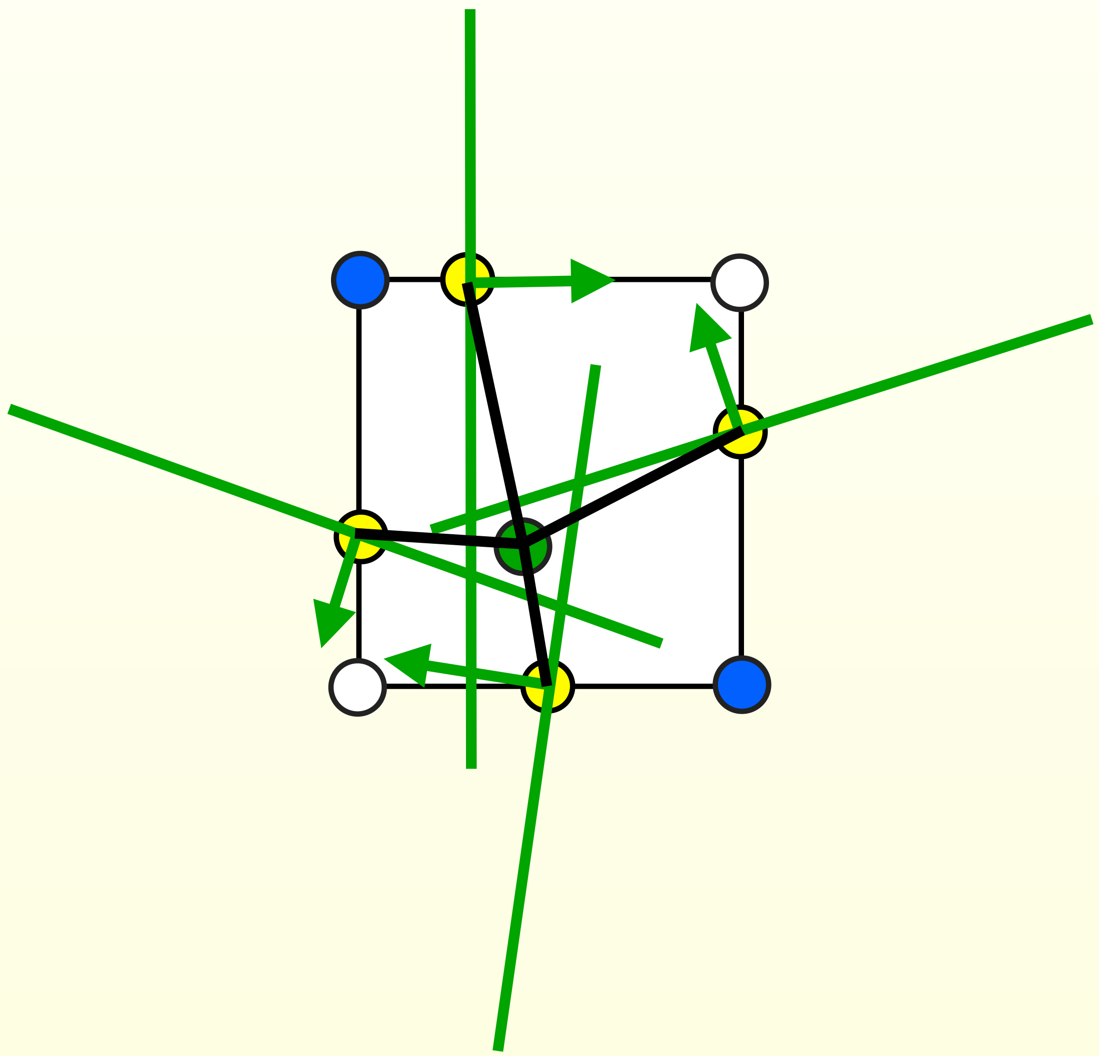
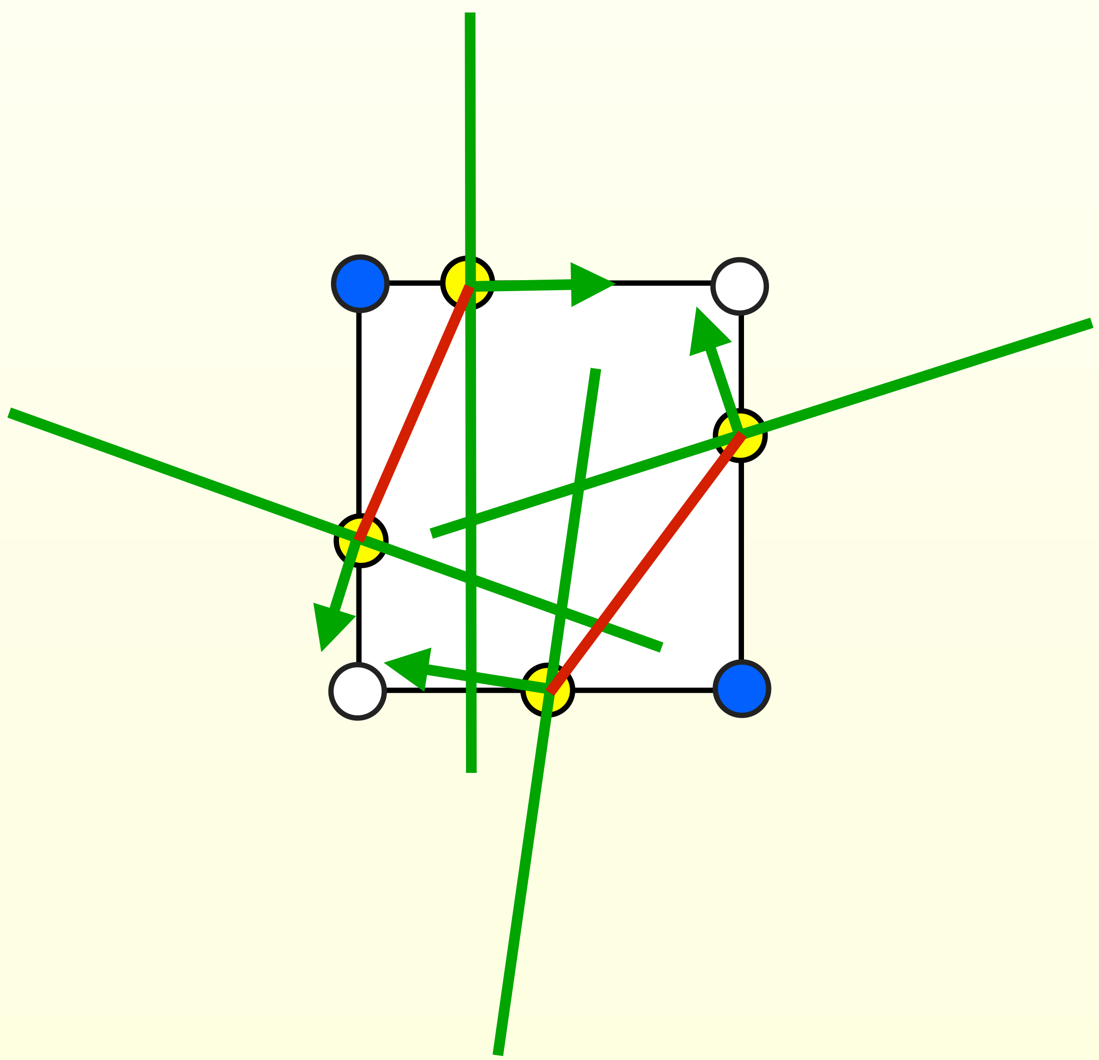


Extended Marching Cubes

- Reintroduce ambiguity:
  - Use the regular marching cubes connectivity
  - Compute several points



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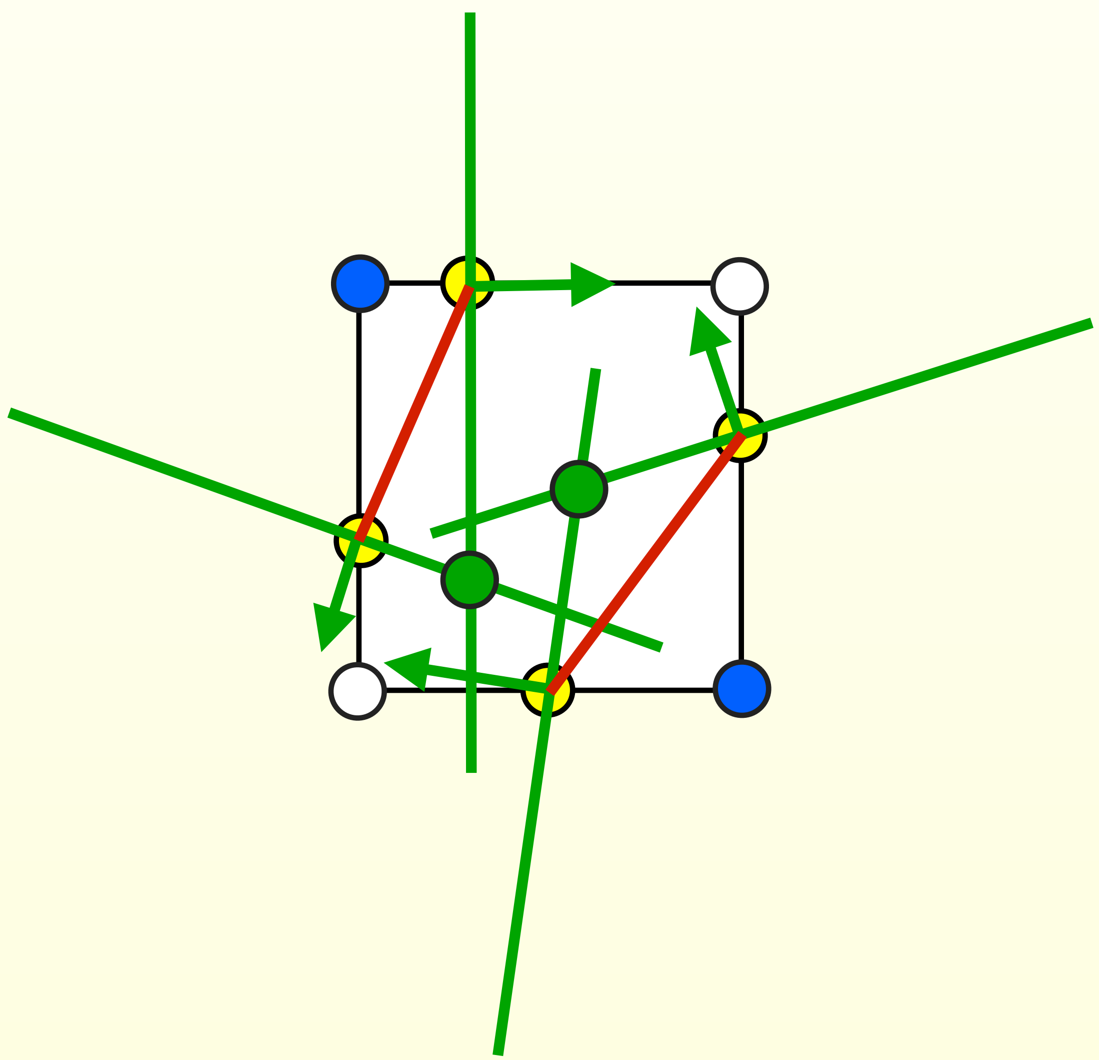

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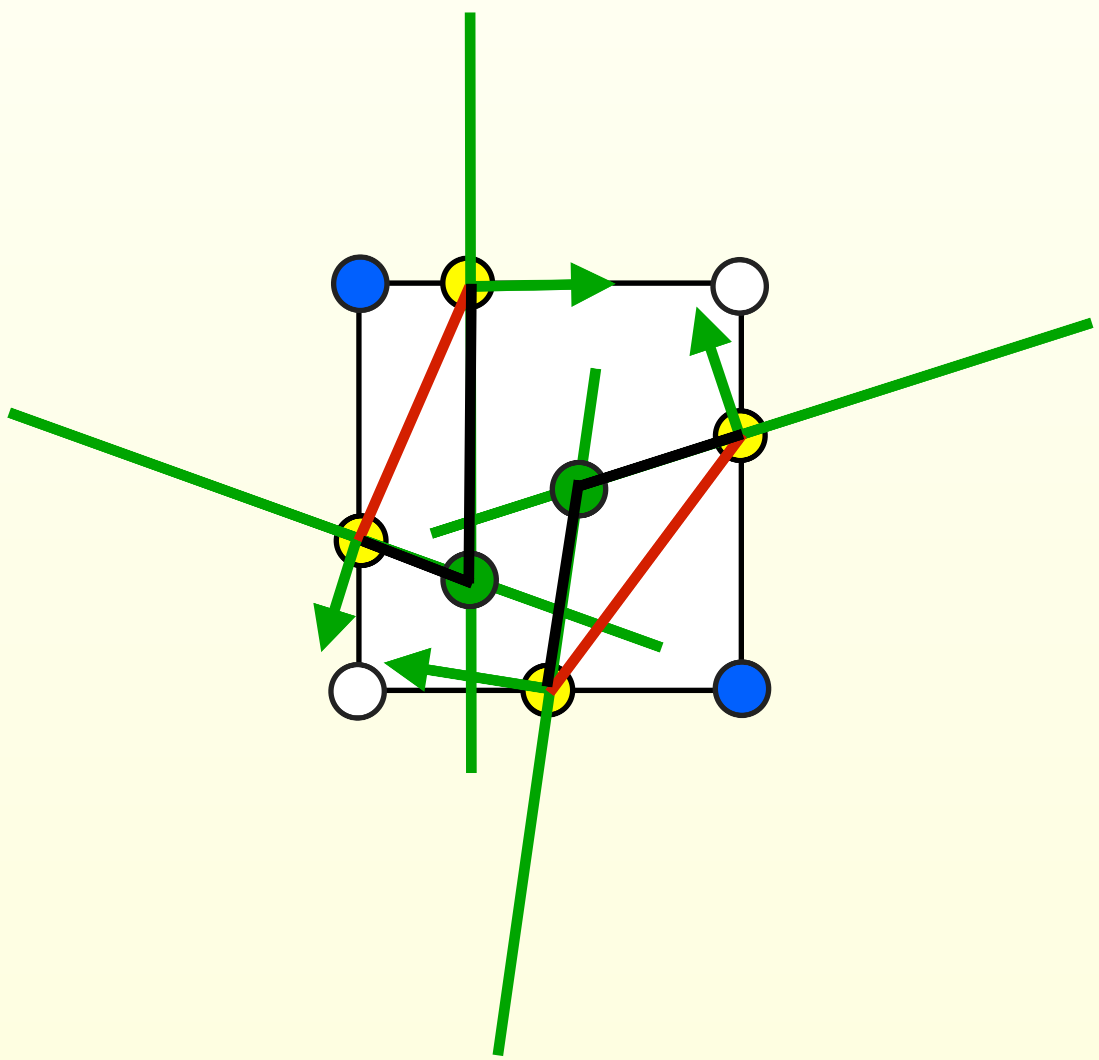
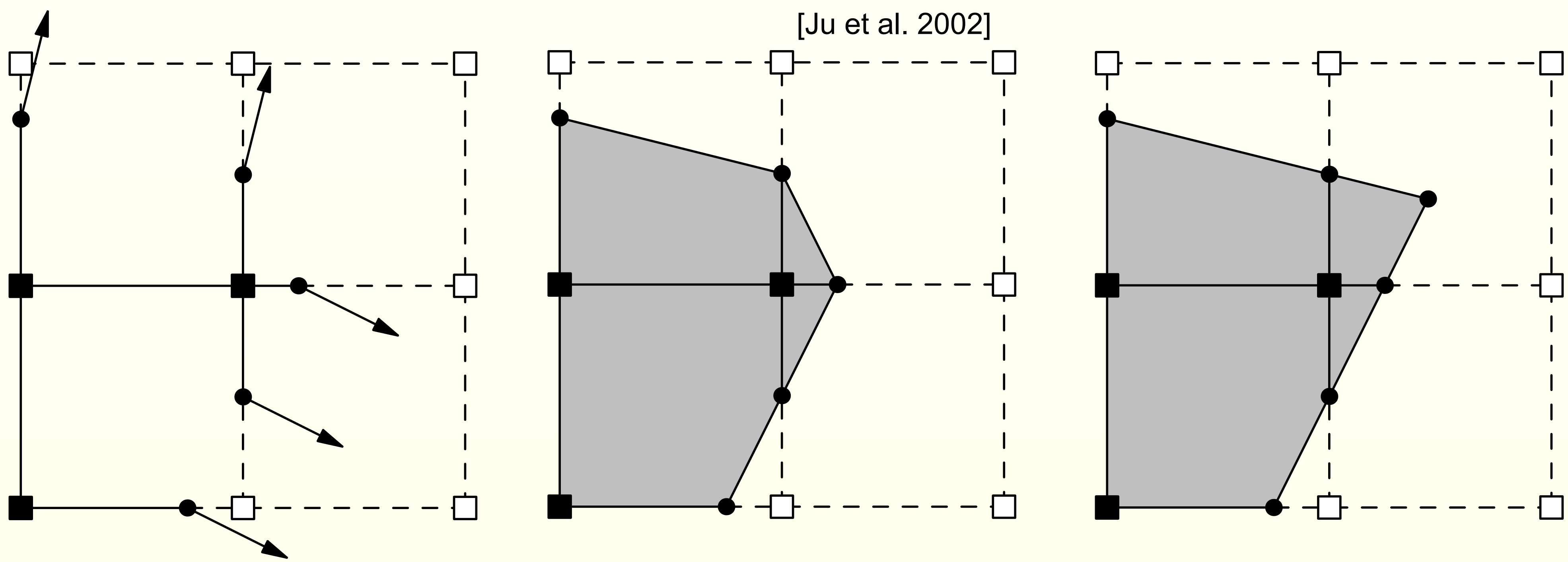


