Point set topology is something that every analyst should know something about, but it's easy to get carried away and do too much – it's like candy!
— Ron Getoor (UCSD), 1997 (quoted by Jason Lee)

1 Point Set Topology

In this lecture, we look at a major branch of topology: point set topology. This branch is devoted to the study of continuity. Developed in the beginning of the last century, point set topology was the culmination of a movement of theorists who wished to place mathematics on a rigorous and unified foundation. The theory is analytical and is therefore not suitable for computational purposes. The concepts, however, are foundational. Therefore, it is important to become familiar with them, as we will see them later, when studying combinatorial topology.

We know that topology is concerned with *connectivity*, and therefore the *neighborhoods* of points. We have actually seen neighborhoods before. In studying high-school calculus, you may have dealt with *epsilon-delta* definition of a limit (or continuity):

Definition 1.1 (Continuity) $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at* $p \in \mathbb{R}$ iff for all $\epsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ such that if $x \in \mathbb{R}$ and $|x - p| < \delta$, then $|f(x) - f(p)| < \epsilon$.

What does this definition mean? It means that if x is *near* p, then $f(x)$ will be *near* $f(p)$. The definition of *near* is *within* δ and *within* ϵ, respectively. Another way of thinking about this is that the function is continuous if an open set (of size 2ϵ) comes from an open set (of size 2δ). Open intervals and disks are natural neighborhoods in a Euclidean world. We take their existence for granted because we know how to measure distances (the *bars* in the definition), so we know who is *near* to us. Our ability to measure distances (a metric) gives us the neighborhoods, and therefore our topology.

But suppose we didn't have a metric. We still need neighborhoods to talk about connectivity. Topology formalizes this notion using set theory. If you need to brush up on sets and their operations, read Section 1.5 first.

1.1 Topological Spaces

We begin with a set of X objects we call *points*. Both sets and points are primitive notions, that is, we cannot define them. These points are not in any space yet. We endow our set with structure by using a topology to get a topological space.

Definition 1.2 (topology) A *topology* on a set X is a subset $T \subseteq 2^X$ such that:

1. If $S_1, S_2 \in T$, then $S_1 \cap S_2 \in T$.
2. If $\{S_J \mid j \in J\} \subseteq T$, then $\cup_{j \in J} S_j \in T$.
3. $\emptyset, X \in T$.

The definition states implicitly that only finite intersections, and infinite unions, of the sets in T are also in T. A topology is simply a system of sets that describe the connectivity of the set. These sets have names:

Definition 1.3 (open, closed) Let X be a set and T be a topology. $S \in T$ is an *open set*. The complement of an open set is *closed*.

A set may be only closed, only open, both open and closed, or neither. For instance, $\emptyset$ is both open and closed by definition. These sets are precisely the neighborhoods that we will use to define topology. We combine a set with a topology to get the spaces we are interested in.

Definition 1.4 (topological space) The pair (X, T) of a set X and a topology T is a *topological space*.

We often use $\mathbb{X}$ as notation for a topological space X, with T being understood.

Definition 1.5 (continuous) A function $f : \mathbb{X} \to \mathbb{Y}$ is continuous if for every open set A in $\mathbb{Y}$, $f^{-1}(A)$ is open in $\mathbb{X}$. We call a continuous function a *map*.

(a) $A \subseteq \mathbb{X}$ (b) $\overline{A}$ (c) $\mathring{A}$ (d) ∂A

Figure 1. A set $A \subseteq \mathbb{X}$ and related sets.

Compare this definition with Definition 1.1. We next turn our attention to the individual sets.

Definition 1.6 (interior, closure, boundary) Let $A \subseteq \mathbb{X}$. The *closure* $\overline{A}$ *of* A is the intersection of all closed sets containing A. The *interior* $\mathring{A}$ *of* A is the union of all open sets contained in A. The *boundary* ∂A *of* A is $\partial A = \overline{A} - \mathring{A}$.

In Figure 1, we see a set that is composed of a single point and a upside-down teardrop shape. We also see its closure, interior, and boundary. There are other equivalent ways of defining these concepts. For example, we may think of the boundary of a set as the set of points all of whose neighborhoods intersect both the set and its complement. Similarly, the closure of a set is the minimum closed set that contains the set. Using open sets, we can now define neighborhoods.

Definition 1.7 (neighborhoods) Let $\mathbb{X} = (X, T)$ be a topological space. A *neighborhood* of $x \in X$ is any $A \in T$ such that $x \in \mathring{A}$. A *basis of neighborhoods at* $x \in X$ is a collection of neighborhoods of x such that every neighborhood of x contains one of the basis neighborhoods.

Given a topological space $\mathbb{X} = (X, T)$, we may induce topology on any subset $A \subseteq X$. We get the *relative* (or *induced*) topology T_A by defining

$$T_A = \{S \cap A \mid S \in T\}. \tag{1}$$

It is easy to verify that T_A is, indeed, a topology on A, upgrading A to topological space $\mathbb{A}$.

Definition 1.8 (subspace) A subset $A \subseteq X$ with induced topology T_A is a (topological) *subspace* of $\mathbb{X}$.

The important point to keep in mind is that the same set of points may be endowed with different topologies. This is very counter-intuitive at first, but will become clear when we learn about immersions.

1.2 Metric Spaces

As in the definition of continuity earlier, we are more familiar with open sets that come from a metric. Let's look at metric spaces next, as they are useful places within which we shall place other spaces.

Definition 1.9 (metric) A *metric* or *distance function* $d : X \times X \to \mathbb{R}$ is a function satisfying the following axioms:

1. For all $x, y \in X$, $d(x, y) \geq 0$ (positivity).
2. If $d(x, y) = 0$, then $x = y$ (non-degeneracy).
3. For all $x, y \in X$, $d(x, y) = d(y, x)$ (symmetry).
4. For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

Definition 1.10 (open ball) The *open ball* $B(x, r)$ with center x and radius $r > 0$ with respect to metric d is defined to be $B(x, r) = \{y \mid d(x, y) < r\}$.