Comparison



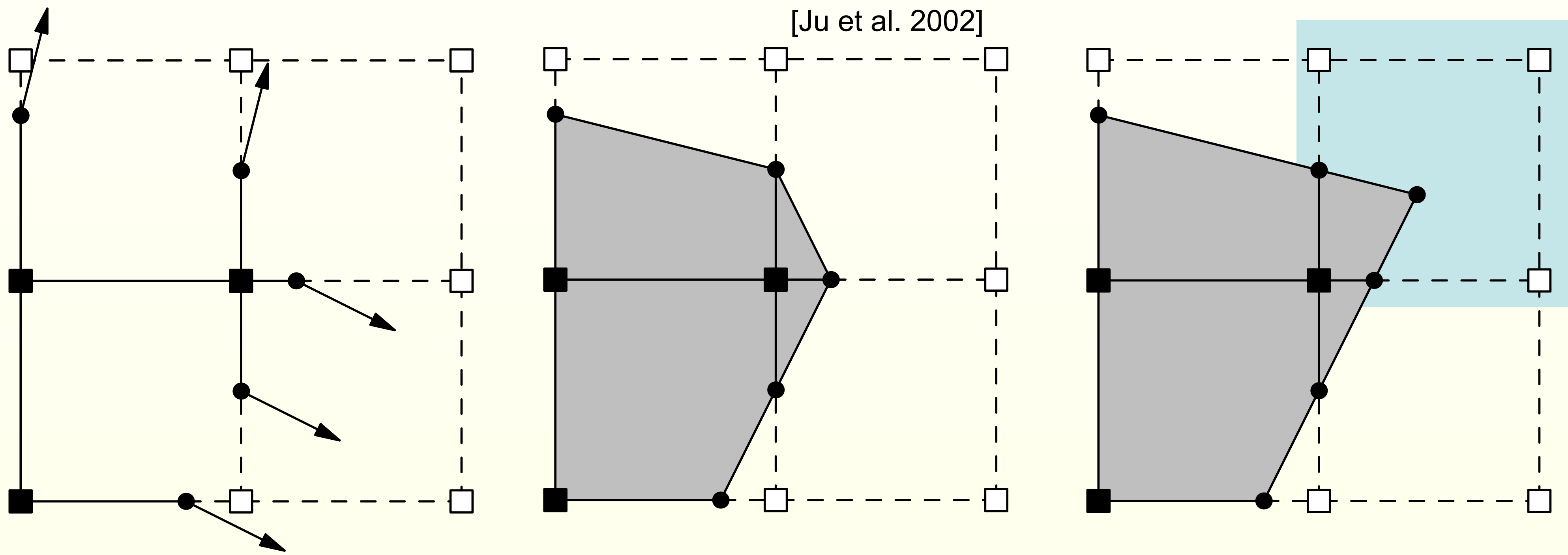
Input data

Marching Cubes

Extended Marching Cubes

Comparison

[Ju et al. 2002]



Input data

Marching Cubes

Extended Marching Cubes

Dual Methods

- Marching Cubes (Primal)
  - 1 point per edge
  - Connecting primitives (triangles) per voxel

- Dual Contouring (Dual)
  - 1 point per voxel
  - Connecting primitives (quads) per edge

Dual Methods

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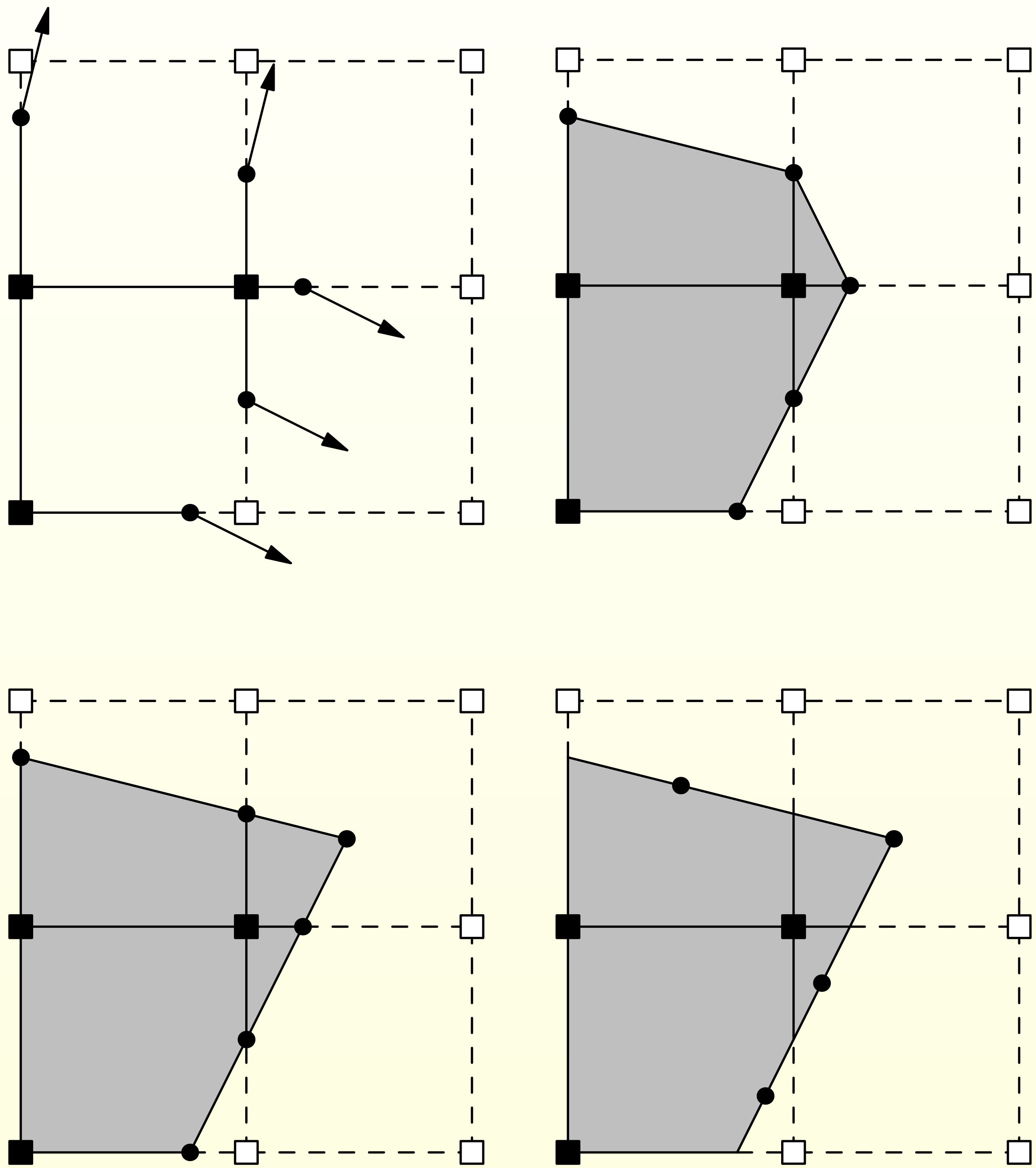
Comparison

Input data

Marching Cubes



Extended Marching Cubes

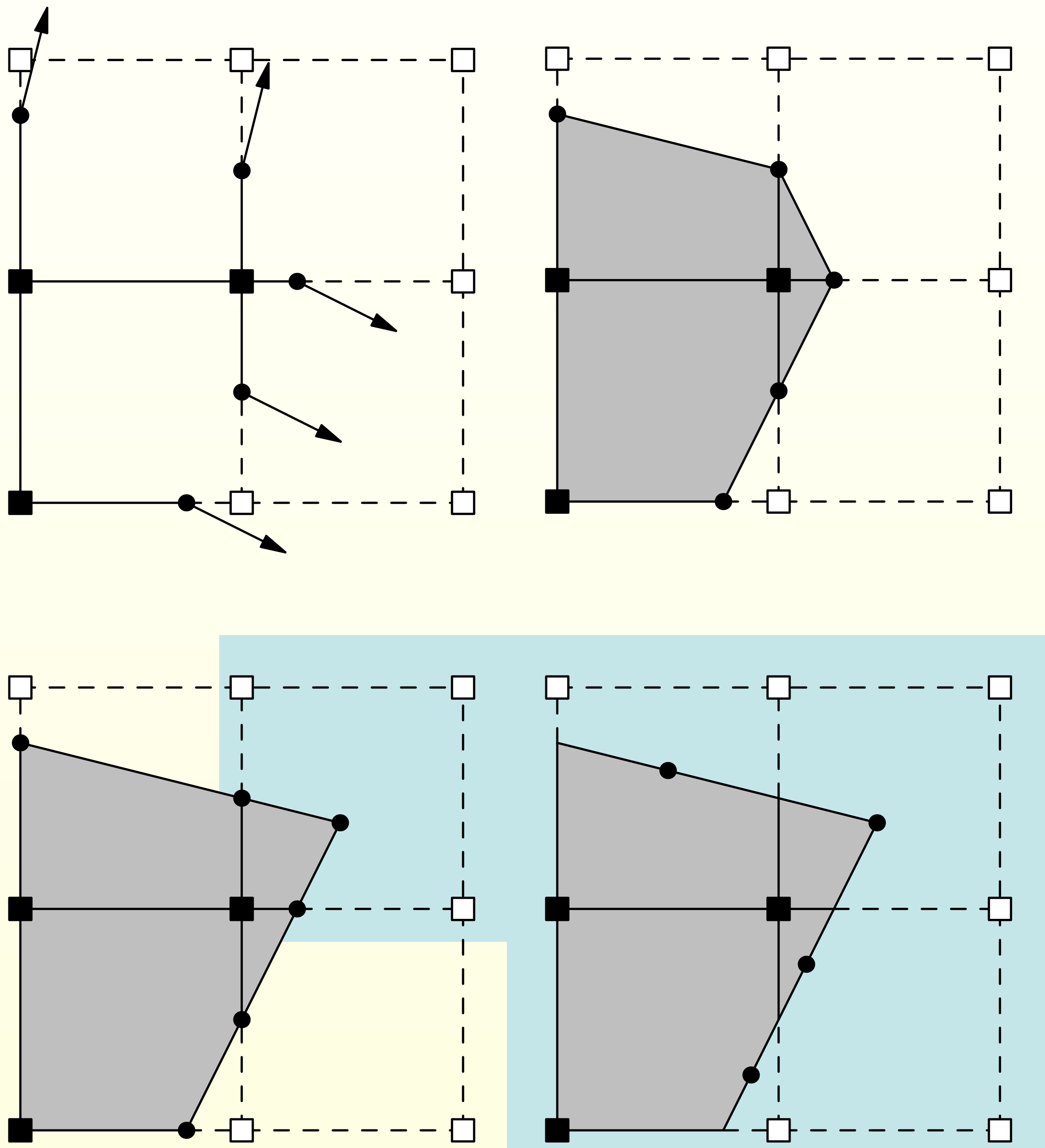
Dual Contouring

[Ju et al. 2002]

Comparison

Input data

Marching Cubes



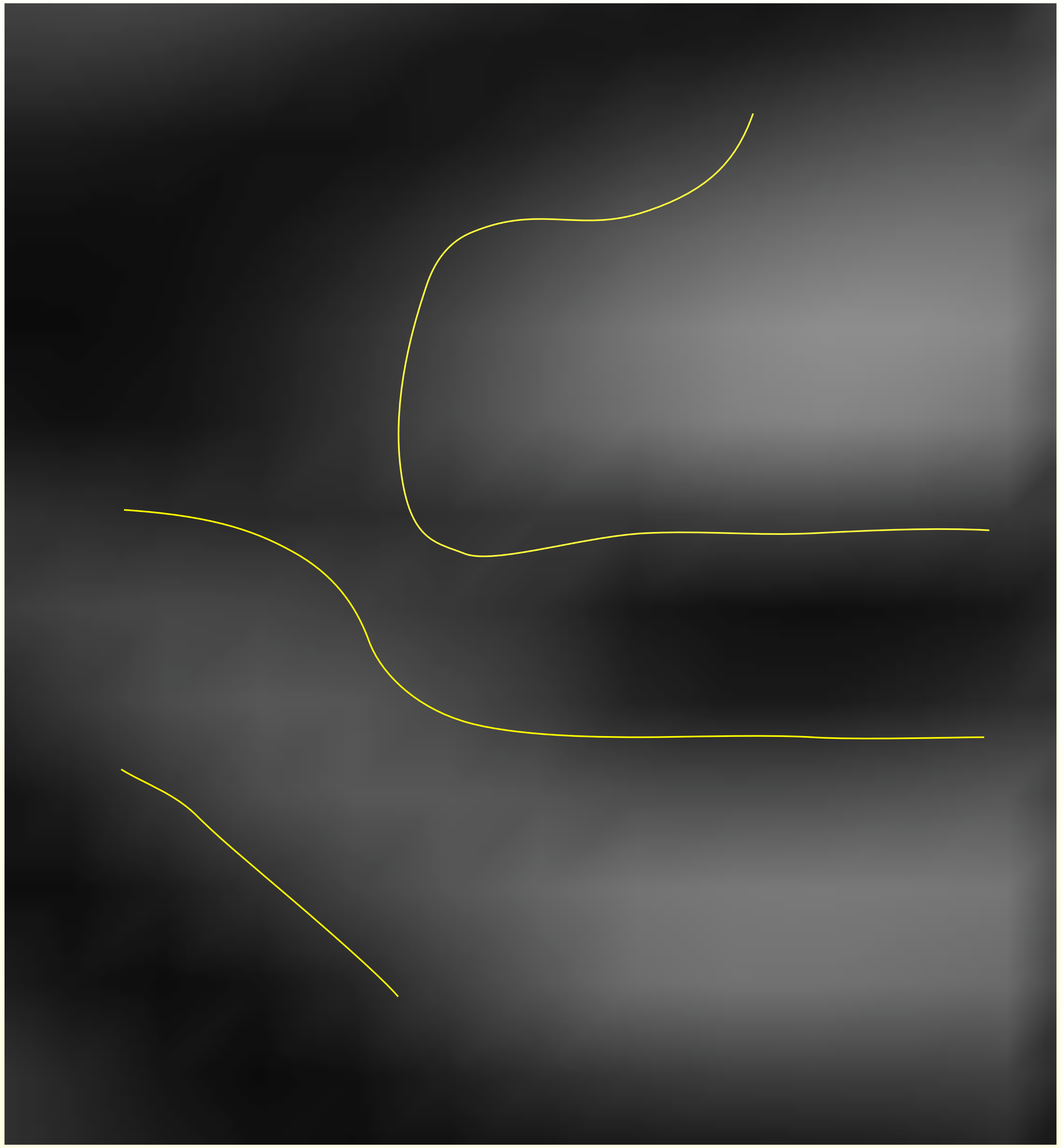
Extended Marching Cubes

Dual Contouring

[Ju et al. 2002]

Marching Squares

- Given $I(\mathbf{x})$
  - $I(\mathbf{x}) < q$: $\mathbf{x}$ outside
  - $I(\mathbf{x}) > q$: $\mathbf{x}$ inside
- Discretize space
- Evaluate on the grid



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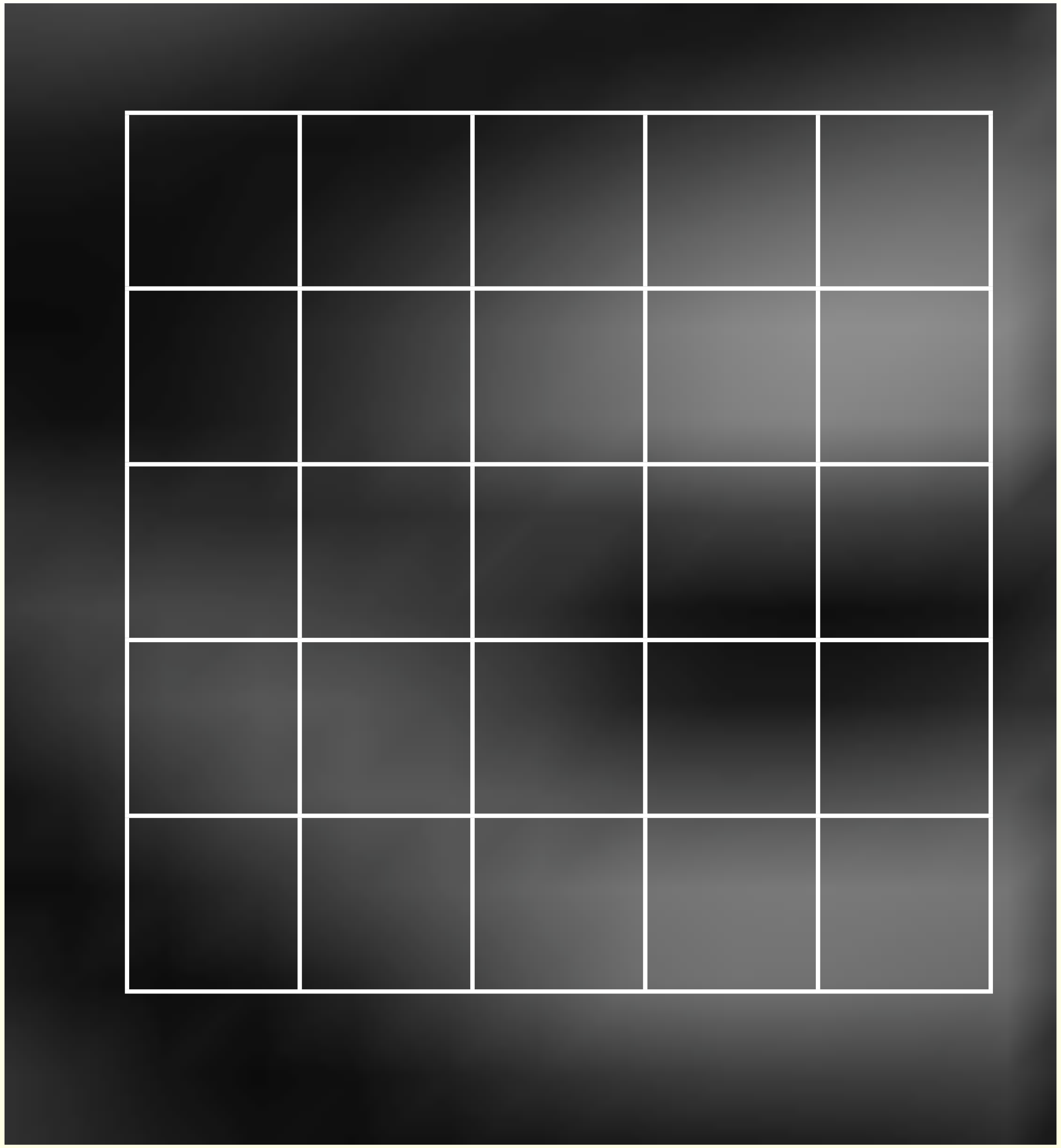

Marching Squares

- Given $I(\mathbf{x})$
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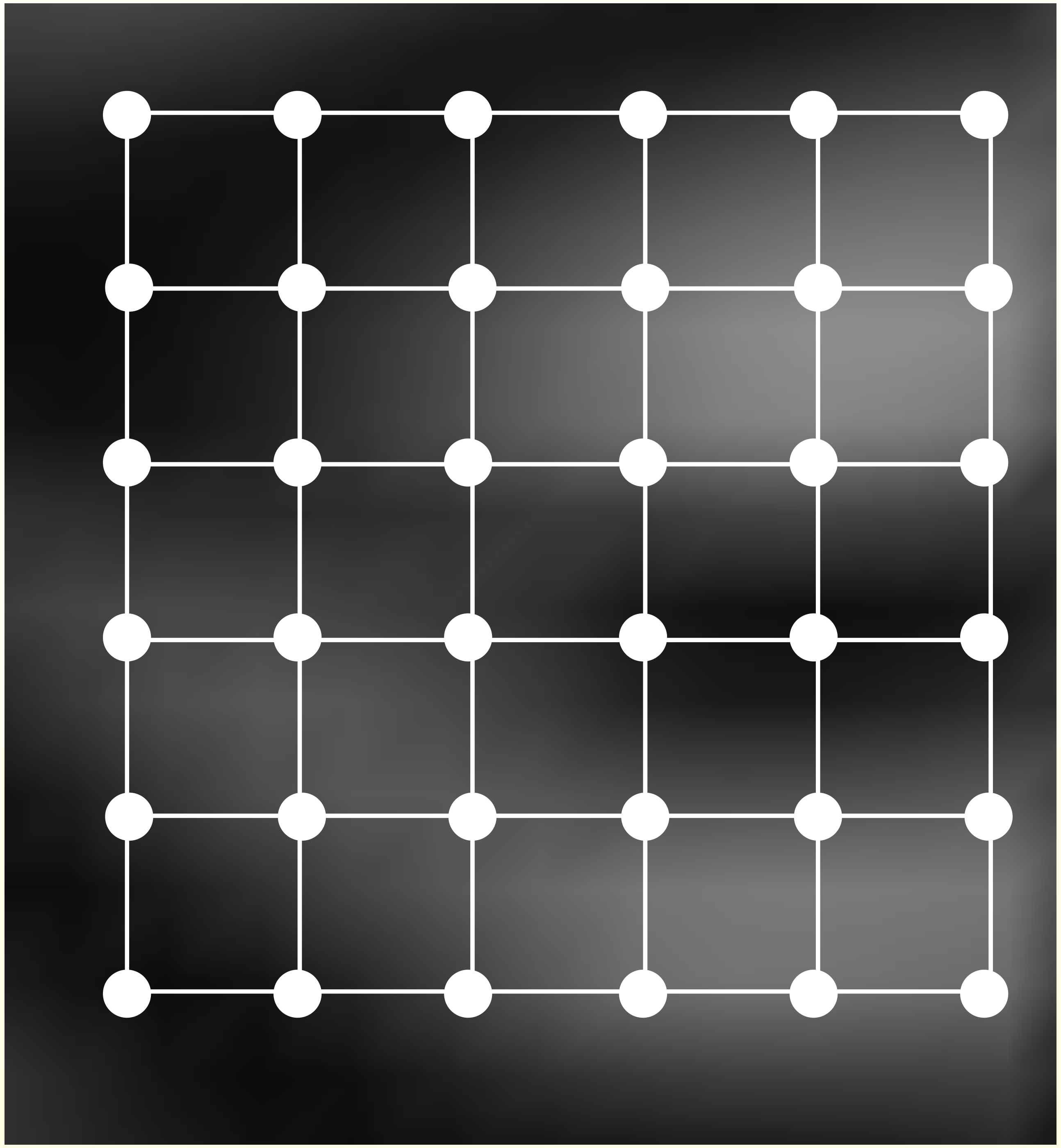
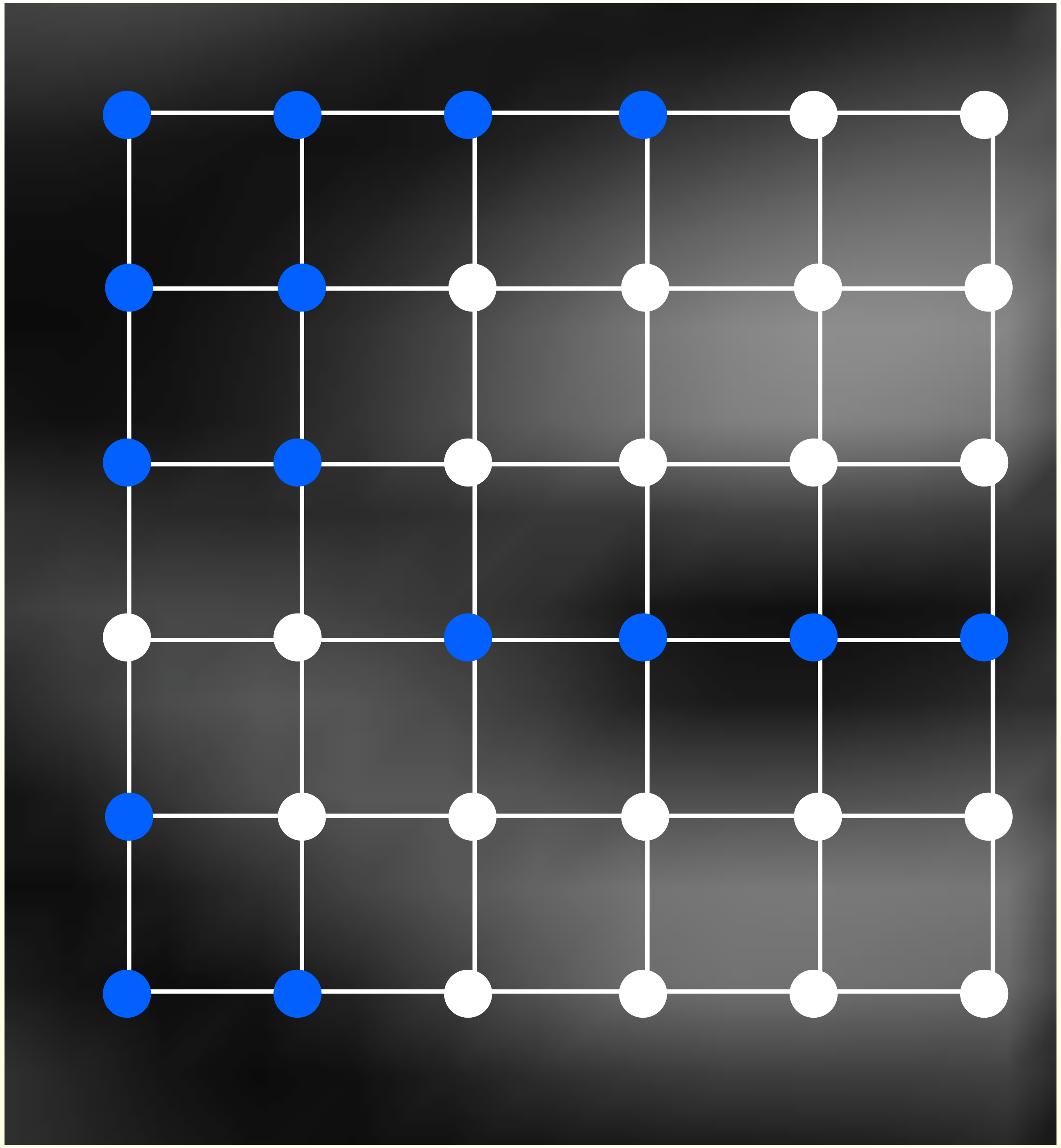
- Evaluate $f(\mathbf{x})$ on the grid