A metric space is a topological space. We can show that open balls can serve as basis neighborhoods for a topology of a set X with a metric.

Definition 1.11 (metric space) A set X with a metric function d is called a *metric space*. We give it the *metric topology* of d, where the set of open balls defined using d serve as basis neighborhoods.

The most familiar of the metric spaces are the Euclidean spaces, where we use the Euclidean metric to measure distances. Below, we use the Cartesian coordinate functions u_i (Definition 1.20 in the appendix.)

Definition 1.12 (Euclidean space) The Cartesian product of n copies of $\mathbb{R}$, the set of real numbers, along with the *Euclidean metric* $d(x, y) = \sqrt{\sum_{i=1}^{n}(u_i(x) - u_i(y))^2}$ is the n*-dimensional Euclidean space* $\mathbb{R}^n$.

We are most familiar with spaces that are subsets of Euclidean spaces. For example, if we have a circle sitting in $\mathbb{R}^2$, we may measure the distance between points on the circle using the metric on $\mathbb{R}^2$. This is the length of the chord connecting the two points. When we do so, we are using the topology induced by $\mathbb{R}^2$ to endow the circle with a topology. We might, however, like to have the distance between the two points on the circle itself. This is a different metric and a different neighborhood basis.

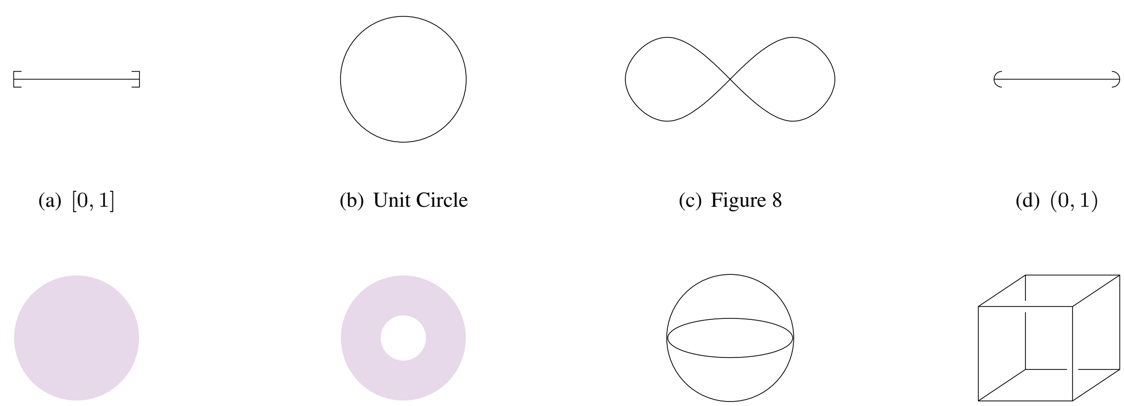
1.3 Homeomorphism

Recall Klein's unifying definition for topology and geometry. We transform a space by allowing a specific set of transformations. We then look at the properties that remain unchanged. If we allow *rigid motion* – translations and rotations – we get properties studied in *Euclidean geometry*. To study how a space is connected, we enrich this set to be much larger: We may allow a space to stretch, twist, expand, or shrink since as long as the transformation does not tear a space apart or sew two portions together, the space has not changed its connectivity. We call such a transformation a homeomorphism.

Definition 1.13 (homeomorphism) A *homeomorphism* $f : \mathbb{X} \to \mathbb{Y}$ is a 1-1 onto function, such that both f, f^{-1} are continuous. We say that $\mathbb{X}$ is *homeomorphic* to $\mathbb{Y}$, $\mathbb{X} \approx \mathbb{Y}$, and that $\mathbb{X}$ and $\mathbb{Y}$ have the same *topological type*.

Later, we will use homeomorphisms to define a classification of spaces. For now, let us look whether some spaces are homeomorphic. In each instance, we give an intuitive explanation and not formal proofs.

Example 1.1 ($[0, 1]$ and $\mathbb{S}^1$) Is the closed interval in Figure 2(a) homeomorphic to the unit circle in Figure 2(b). The closed interval has two endpoints, but the circle has no endpoints. Suppose there was we have a homeomorphism between these two spaces. Then, the homeomorphism must map the endpoints to the same point on the circle. But then the inverse of the homeomorphism is clearly not continuous. So, $[0, 1] \not\approx \mathbb{S}^1$.



(a) $[0, 1]$ (b) Unit Circle (c) Figure 8 (d) $(0, 1)$

(e) Unit Open Disc (f) Annulus (g) Sphere (h) Cube

Figure 2. Some simple spaces.

Example 1.2 (Circle and Figure 8) Is the circle in Figure 2(b) homeomorphic to the *Figure 8* in Figure 2(c)? Intuitively, we have to map the points of the circle to one of the two "circles" in the Figure 8, so it seems like we will not cover the other circle. Alternatively, suppose a homeomorphism exists. But then, the neighborhood of the crossing in Figure 8 cannot be mapped back to any point on the circle, as no point has a similar neighborhood. So, $\mathbb{S}^1 \not\approx$ Figure 8.

Example 1.3 ($(0, 1)$ and $\mathbb{R}$) Is the open interval in Figure 2(d) homeomorphic to the real numbers $\mathbb{R}$? We can stretch the interval until it covers all of $\mathbb{R}$. Any continuous function that maps 0 to $-\infty$ and 1 to $+\infty$ works. For example, we can use rescale the tangent function. Consider $h\colon [0, 1] \to \mathbb{R}$, where $h(x) = \tan(\pi x - \pi/2)$. Therefore, $(0, 1) \approx \mathbb{R}$.

Example 1.4 (Open disc and $\mathbb{R}^2$) Is the unit open disc in Figure 2(e) homeomorphic to $\mathbb{R}^2$? Again, we can stretch the open disc like a pizza dough until it covers all of $\mathbb{R}^2$. Alternatively, remember that a homeomorphism is a bijection, so we can shrink $\mathbb{R}^2$ until it fits inside the open disc using homeomorphism $h\colon \mathbb{R}^2 \to$ open disc, where $h(x) = x/(1 + ||x||)$, where $|| \cdot ||$ is the Euclidean norm. The homeomorphism maps the open disc to a disc of radius $1/2$, and then fits the rest of $\mathbb{R}^2$ in the remaining annulus. So, open disc $\approx \mathbb{R}^2$.