2D Dual Contouring

- Given $I(\mathbf{x})$
  - $I(\mathbf{x}) < q$: $\mathbf{x}$ outside
  - $I(\mathbf{x}) > q$: $\mathbf{x}$ inside
- Discretize space
- Evaluate $f(\mathbf{x})$ on the grid
- Classify grid points



2D Dual Contouring

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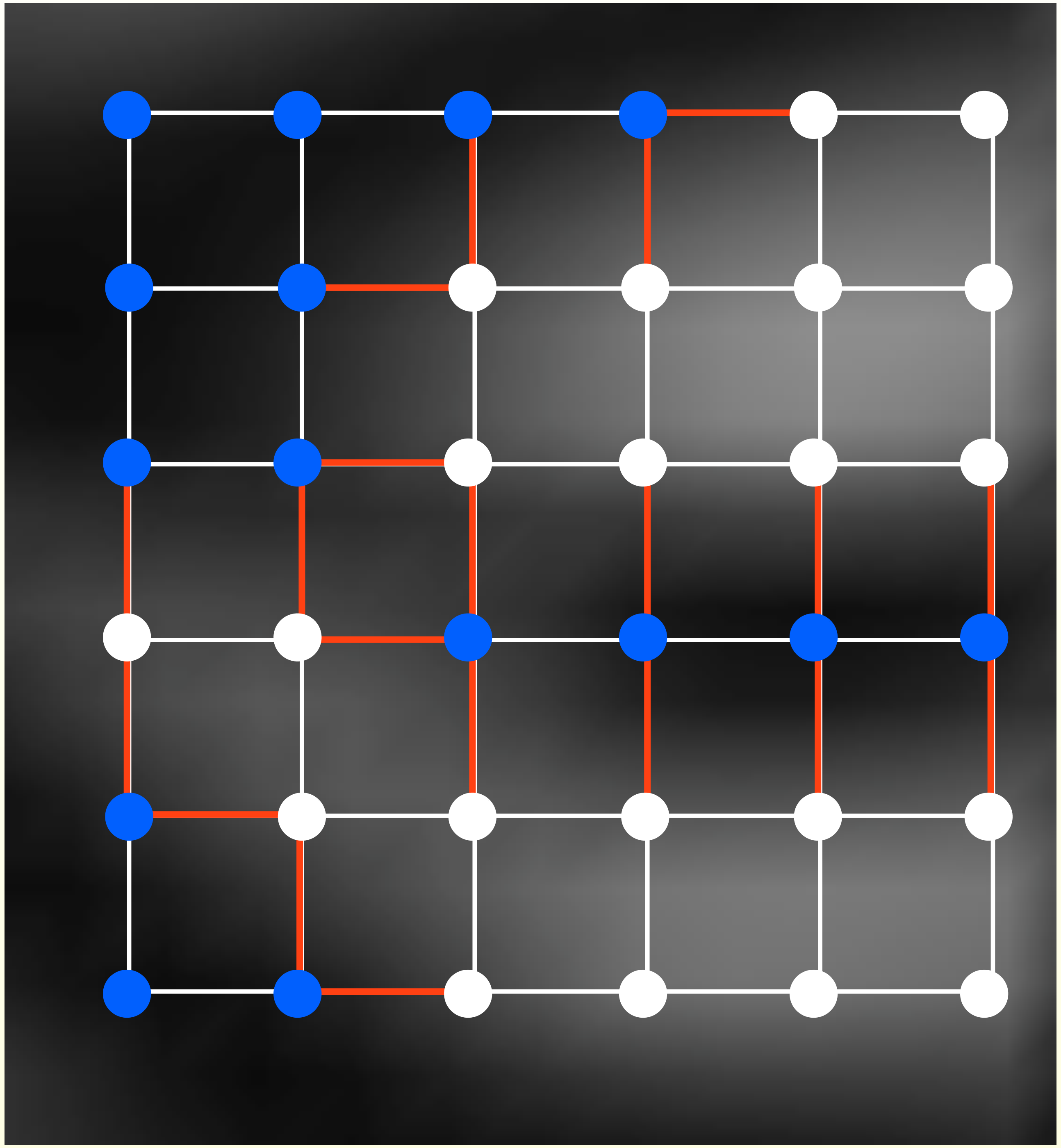
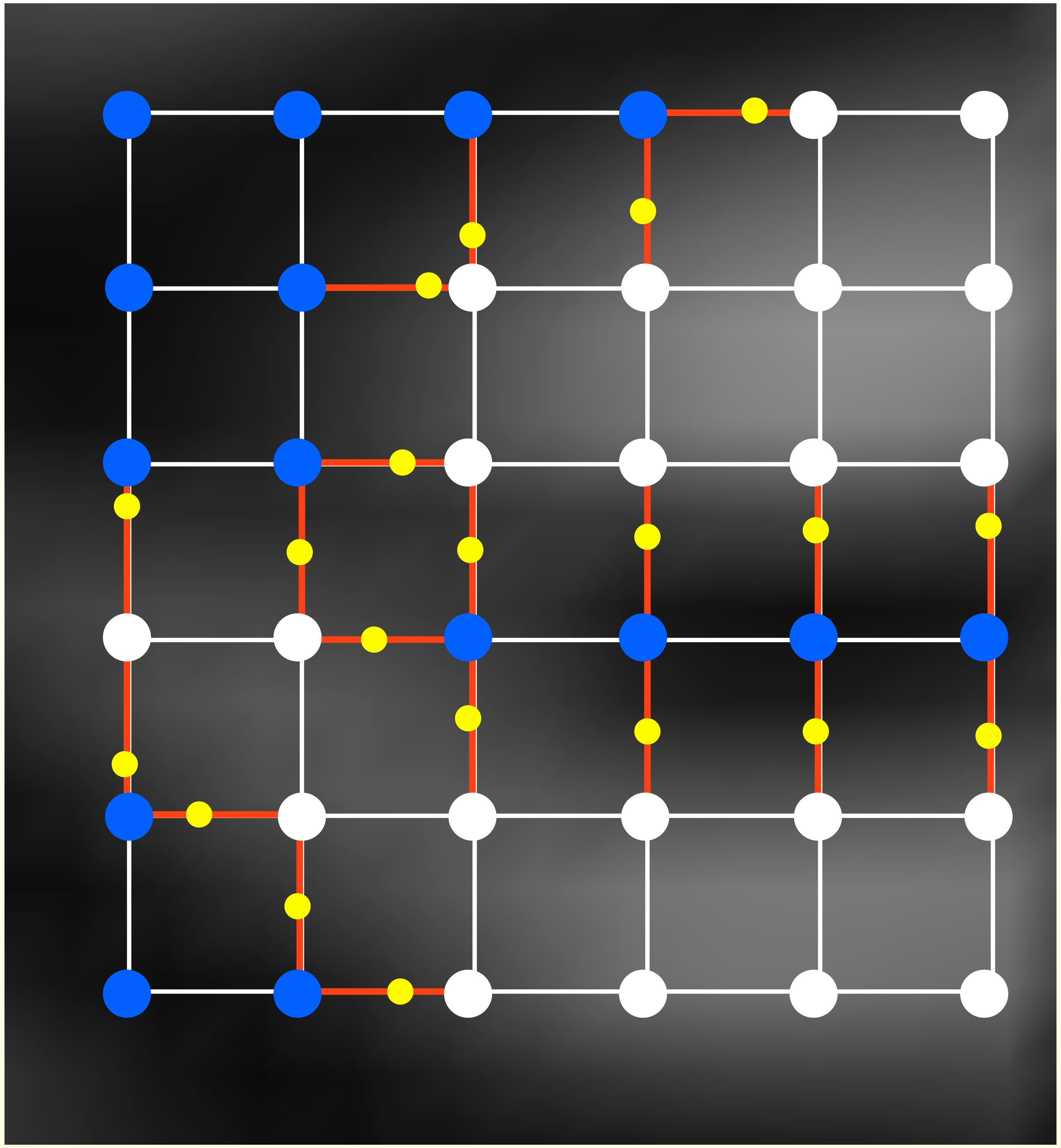
- Discretize space
- Evaluate $f(\mathbf{x})$ on the grid
- Classify grid points
- Classify grid edges



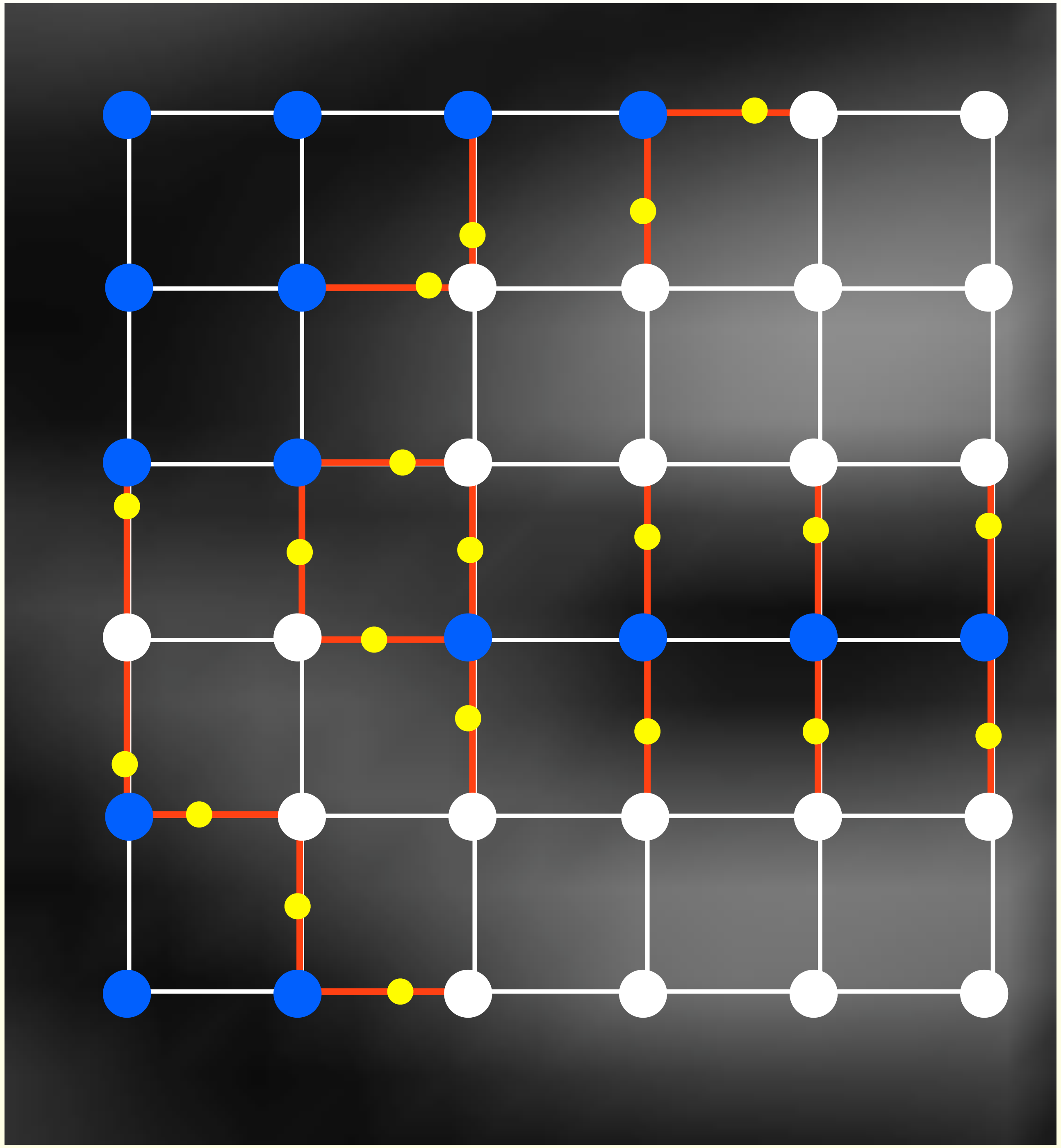
2D Dual Contouring

- Given $I(\mathbf{x})$
  - $I(\mathbf{x}) < q$: $\mathbf{x}$ outside
  - $I(\mathbf{x}) > q$: $\mathbf{x}$ inside
- Discretize space
- Evaluate $f(\mathbf{x})$ on the grid
- Classify grid points
- Classify grid edges
- Compute intersections