Example 1.5 (Open disc and Annulus) Is the open disc in Figure 2(e) homeomorphic to the *annulus* in Figure 2(f)? There is a topological difference: the annulus has a hole in its center. No matter how much we shrink or expand it, we cannot get rid of it. So, open disc $\not\approx$ annulus.

Example 1.6 (Circle and Annulus) Is the circle in Figure 2(b) homeomorphic to the annulus in Figure 2(f)? They both seem to share the topological characteristic in having a hole. We can certainly shrink the annulus to the circle. But we cannot expand the circle to the annulus, so we have problems in finding a bijection. In other words, an annulus has a "two-dimensionality" that the circle lacks. So, circle $\not\approx$ annulus.

Example 1.7 (Annulus and $\mathbb{R}^2 - \{0\}$) Is the annulus in Figure 2(f) homeomorphic to $\mathbb{R}^2$, provided we remove the origin? We can shrink the hold of the annulus to a single point at the origin. We can then use our trick in showing that an open disc was homeomorphic to $\mathbb{R}^2$ to expand the annulus to cover the rest of $\mathbb{R}^2$. Therefore, annulus $\approx \mathbb{R}^2 - \{0\}$.

Example 1.8 (Sphere and Cube) Is the sphere in Figure 2(g) homeomorphic to the cube in Figure 2(h)? In both cases, you should imagine these objects as being surfaces, with the sphere being like a basketball but not like a bowling ball. If we imagine the cube of being made from a stretchable material, we can blow it up until it looks like a sphere. Mathematically, the map $x/||x||$ is the homeomorphism we need. Therefore, sphere $\approx$ cube.

Example 1.9 (Sphere and $\mathbb{R}^2$) Is the sphere in Figure 2(g) homeomorphic to $\mathbb{R}^2$? To begin with, a sphere is a closed surface, but $\mathbb{R}^2$ is open, so we already have a problem! A classic method for stretching a sphere onto the plane of $\mathbb{R}^2$ is by the *stereographic projection*: we place a sphere onto $\mathbb{R}^2$ at the origin and connect a line segment from the sphere's maximum point (the north pole), through the sphere, to the plane. Each line segment maps its point of intersection with the sphere to the plane. Clearly, the method associates every point in $\mathbb{R}^2$ with a point on the sphere except for the north pole. This implies that sphere $\not\approx \mathbb{R}^2$. But more importantly, it gives us sphere - a point $\approx \mathbb{R}^2$. That is, a punctured sphere is homeomorphic to $\mathbb{R}^2$. Every time we pop a balloon, we are using this homeomorphism to flatten the balloon! Similarly, the example tells us that if we were to *add* a point to $\mathbb{R}^2$ – call it the point ∞ – then the new space will be homeomorphic to the sphere. In other words, sphere $\approx \mathbb{R}^2 \cup \infty$. This process is called the *one-point compactification of* $\mathbb{R}^2$. We will soon learn what *compact* means.

1.4 Manifolds

We are very familiar with Euclidean spaces, but we would like to expand the type of spaces we may work with to non-metric spaces. For example, we saw in Example 1.9 that the sphere is not topologically the same as the plane $\mathbb{R}^2$, and yet, it feels Euclidean *locally*: If we were living on a space like a sphere, we would think that we are living on a flat plane. In fact, we did!

The types of spaces we would like to define generalize Euclidean spaces: they *look* Euclidean locally, but are connected differently globally. These spaces are called *manifolds*. We begin by using a homeomorphism to formally define what we mean by locally Euclidean, as shown in Figure 3.

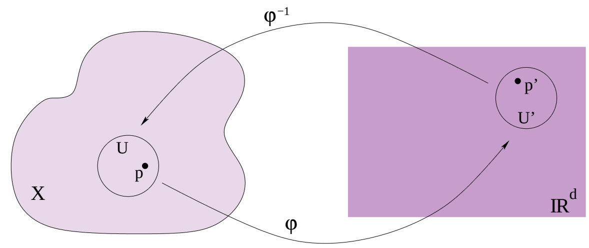


Figure 3. A chart at $p \in \mathbb{X}$. φ maps $U \subset \mathbb{X}$ containing p to $U' \subseteq \mathbb{R}^d$. As φ is a homeomorphism, φ^{-1} also exists and is continuous.

Definition 1.14 (chart) A *chart* at $p \in \mathbb{X}$ is a function $\varphi : U \to \mathbb{R}^d$, where $U \subseteq \mathbb{X}$ is an open set containing p and φ is a homeomorphism onto an open subset of $\mathbb{R}^d$. The *dimension* of the chart φ is d. The *coordinate functions* of the chart are $x^i = u^i \circ \varphi : U \to \mathbb{R}$, where $u^i : \mathbb{R}^n \to \mathbb{R}$ are the standard coordinates on $\mathbb{R}^d$.

We need two additional technical definitions, before we may define manifolds. These definitions rule out really strange spaces which we will never see. I include them so that they do not get endowed with a sense of magic and mystery.

Definition 1.15 (Hausdorff) A topological space $\mathbb{X}$ is *Hausdorff* if for every $x, y \in X, x \neq y$, there are neighborhoods U, V of x, y, respectively, such that $U \cap V = \emptyset$.

The classic example of a non-Hausdorff space is the real line with the origin duplicated as a different point. *All* the neighborhoods of the two origins intersect, but they are different points! A metric space, however, is always Hausdorff.

Definition 1.16 (separable) A topological space $\mathbb{X}$ is *separable* if it has a countable basis of neighborhoods.

Countable means having the same cardinality as integers, that is, the infinity all of us are familiar with (there are bigger ones, such as the cardinality of real numbers.) Again, metric spaces are separable (it's relatively easy to see this in Euclidean space, as an irrational point is always near a rational one.) Finally, we can formally define a manifold.