2D Dual Contouring

- Compute intersections



2D Dual Contouring

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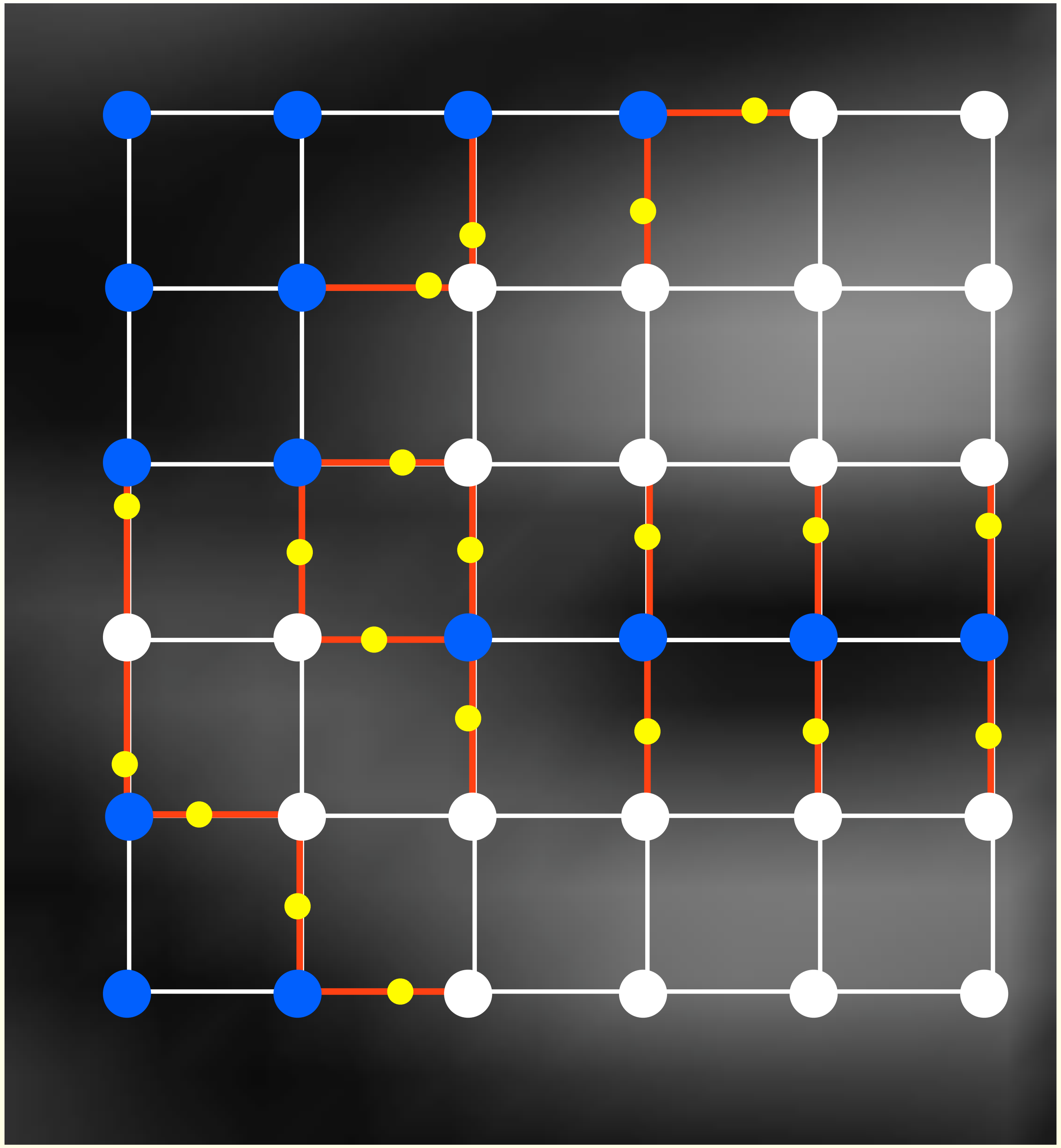
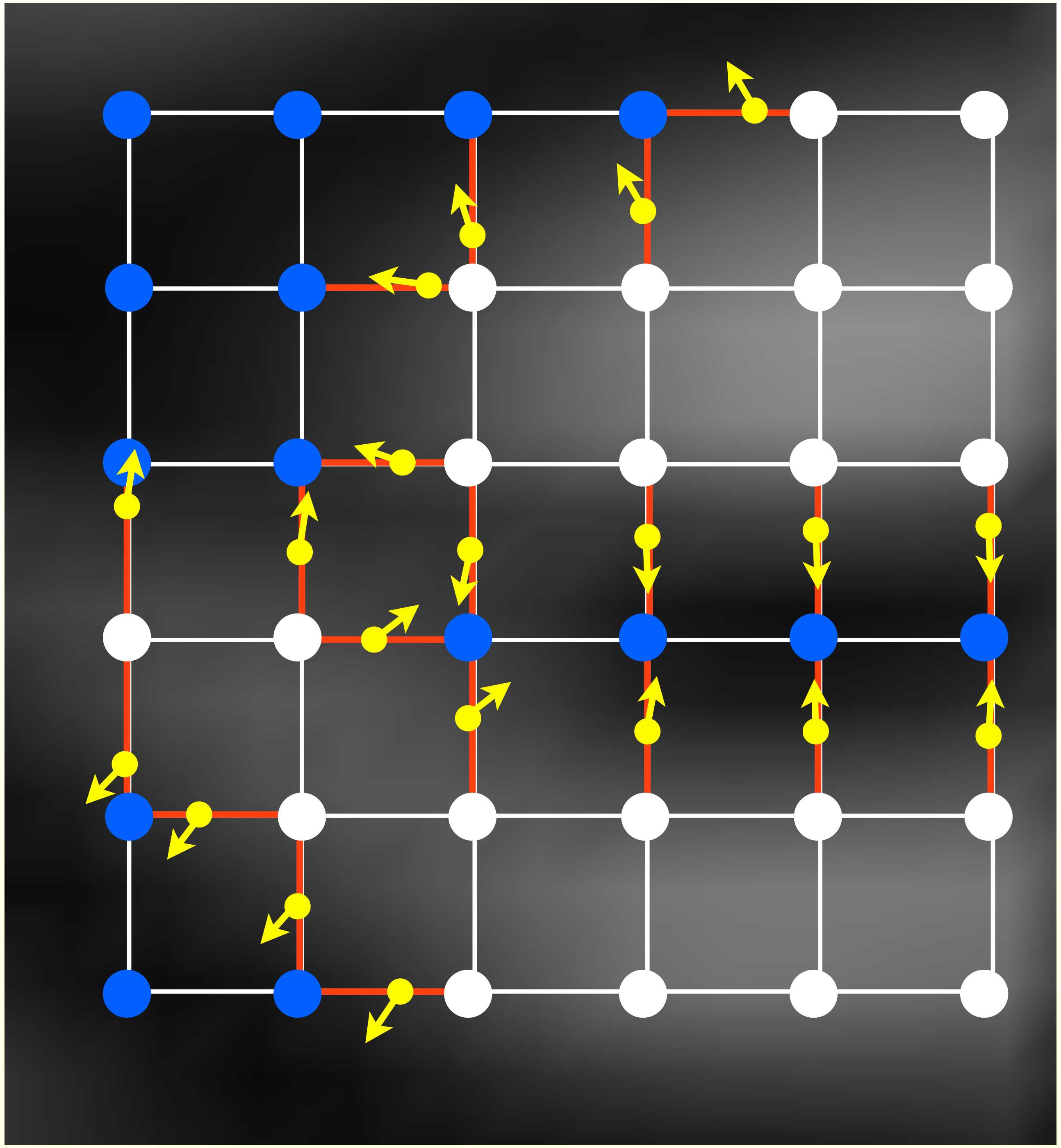


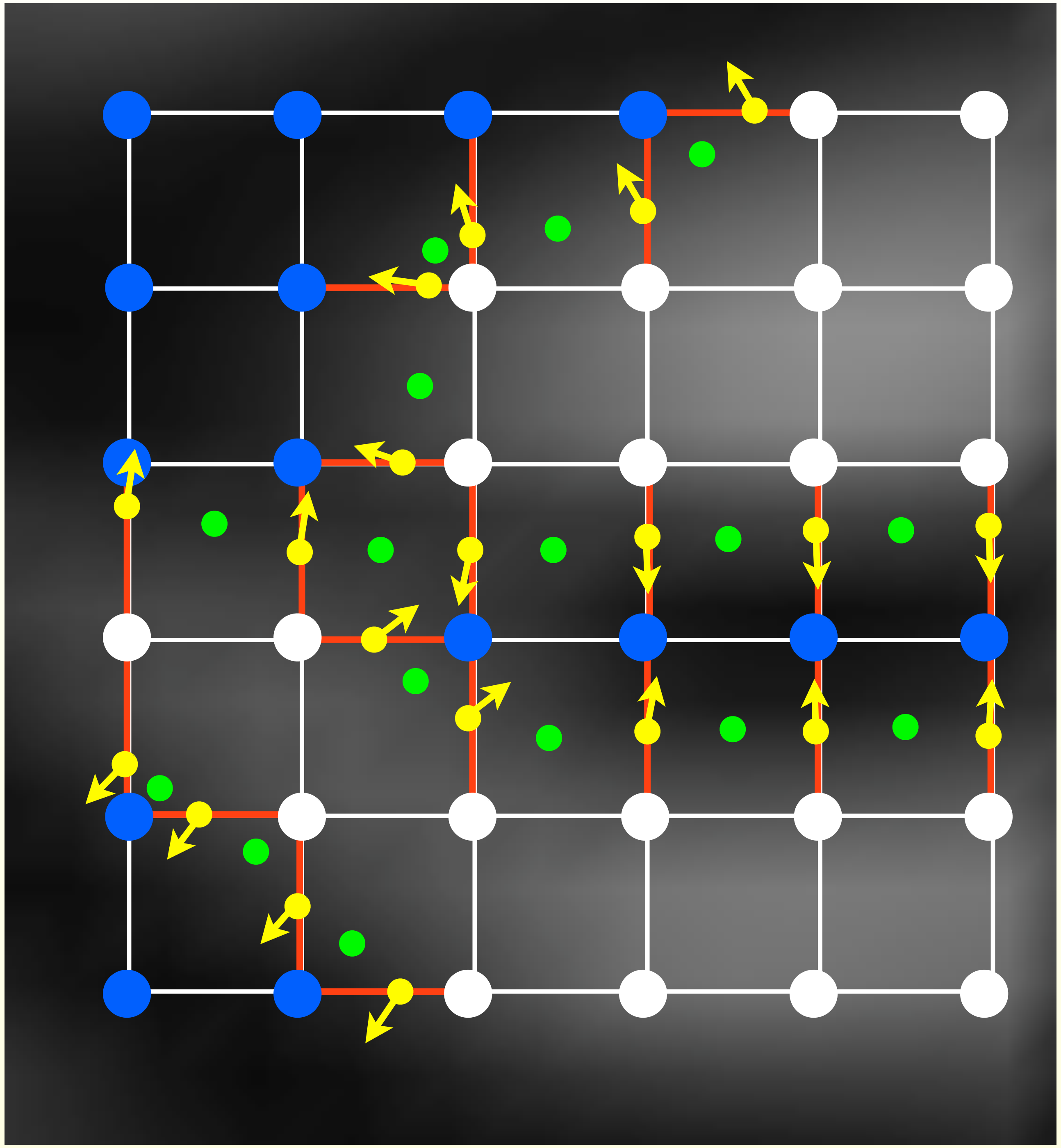
2D Dual Contouring

- Compute intersections
- Compute normals



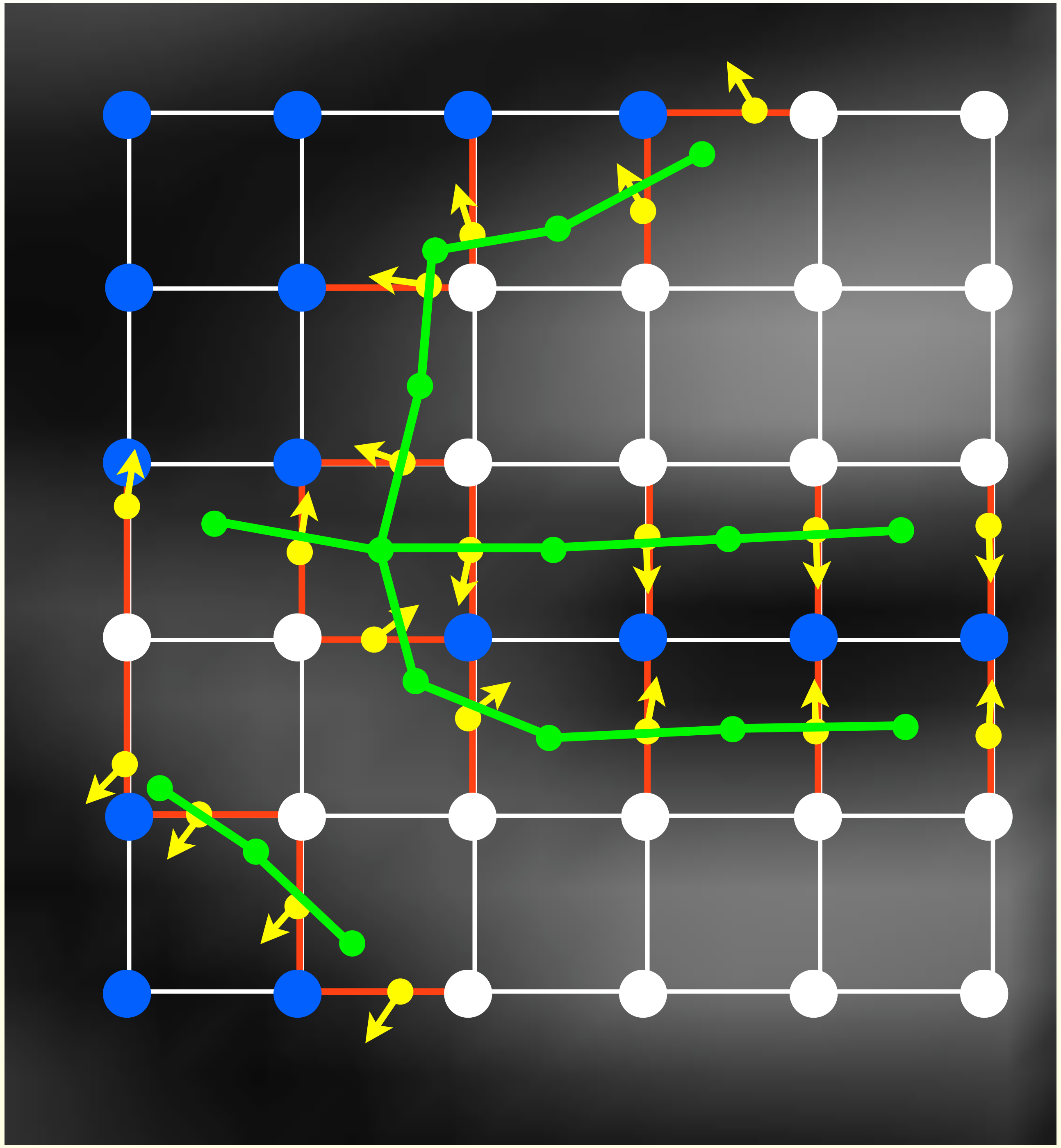
2D Dual Contouring

- Compute intersections
- Compute normals
- Compute dual points
  - for each voxel with sign change



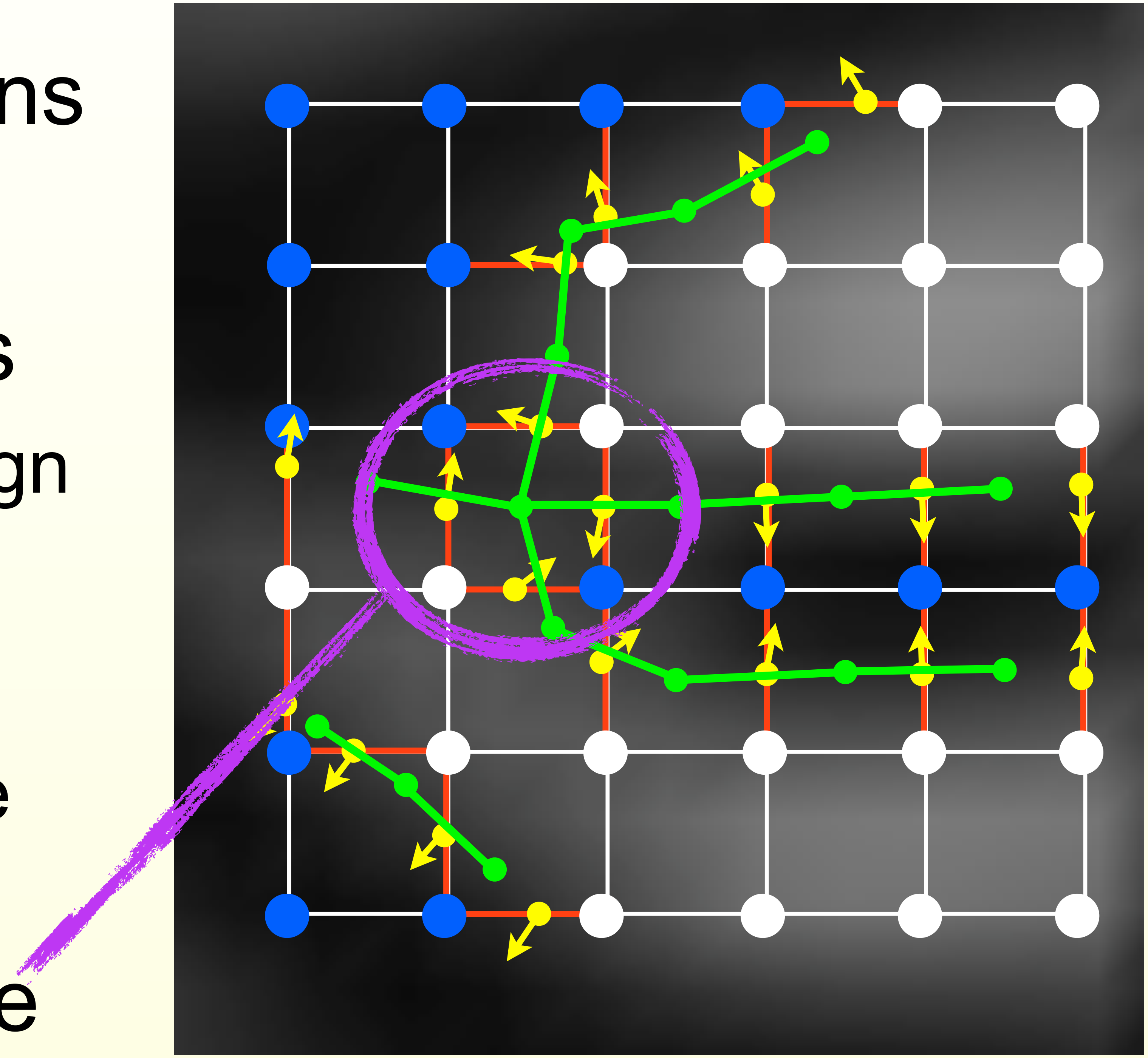
2D Dual Contouring

- Compute intersections
- Compute normals
- Compute dual points
  - for each voxel with sign change
- Connect dual points
  - across each red edge



2D Dual Contouring

- Compute intersections
- Compute normals
- Compute dual points
  - for each voxel with sign change
- Connect dual points
  - across each red edge
- We could reintroduce ambiguity

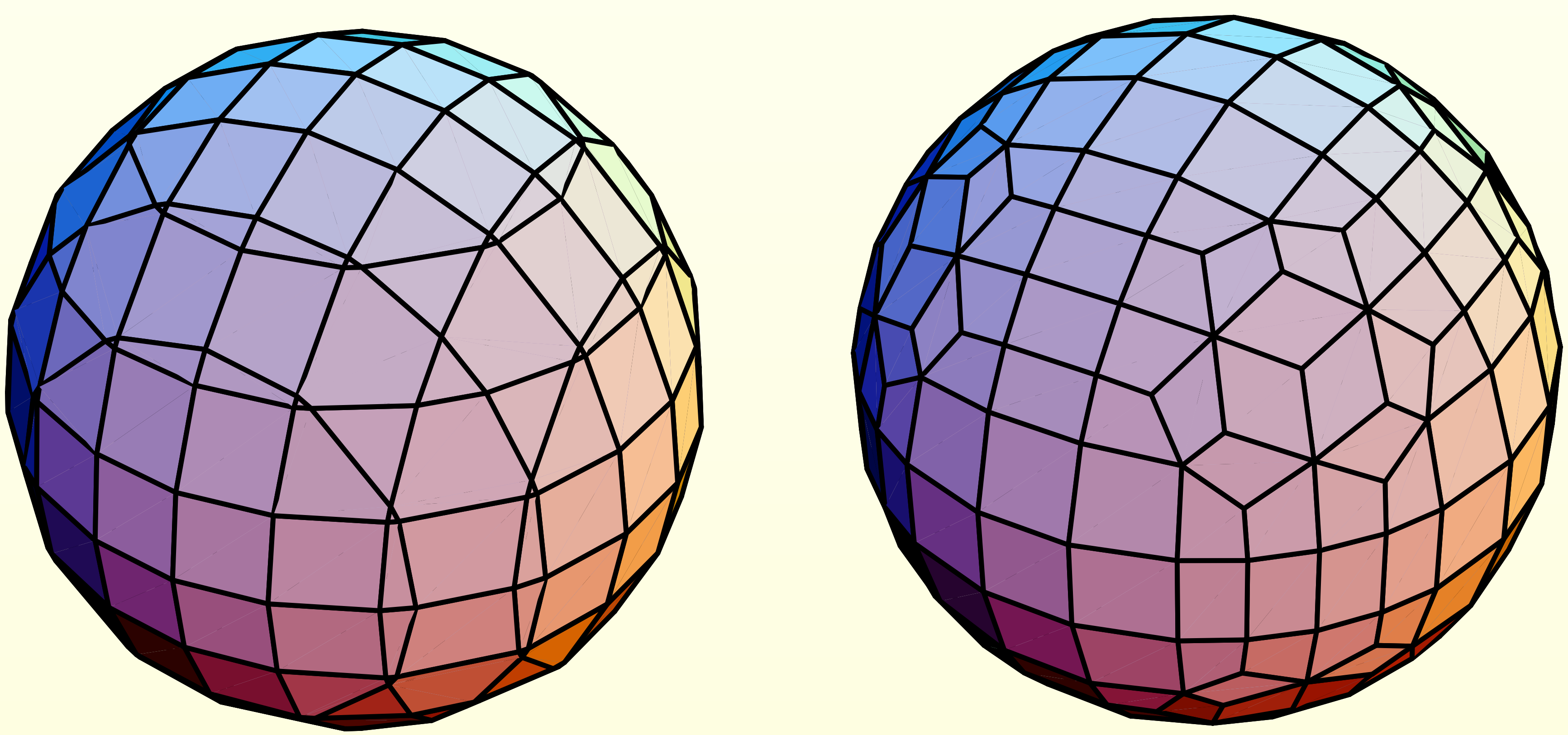


Dual Contouring

- No ambiguities
- Does not interpolate known surface points/ normals
- No special cases for features
- No lookup tables
- Some trickiness in matrix pseudo-inverse
- Naturally produces quads, not triangles (in 3D)
  - What would dual marching tetrahedra produce?

Mesh quality

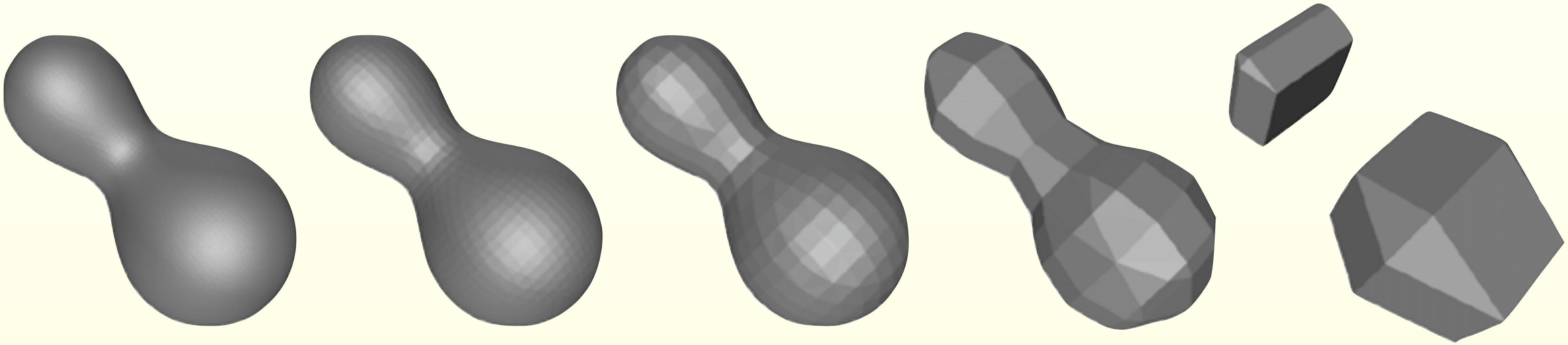
- Dual contouring produces higher-quality meshes
  - Flat surfaces never have bad triangles/quads
- Control over placement of samples



[Ju et al. 2002]

No Topological Guarantees

- Discrete Sampling: Expect Resolution Issues



[Paul Bourke]

Isotopic Meshing

- Reduce mesh size until all features resolved
- Within each voxel that contains the surface:

$$\nabla f(\mathbf{x}) \cdot \nabla f(\mathbf{y}) > 0 \quad \forall \mathbf{x}, \mathbf{y}$$

- Normals in a -cone

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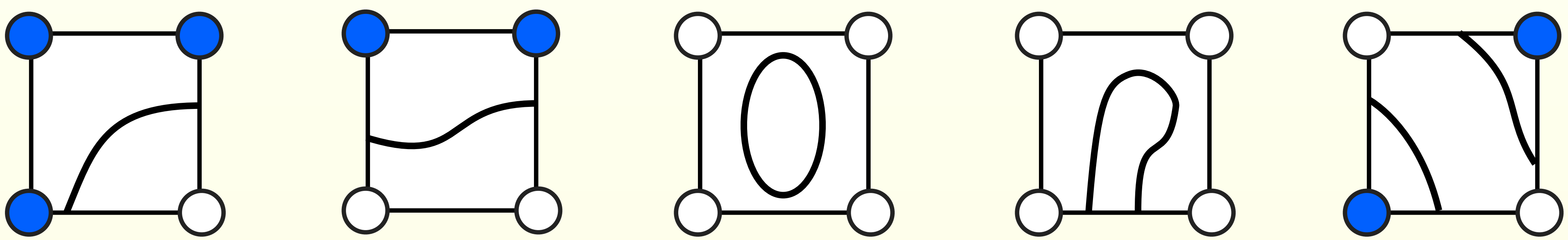

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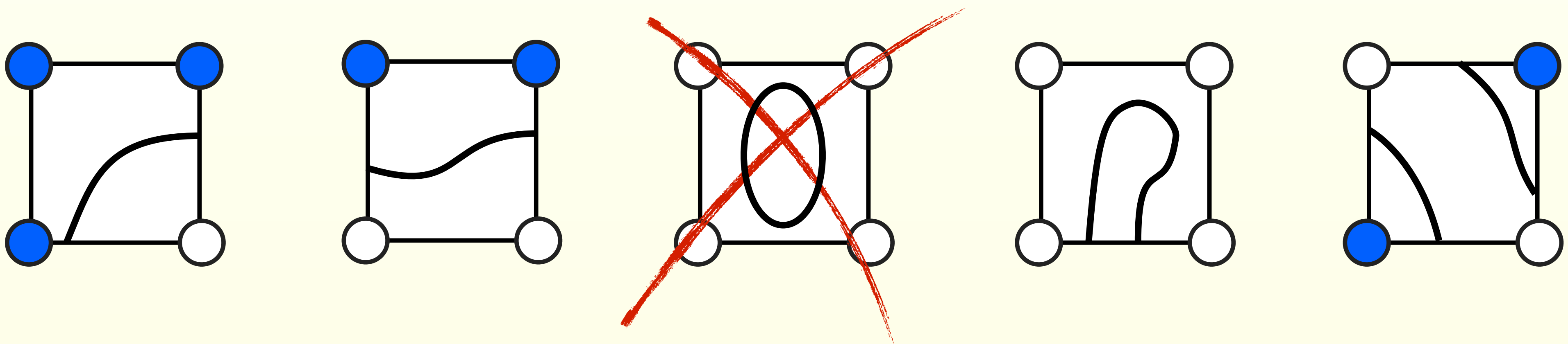

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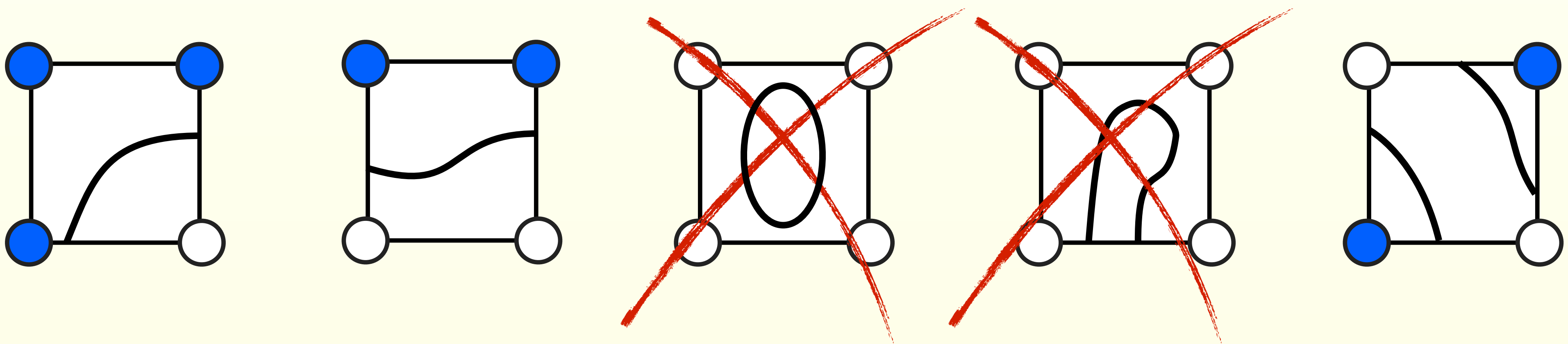

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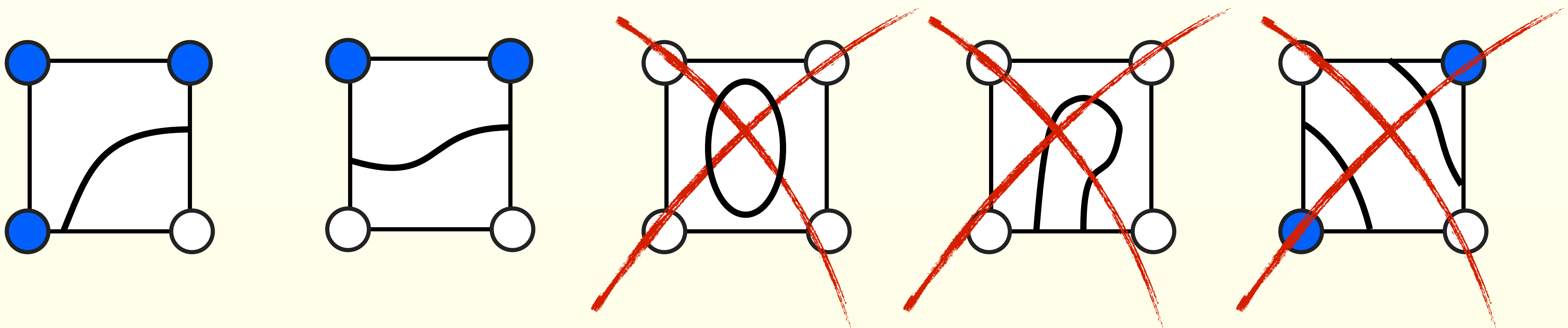