Definition 1.17 (manifold) A separable Hausdorff space $\mathbb{X}$ is called a *(topological, abstract) d-manifold* if there is a d-dimensional chart at every point $x \in \mathbb{X}$, that is, if $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$. It is called a *d-manifold with boundary* if $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ or the Euclidean half-space $\mathbb{H}^d = \{x \in \mathbb{R}^d \mid x_1 \geq 0\}$. The *boundary $\partial\mathbb{X}$ of* $\mathbb{X}$ is the set of points with neighborhood homeomorphic to $\mathbb{H}^d$. The manifold has *dimension d*.

Figure 4 displays a 2-manifold, and a 2-manifold with boundary.

Figure 4. The sphere (left) is a 2-manifold. The torus with two holes (right) is a 2-manifold with boundary. Its boundary, a 1-manifold, is composed of the two circles.

Theorem 1.1 *The boundary of a d-manifold with boundary is a $(d-1)$-manifold without boundary.*

The manifolds shown are compact.

Definition 1.18 (compact) A *covering of* $A \subseteq X$ is a family $\{C_j \mid j \in J\}$ in 2^X, such that $A \subseteq \bigcup_{j \in J} C_j$. An *open covering* is a covering consisting of open sets. A *subcovering* of a covering $\{C_j \mid j \in J\}$ is a covering $\{C_k \mid k \in K\}$, where $K \subseteq J$. $\mathbb{A} \subseteq \mathbb{X}$ is *compact* if every open covering of A has a finite subcovering.

Intuitively, you might think any finite area manifold is compact. However, a manifold can have finite area and not be compact, such as the cusp in Figure 5.

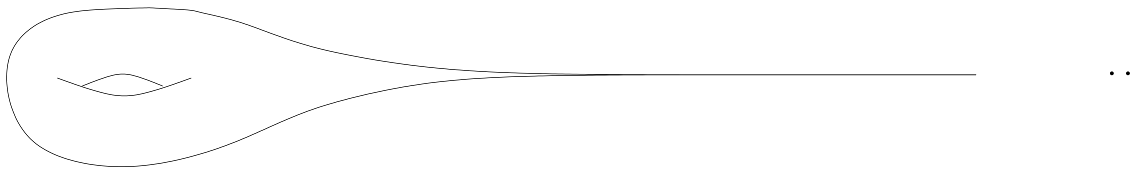


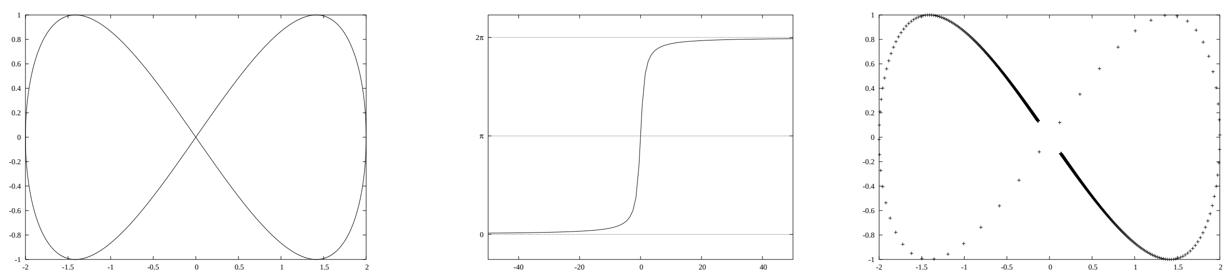
Figure 5. The cusp has finite area, but is not compact

A homeomorphism allows us to place one manifold within another.

Definition 1.19 (embedding) An *embedding* $g : \mathbb{X} \to \mathbb{Y}$ is a homeomorphism onto its image $g(\mathbb{X})$. The image is called an *embedded submanifold* and it is given its relative topology in $\mathbb{Y}$.

Most of our interaction with manifolds in our lives has been with embedded manifolds in Euclidean spaces. Consequently, we always think of manifolds in terms of an embedding. It is important to remember that a manifold exists independently of any embedding: a sphere does not have to sit within $\mathbb{R}^3$ to be a sphere. This is, by far, the biggest shift in the view of the world required by topology.

Example 1.10 Figure 6(a) shows an map of $\mathbb{R}$ into $\mathbb{R}^2$. Note that while the map is 1-1 locally, it is not 1-1 globally. The map F wraps $\mathbb{R}$ over the figure-eight over and over. Using the monotone function g in Figure 6(b), we first fit all of $\mathbb{R}$ into the interval $(0, 2\pi)$ and then map it using F once again. We get the same image (figure-eight) but cover it only once, making $\hat{F}$ 1-1 in Figure 6(c). However, the graph of $\hat{F}$ approaches the origin in the limit, at both ∞ and $-\infty$. Any neighborhood of the origin within $\mathbb{R}^2$ will have four pieces of the graph within it and will not be homeomorphic to $\mathbb{R}$. Therefore, the map is not homeomorphic to its image and not an embedding.



(a) $F(t) = 2\cos(t - \pi/2), \sin(2(t - \pi/2))$ (b) $g(t) = \pi + 2\tan^{-1}(t)$ (c) $\hat{F}(t) = F(g(t))$

Figure 6. Different placements of $\mathbb{R}$ into $\mathbb{R}^2$.

The maps shown on the left and right of Figure 6 are both immersions. Immersions are defined for smooth manifolds, which are described in further detail in the second appendix (for those of you who think differential manifolds are like candy.) If our original manifold $\mathbb{X}$ is compact, nothing "nasty" can happen. an *immersion* $F : \mathbb{X} \to \mathbb{Y}$ is simply a *local* embedding. In other words, for any point $p \in \mathbb{X}$, there exists a neighborhood U containing p such that $F|_U$ is an embedding. However, F need not be an embedding within the neighborhood of $F(p)$ in $\mathbb{Y}$. That is, immersed compact spaces may self-intersect.

Acknowledgments

Most of the material of this section is from Bishop and Goldberg [1] and Boothby [2]. I also used Henle [3] and McCarthy [4] for reference and inspiration.