- Normals in a -cone

Isotopic Meshing

- Reduce mesh size until all features resolved
- Within each voxel that contains the surface:

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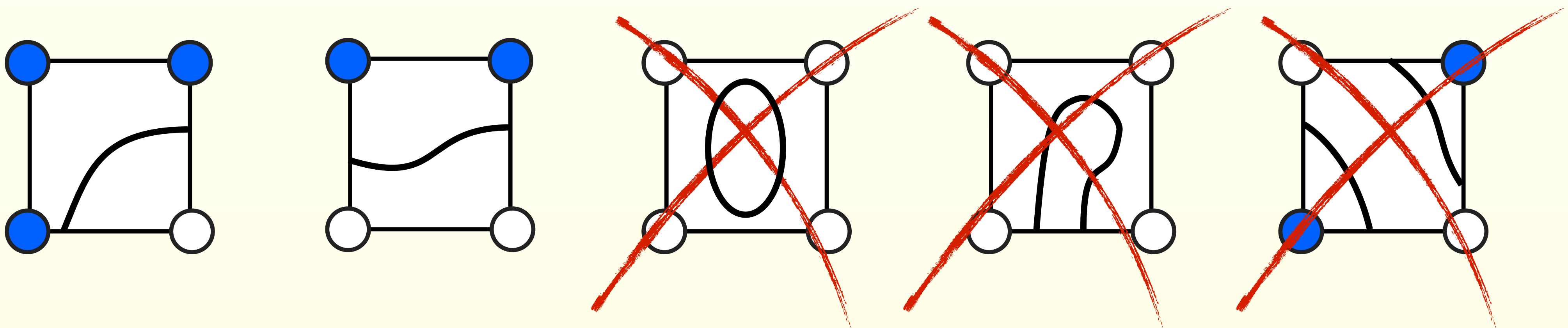
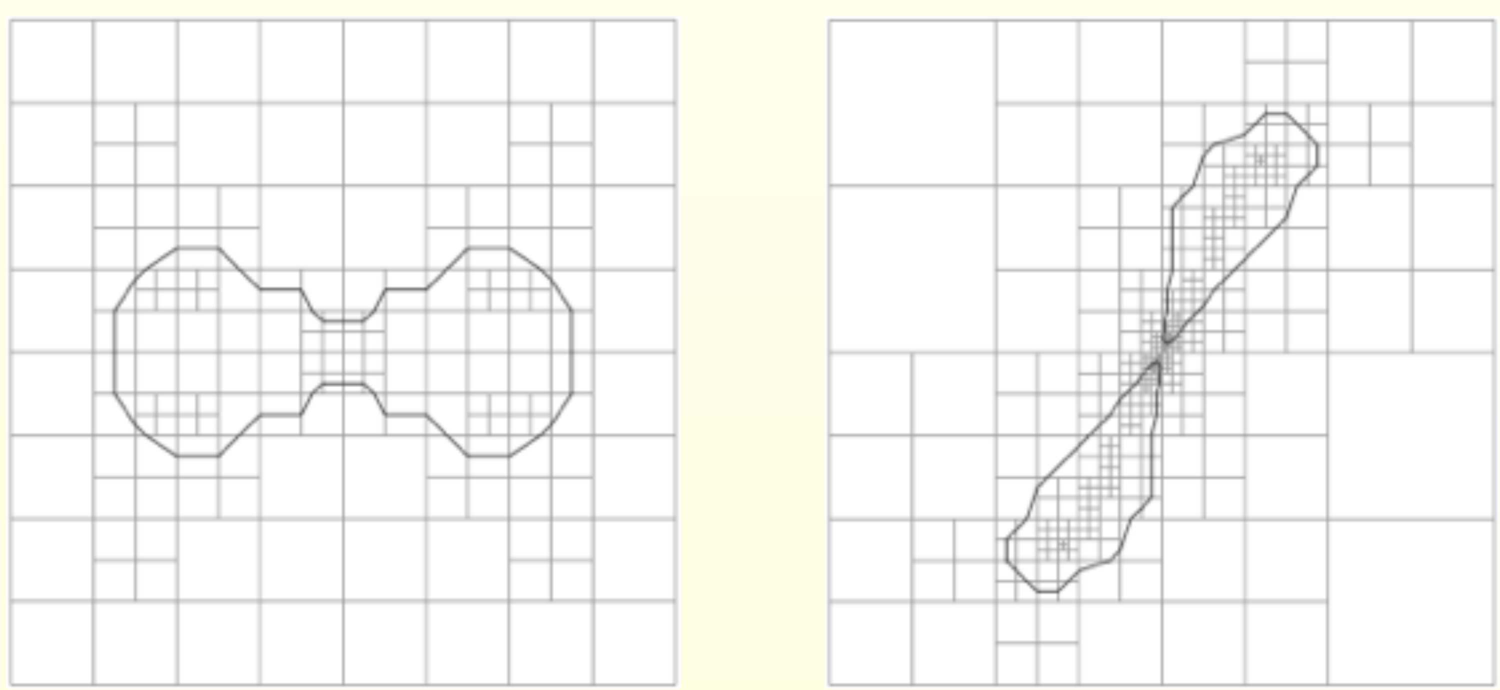


- Normals in a $\frac{\pi}{2}$ -cone

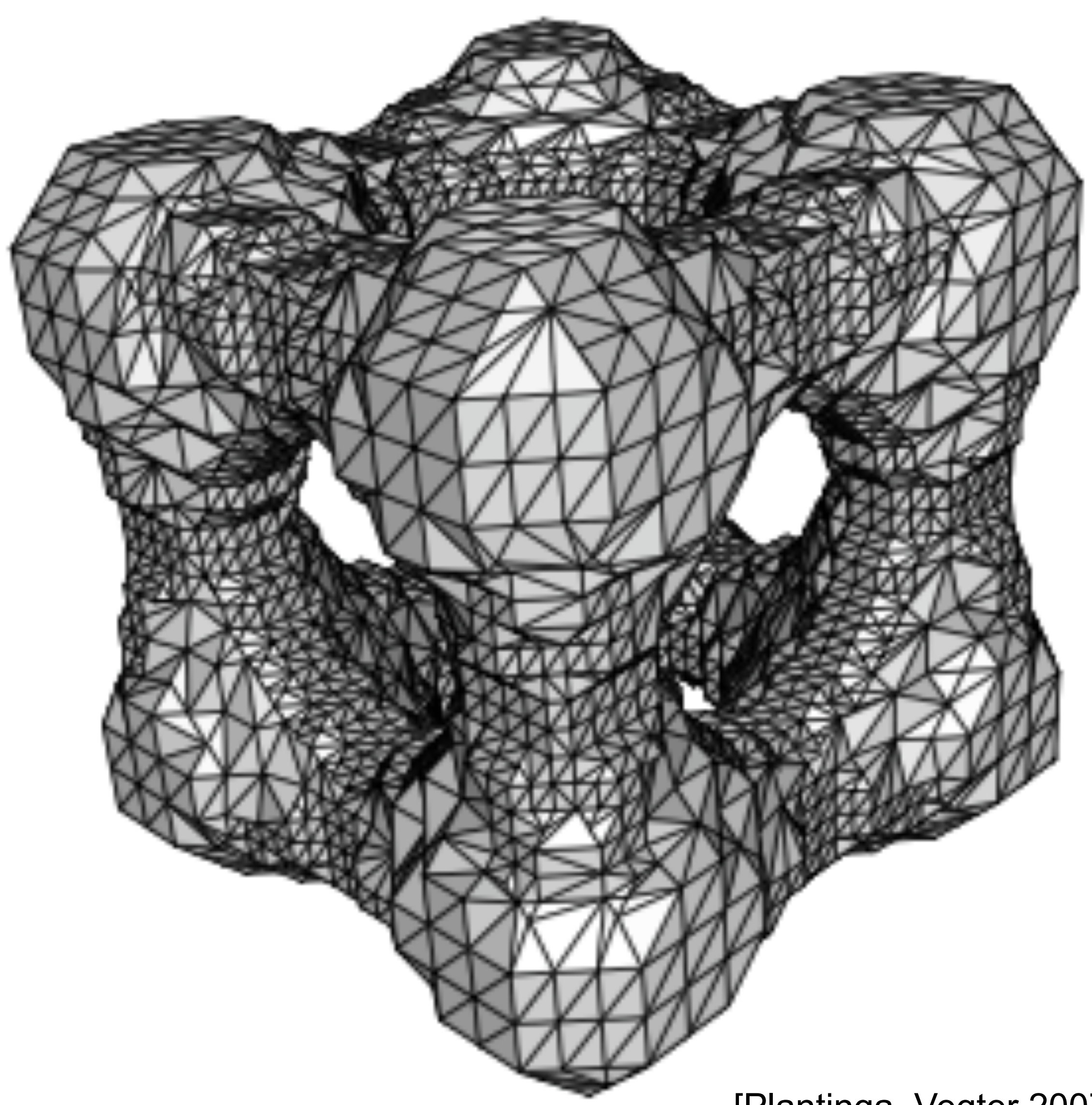
Isotopic Meshing

- Check the condition using interval arithmetic
  - Good for analytic surfaces
  - Expensive for sampled volume data
- Refine hierarchically: Use quadtree/octree
  - Less painful: Use tetrahedral subdivision



[Plantinga, Vegter 2007]

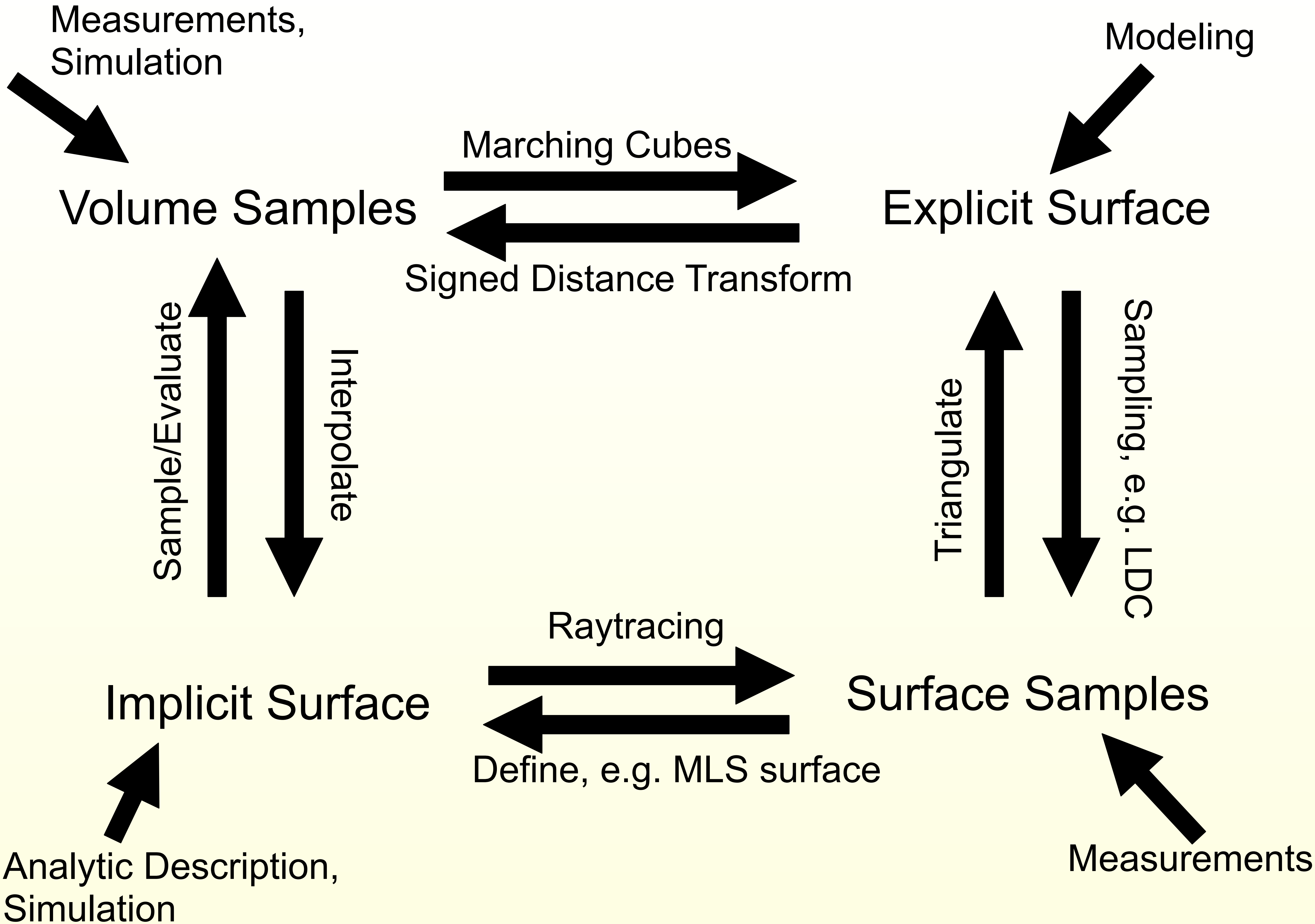
Isotopic Meshing



[Plantinga, Vegter 2007]