References

[1] BISHOP, R. L., AND GOLDBERG, S. I. *Tensor Analysis on Manifolds*. Dover Publications, Inc., New York, 1980.

[2] BOOTHBY, W. M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*, second ed. Academic Press, San Diego, CA, 1986.

[3] HENLE, M. *A Combinatorial Introduction to Topology*. Dover Publications, Inc., New York, 1997.

[4] MCCARTHY, G. *Topology: An Introduction with Application to Topological Groups*. Dover Publications, Inc., New York, 1988.

$R = \{x : x \notin x\}$. Then, $R \in R$ iff $R \notin R$.
— Bertrand Russell (1872–1970)

1.5 Sets and Functions (Appendix)

We cannot define a set formally, other than stating that a *set* is a well-defined collection of objects. We also assume the following:

1. Set S is made up of *elements* $a \in S$.
2. There is only one empty set $\emptyset$.
3. We may describe a set by characterizing it ($\{x \mid \mathrm{P}(x)\}$), or by enumerating elements ($\{1, 2, 3\}$). Here P is a predicate.
4. A set S is *well-defined* if for each object a, either $a \in S$ or $a \notin S$.

Note that "well-defined" really refers to the definition of a set, rather than the set itself. $|S|$ or $\mathrm{card}\, S$ is the size of the set. We may multiply sets in order to get larger sets.

Definition 1.20 (Cartesian) *Cartesian product of sets* $S_1, S_2, \ldots, S_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in S_i$. The Cartesian product is denoted by either $S_1 \times S_2 \times \ldots \times S_n$ or by $\oplus_{i=1}^n S_i$. The i*-th Cartesian coordinate function* $u_i : \oplus_{i=1}^n S_i \to S_i$ is defined by $u_i(a_1, a_2, \ldots, a_n) = a_i$.

Having described sets, we define subsets.

Definition 1.21 (subsets) A set B is a *subset* of a set A, denoted $B \subseteq A$ or $A \supseteq B$, if every element of B is in A. $B \subset A$ or $A \supset B$ is generally used for $B \subseteq A$ and $B \neq A$. If A is any set, then A is the *improper subset* of A. Any other subset is *proper*. If A is a set, we denote by 2^A, the *power set of* A, the collection of all subsets of A, $2^A = \{B \mid B \subseteq A\}$.

We also have a couple of fundamental set operations.

Definition 1.22 (intersection, union) The *intersection* $A \cap B$ of sets A and B is the set consisting of those elements which belong to both A and B, that is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The *union* $A \cup B$ of sets A and B is the set consisting of those elements which belong to A or B, that is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

We indicate a collection of sets by labeling them with subscripts from an index set J, e.g. A_j with $j \in J$. For example, we use $\bigcap_{j \in J} A_j = \bigcap\{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for all } j \in J\}$ for general intersection. The next definition summarizes functions, maps relating sets to sets.

Definition 1.23 (relations and functions) A *relation* φ between sets A and B is a collection of ordered pairs (a, b) such that $a \in A$ and $b \in B$. If $(a, b) \in \varphi$, we often denote the relationship by $a \sim b$. A *function* or *mapping* φ *from a set A into a set B* is a rule that assigns to each element a of A exactly one element b of B. We say that φ *maps* a *into* b, and that φ *maps* A *into* B. We denote this by $\varphi(a) = b$. The element b is the *image of* a *under* φ. We show the map as $\varphi : A \to B$. The set A is the *domain of* φ, the set B is the *codomain of* φ, and the set $\mathrm{im}\, \varphi = \varphi(A) = \{\varphi(a) \mid a \in A\}$ is the *image of* A *under* φ. If φ and ψ are functions with $\varphi : A \to B$ and $\psi : B \to C$, then there is a natural function mapping A into C, the *composite function*, consisting of φ followed by ψ. We write $\psi(\varphi(a)) = c$ and denote the composite function by $\psi \circ \varphi$. A function from a set A into a set B is *one to one (1-1)* (*injective*) if each element B has at most one element mapped into it, and it is *onto* B (*surjective*) if each element of B has at least one element of A mapped into it. If it is both, it's a *bijection*. A bijection of a set onto itself is called a *permutation*.

A permutation of a finite set is usually specified by its action on the elements of the set. For example, we may denote a permutation of the set $\{1, 2, 3, 4, 5, 6\}$ by $(6, 5, 2, 4, 3, 1)$, where the notation states that the permutation maps 1 to 6, 2 to 5, 3 to 2, and so on. We may then obtain a new permutation by a transposition: switching the order of two neighboring elements. In our example, $(5, 6, 2, 4, 3, 1)$ is a permutation that is one transposition away from $(6, 5, 2, 4, 3, 1)$. We may place all permutations of a finite set in two sets.

Theorem 1.2 (Parity) *A permutation of a finite set can be expressed as either an even or an odd number of transpositions, but not both. In the former case, the permutation is* even*. In the latter, it is* odd*.*

1.6 Smooth Manifolds (Appendix)

We will next look at smooth manifolds. We know what smooth means within the Euclidean domain. It's easy to extend the notion of smoothness to manifolds because we know that they are locally flat; that is, there is a local chart that maps the neighborhood of a point to the Euclidean space.

Definition 1.24 (C^∞) Let $U, V \subseteq \mathbb{R}^d$ be open. A function $f : U \to \mathbb{R}$ is *smooth* or C^∞ (*continuous of order* ∞) if f has partial derivatives of all orders and types. A function $\varphi : U \to \mathbb{R}^e$ is a C^∞ *map* if all its components $e^i \circ \varphi : U \to \mathbb{R}$ are smooth Two charts $\varphi : U \to \mathbb{R}^d, \psi : V \to \mathbb{R}^e$ are C^∞*-related* if $d = e$ and either $U \cap V = \emptyset$ or $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are C^∞ maps. A C^∞ *atlas* is one for which every pair of charts is C^∞-related. A chart is *admissible* to a C^∞ atlas if it is C^∞-related to every chart in the atlas.

C^∞-related charts allow us to pass from one coordinate system to another smoothly in the overlapping region, so we may extend our notions of curves, functions, and differentials easily to manifolds.

Definition 1.25 (C^∞ manifold) A *smooth (C^∞) manifold* is a topological manifold together with all the admissible charts of some C^∞ atlas.

The map used between smooth manifolds is called a diffeomorphism.

Definition 1.26 (diffeomorphism) A *diffeomorphism* $g : \mathbb{X} \to \mathbb{Y}$ is a C^∞ map that is a homeomorphism and whose inverse g^{-1} is C^∞. We say that $\mathbb{X}$ is *diffeomorphic* to $\mathbb{Y}$.

A diffeomorphism g allows us to place a smooth manifold $\mathbb{X}$ within another smooth manifold $\mathbb{Y}$. We would like to know more about the image $g(\mathbb{X}) \subseteq \mathbb{Y}$. To do so, we take advantage of the atlas on each manifold. Suppose that U, φ is a chart at $p \in \mathbb{X}$ and V, ψ is a chart at $g(p) \in \mathbb{Y}$. This allows us to get an expression for g in terms of local coordinates:

$$\begin{array}{ccc} U \subseteq \mathbb{X} & \xrightarrow{g} & V \subseteq \mathbb{Y} \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ \varphi(U) \subseteq \mathbb{R}^d & \xrightarrow{\hat{g}} & \psi(V) \subseteq \mathbb{R}^e \end{array}$$

That is, $\hat{g} = \psi \circ g \circ \varphi^{-1}$.

Definition 1.27 (Jacobian) The *Jacobian matrix* Dg of a map $g : \mathbb{X} \to \mathbb{Y}$ with local charts U, φ at $p \in \mathbb{X}$ and V, ψ is a chart at $g(p) \in \mathbb{Y}$ is:

$$\frac{\partial(g^1, \ldots, g^e)}{\partial(x^1, \ldots, x^d)} = \begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \cdots & \frac{\partial g^1}{\partial x^d} \\ \vdots & & \vdots \\ \frac{\partial g^e}{\partial x^1} & \cdots & \frac{\partial g^e}{\partial x^d} \end{pmatrix}$$

Dg is defined at each point of U, its $d \cdot e$ entries being functions on U.

The rank of the Jacobian tells us what the diffeomorphism does to its domain space.

Definition 1.28 (rank) The rank of g is the rank of Dg.

This rank is independent of the coordinate system we use (and can be defined independently, too, but that's beyond the scope of this class.)

Definition 1.29 (immersion) $g : \mathbb{X} \to \mathbb{Y}$ is an *immersion* if $\operatorname{rank} g = \dim \mathbb{X}$.

Intuitively, An immersion places a space within another one so that its dimension does not change, and it doesn't develop any kinks. The immersed space, however, can intersect itself or behave in otherwise unappetizing ways, as we saw in Example 1.10. What we are really after are nice immersions, or embeddings.

Definition 1.30 (embedding) An *embedding* $g : \mathbb{X} \to \mathbb{Y}$ is a 1-1 immersion that is a homeomorphism onto its image $g(\mathbb{X})$ considered as a subspace of $\mathbb{Y}$. The image is called an *embedded submanifold* and is given the relative topology.

The definition of smooth manifolds also allows us to give a point-set theoretic definition of orientability. We will see later that the following definitions also apply in non-smooth spaces, such as simplicial spaces.

Definition 1.31 (orientability) A pair of charts x^i and y^i is *consistently oriented* if the Jacobian determinant $\det(\partial x^i / \partial y^j)$ is positive whenever defined. A manifold M is *orientable* if there exists an atlas such that every pair of coordinate systems in the atlas is consistently oriented. Such an atlas is *consistently oriented* and determines an *orientation on* M. If a manifold is not orientable, it is *unorientable*.

In other words, a manifold of any dimension falls into two classes, depending on whether it is orientable or not.

Klein bottle for rent – inquire within.
— Anonymous

2 Surface Topology

Last lecture, we spent a considerable amount of effort defining *manifolds*. We like manifolds because they are locally Euclidean. So, even though it is hard for us to reason about them globally, we know what to do in small neighborhoods. It turns out that this ability is all we really need. This is rather fortunate, because we suddenly have spaces with more interesting structure than the Euclidean spaces to study.

Recall that topology, like Euclidean geometry, is a study of the properties of spaces that remain invariant (do not change) under a fixed set of transformations. In topology, we expand the transformations that are allowed from rigid motions (Euclidean geometry) to *homeomorphisms*: bijective bi-continuous maps. In this lecture, we ask whether we may classify manifolds under this set of transformations, and we see that such a classification is possible for two-dimensional manifolds or *surfaces*.

2.1 Topological Type

To begin with, we should indicate what we mean by a *classification*. This notion has a nice mathematical definition, which you may have seen in high school.

Definition 2.1 (partition) A *partition of a set* is a decomposition of the set into subsets (*cells*) such that every element of the set is in one and only one of the subsets.

We wish to partition the set of manifolds according to their connectivity. We are forced to look at different partitioning schemes in our search for one which is computationally feasible. Each scheme depends on an equivalence relation.

Definition 2.2 (equivalence) Let S be a nonempty set and let $\sim$ be a relation between elements of S that satisfies the following properties for all $a, b, c \in S$:

1. (Reflexive) $a \sim a$.
2. (Symmetric) If $a \sim b$, then $b \sim a$.
3. (Transitive) If $a \sim b$ and $b \sim c$, $a \sim c$.

Then, the relation $\sim$ is an *equivalence relation* on S.