$$f(x, y, z) = x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 10$$

Summary


Measurements,
Simulation
Modeling
Marching Cubes
Volume Samples
Explicit Surface
Signed Distance Transform
Sample/Evaluate
Interpolate
Triangulate
Sampling, e.g. LDC
Raytracing
Implicit Surface
Surface Samples
Define, e.g. MLS surface
Analytic Description,
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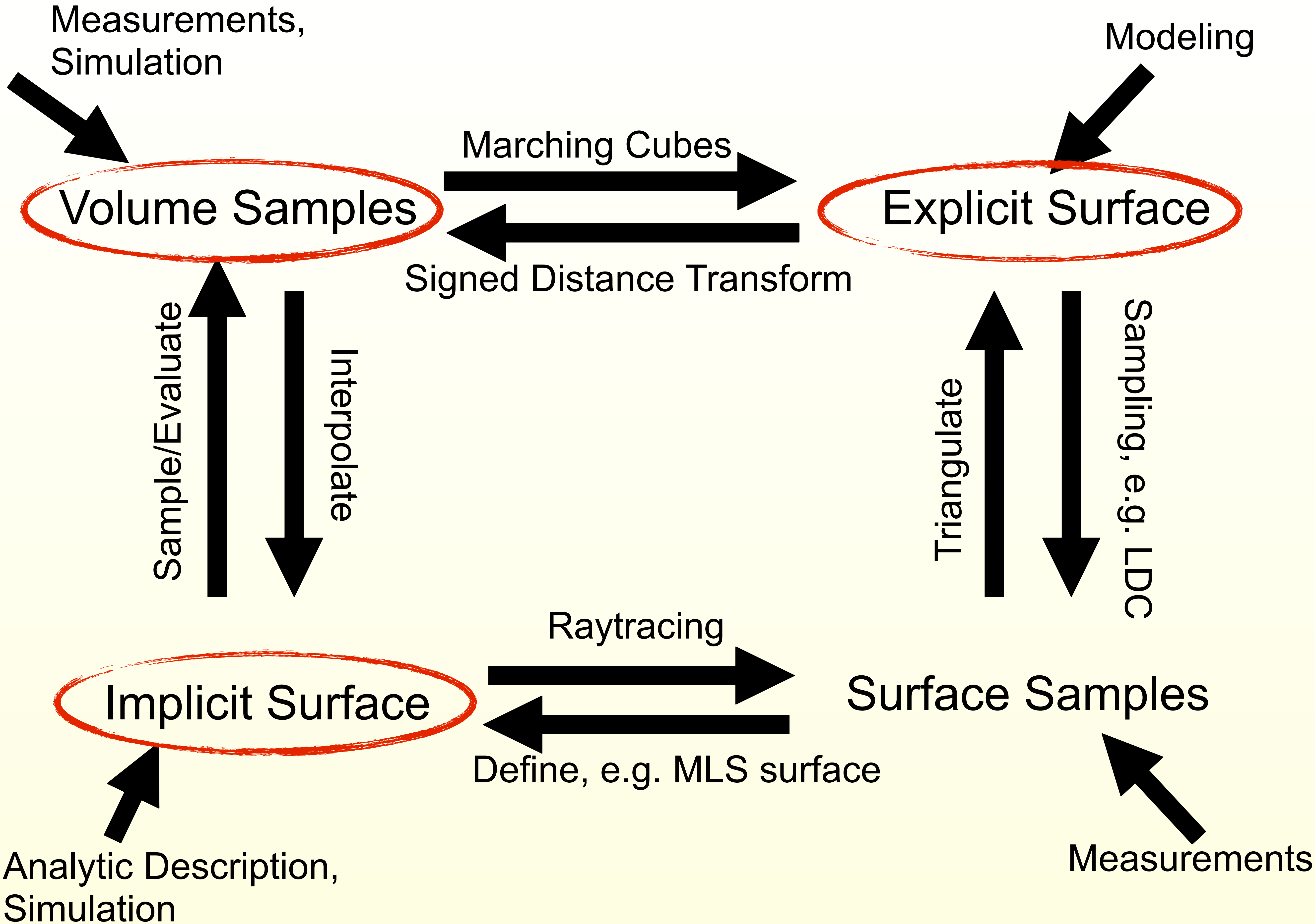
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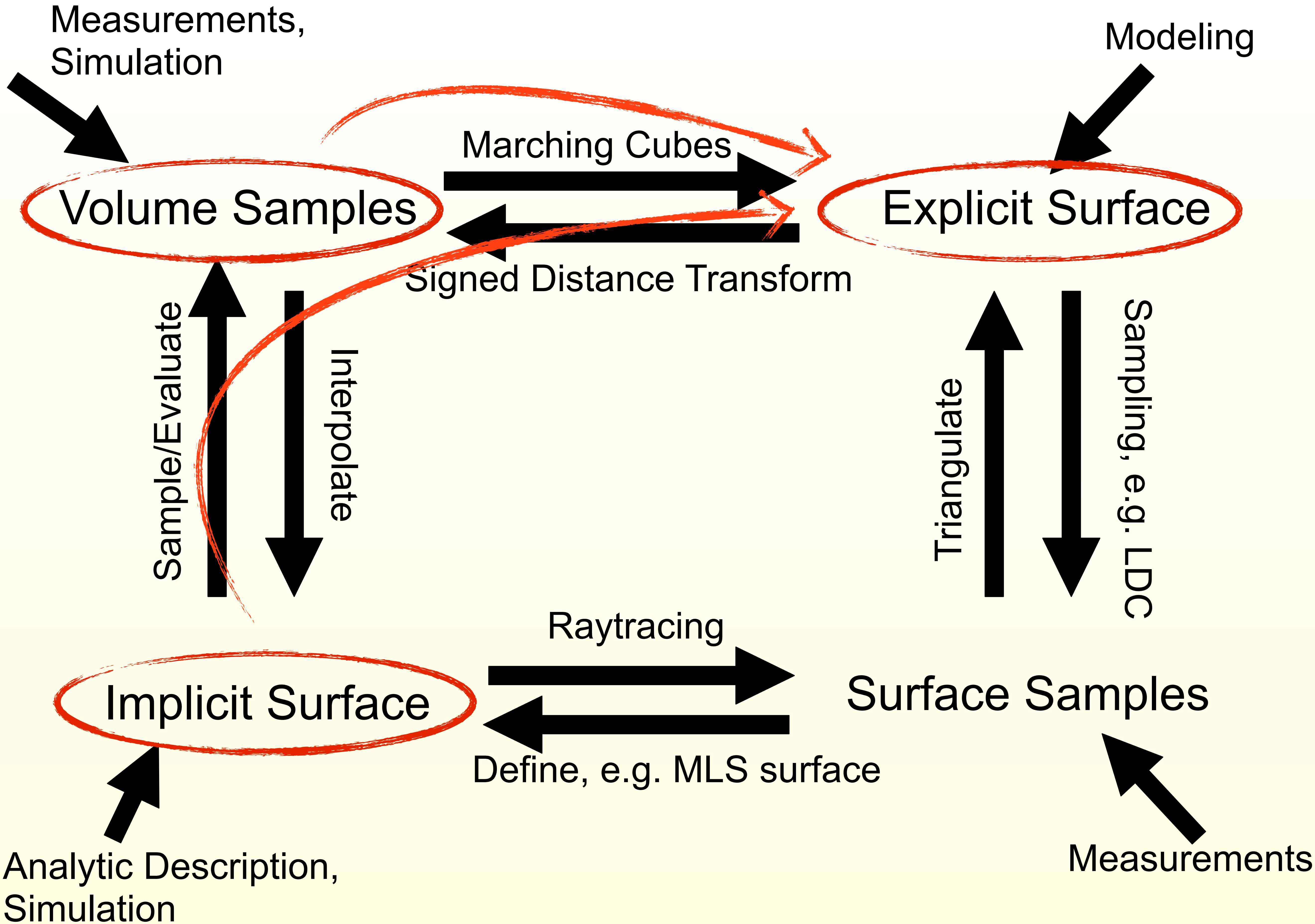

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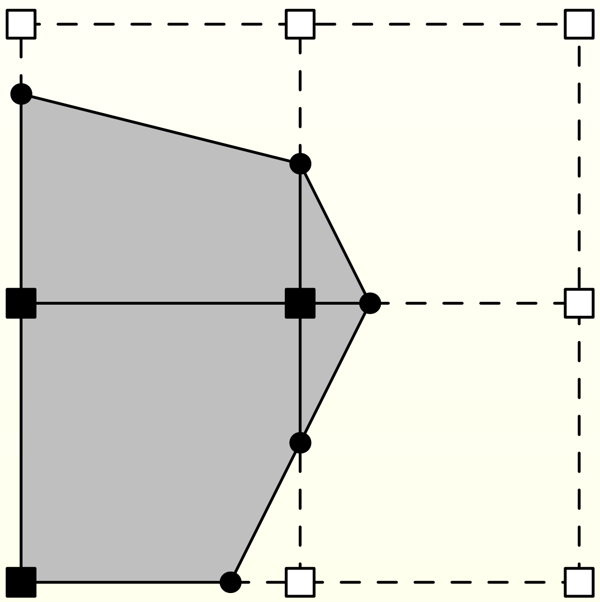
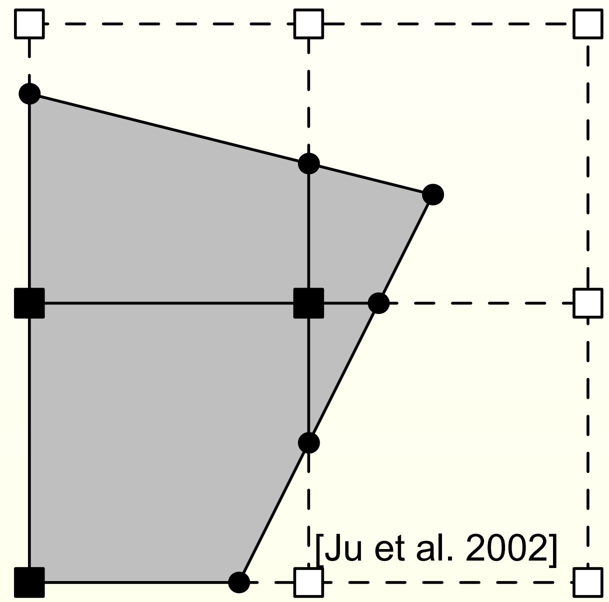
Marching Cubes Variants

Marching Cubes



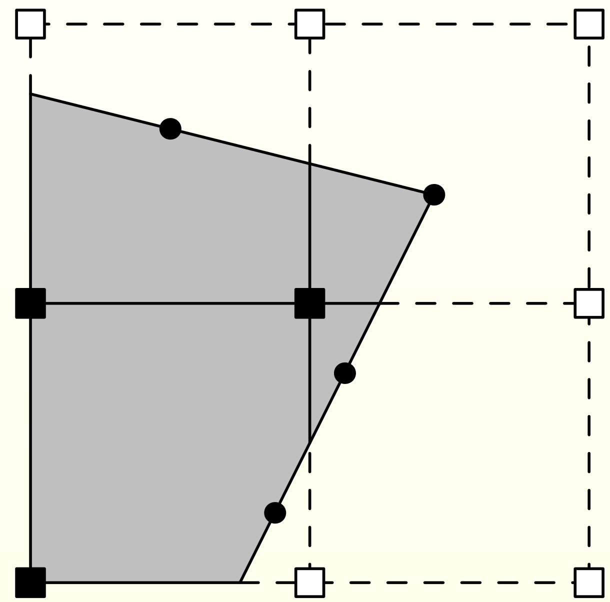
- Lookup tables
- No sharp features

Extended Marching Cubes



- Hybrid method
- Edges/ Corners are special cases
- Need feature threshold

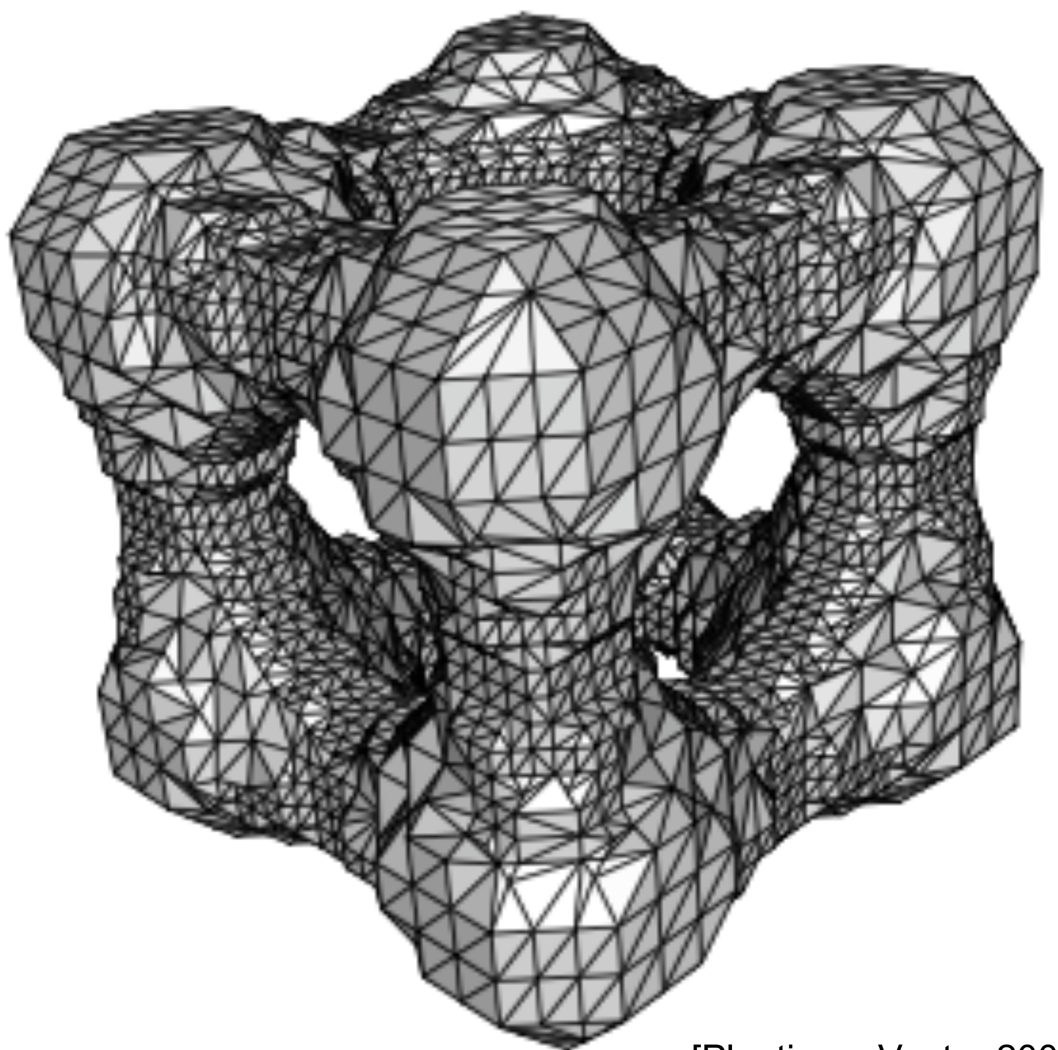
Dual Contouring



- Dual method
- No special cases
- No feature threshold

Topological Guarantees

- Enforced by subdivision
- Best for analytic surfaces
- Hierarchical refinement



[Plantinga, Vegter 2007]

Literature

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- S. Plantinga and G. Vegter: "**Isotopic Meshing of Implicit Surfaces**", The Visual Computer 23, 2007
- James Sharman: http://www.exaflop.org/docs/marchcubes/
- Paul Bourke: http://local.wasp.uwa.edu.au/~pbourke/geometry/polygonise/