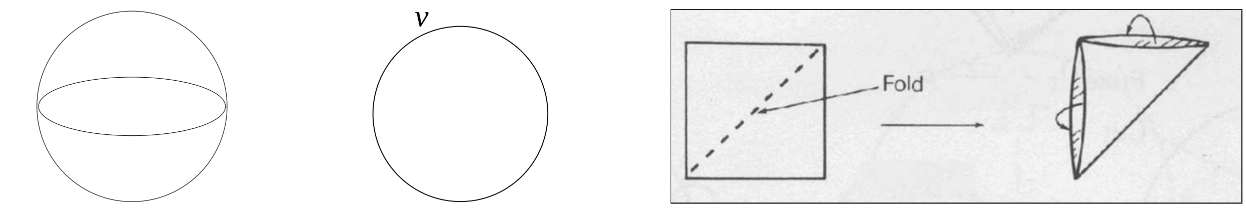
It is clear from the definition of homeomorphism that it is an equivalence relation. The following theorem allows us to derive a partition from an equivalence relation.

Theorem 2.1 *Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Then, $\sim$ yields a natural partition of S, where $\bar{a} = \{x \in S \mid x \sim a\}$. $\bar{a}$ represents the subset to which a belongs to. Each cell $\bar{a}$ is an* equivalence class.

As homeomorphism is an equivalence relation, we may use it to partition manifolds by this theorem. If two manifolds are placed in the same subset, they are connected the same way, and we say that they have the same *topological type*. One of the fundamental questions in topology is whether this partition is computable. In this lecture, we focus on the solution to this problem in two dimensions.

2.2 Basic 2-Manifolds

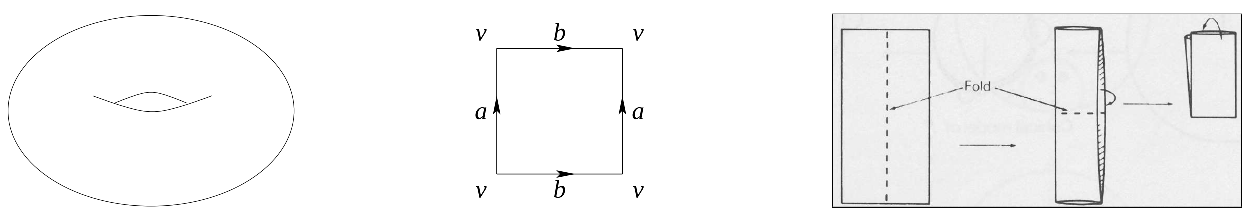
Before classifying 2-manifolds, however, it would be nice to meet a few of them. In this section, we look at a few basic 2-manifolds.



(a) $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ (b) Identify boundary to v (c) Instructions for a flat sphere

Figure 1. The sphere $\mathbb{S}^2$

The sphere. Topologically, the sphere $\mathbb{S}^2$ is the simplest surface. We are most comfortable with the implicit surface definition in Figure 1(a), that defines the unit sphere as a subspace of $\mathbb{R}^3$. The sphere may be defined, however, using a diagram in Figure 1(b), which asks us to make the entire boundary of a disc to a single point. This process is called *identification*: this means that all the points in the boundary should be treated as if they were the same point. The identification here gives us a topological sphere. We may also make a sphere out of paper as shown in Figure 1(c). Paper has no curvature, so it has flat geometry, and we get a *flat* sphere. The abstract sphere defined by the diagram (b), along with the flat sphere, highlight the difference between the sphere as a topological concept, and a sphere as a geometric entity. It is important for you to consider the difference carefully. We only care about connectivity in topology, and what is connected *like* the geometric sphere *is* a sphere, no matter its geometry.



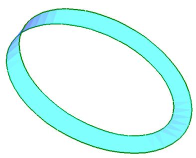
(a) Donut surface (b) Diagram (c) Instructions for a flat torus

Figure 2. The torus $\mathbb{T}^2$

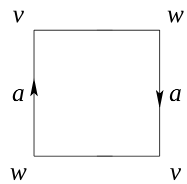
The torus. The torus is familiar to us as the surface of a bagel or a donut, as shown in Figure 2(a). We may describe a torus as a subspace of $\mathbb{R}^3$ geometrically. For example, a *torus of revolution* is created when we sweep a circle around the z-axis: $T(u, v) = ((1 + \cos u)\cos v, (1 + \cos u)\sin v, \sin(u))$. The torus may also be described via a diagram in Figure 2(b), in which the edges are glued according to their direction of their arrows. Finally, we can build a flat torus using the directions in Figure 2(c)

The Möbius strip. Figure 3(a) shows an embedded Möbius strip: a 2-manifold with boundary. It is easy to construct one by gluing one end of a strip of paper to the other end with a single twist, as shown in the diagram in Figure 3(b). This manifold is not *orientable*. The notes for last lecture included a definition of orientability for smooth manifolds in an appendix. We will see another formal definition of orientability in the next lecture. For now, orientability means that the surface has two sides. In Figure 3(c), M. C. Escher establishes that the Möbius strip is one-sided by marching ants on the strip Note that the boundary of the Möbius strip is a single cycle. This cycle corresponds to the two unglued edges in the diagram 3(b) which we may now glue with or without a twist.

The projective plane. If we put non-matching arrows on the remaining two edges of the Möbius diagram as in Figure 4(a), we get the projective plane $\mathbb{RP}^2$. This action corresponds to gluing the boundary of a disk to the boundary



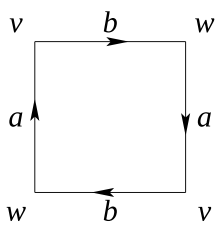
(a) Embedded



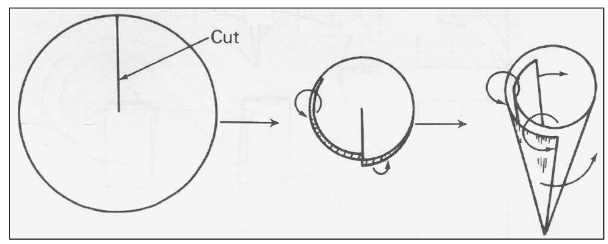
(b) Diagram

(c) Escher's *Möbius Strip II*

Figure 3. The Möbius strip is a non-orientable manifold with boundary.



(a) Diagram

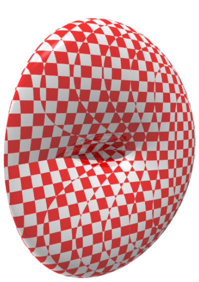


(b) Instructions for a flat $\mathbb{R}\mathrm{P}^2$

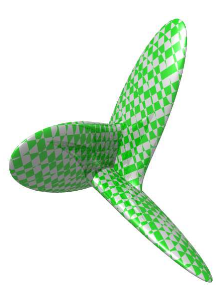
Figure 4. The projective plane $\mathbb{R}\mathrm{P}^2$

of the Möbius strip. This manifold has this name because of its association with *projective geometry* used in art and computer graphics for representing what we see on a flat canvas. For example, we know that railway lines never intersect, as they are parallel. When we look at them in real life, however, we see that they come together at the horizon, or at "infinity". They also intersect at horizon behind us. We would like any two lines to intersect at most once, so we identify the two intersecting points as the same point. Imagine the boundary of the diagram in 4(a) is the horizon. The arrows on the diagram identify a point and its reflected image around the origin. These points are called *anti-podal* points. This manifold is non-orientable as it contains a Möbius strip. It cannot be embedded in $\mathbb{R}^3$, so we have to be content with immersions. Figure 5 shows three immersions of the projective plane, all of which self-intersect. These immersions are famous as they contain interesting geometry in addition to their shared topology. To make an paper model, we have to cut the paper to allow for the self-intersection.

The Klein bottle. If we glue the free edges of the Möbius strip in the same direction, we get the Klein bottle $\mathbb{K}^2$, as shown in Figure 6(a). The Klein bottle is therefore equivalent to gluing two Möbius strips to each other along their boundary. Like the projective plane, it is a closed non-orientable surface. It is not embeddable in $\mathbb{R}^3$, and its immersions in Figures 6(b) and 6(c) self-intersect with the intersecting triangle colored in red. Once again, we need to cut paper in order to make a flat model, as shown in FIgure 6(d).



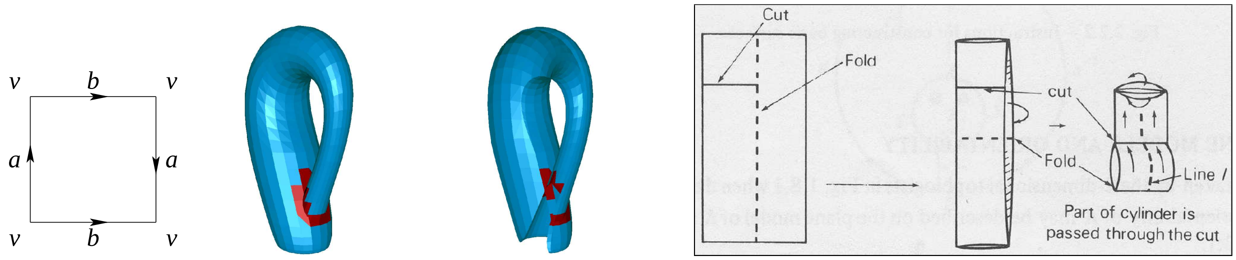
(a) Cross cap



(b) Boy's Surface

(c) Steiner's Roman Surface

Figure 5. Models of the projective plane $\mathbb{R}\mathrm{P}^2$



(a) Diagram (b) An immersion (c) Cut in half (a Möbius strip) (d) Instructions for a flat $\mathbb{K}^2$

Figure 6. The Klein bottle $\mathbb{K}^2$

2.3 Connected Sum

We may use the surfaces we just defined to form larger manifolds. To do this, we form connected sums.

Definition 2.3 (connected sum) The *connected sum* of two n-manifolds $\mathbb{M}_1, \mathbb{M}_2$ is

$$\mathbb{M}_1 \# \mathbb{M}_2 = \mathbb{M}_1 - \mathring{D}_1^n \bigcup_{\partial \mathring{D}_1^n = \partial \mathring{D}_2^n} \mathbb{M}_2 - \mathring{D}_2^n,$$

where D_1^n, D_2^n are n-dimensional closed disks in $\mathbb{M}_1, \mathbb{M}_2$, respectively.

In other words, we cut out two disks and glue the manifolds together along the boundary of those disks using a homeomorphism. In Figure 7, for example, we connect two tori to form a sum with two *handles*.

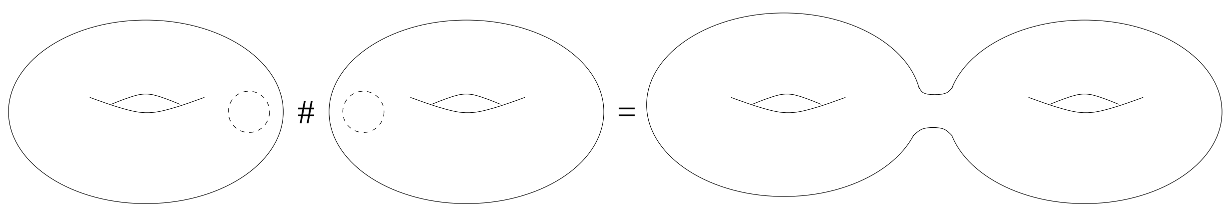


Figure 7. The connected sum of two tori is a genus 2 torus.

2.4 The Classification Theorem

We are now able to state a result that gives a complete classification of compact 2-manifolds.

Theorem 2.2 (classification of compact 2-manifolds) *Every closed compact surface is homeomorphic to a sphere, the connected sum of tori, or connected sum of projective planes.*

We will see in the next lecture that this classification is easily computable. In the remainder of this lecture, we will look at Conway's ZIP proof [2] of this theorem. The paper is provided on the website as the notes for the rest of the lecture.

The theorem answers the homeomorphism question for compact manifolds in two dimensions. After learning about groups, we will see that this question is undecidable for dimensions four and higher. This problem is still open in three dimensions, and *three-manifold topology* is an active area of research. For a very accessible overview, see Weeks [3].

Acknowledgments

The instruction for making flat 2-manifolds are from Firby and Gardiner [1]. I rendered the models of projective plane in Figure 5 in POV-Ray using descriptions by Tore Nordstrand. Figures 6(b) and 6(c) are from [4].

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