Combinatorics is the slums of topology.
— J. H. C. Whitehead (attr.)

3 Simplicial Complexes

In the first lecture, we looked at concepts from *point set topology*, the branch of topology that studies continuity from an analytical point of view. This view does not have a computational nature: we cannot represent infinite point sets or their associated infinite open sets on a computer. Starting with this lecture, we will look at concepts from another major branch of topology: *combinatorial topology*. This branch also studies connectivity, but does so by examining constructing complicated objects out of simple blocks, and deducing the properties of the constructed objects from the blocks. While our view of the world–our *ontology*–will be mostly combinatorial in nature, we will see concepts from point set topology reemerging under disguise, and we will be careful to expose them!

In this lecture, we begin by learning about simple building blocks from which we may construct complicated spaces. Simplicial complexes are combinatorial objects that represent topological spaces. With simplicial complexes, we separate the topology of a space from its geometry, much like the separation of syntax and semantics in logic. Given the finite combinatorial description of a space, we are able to count, and the miracle of combinatorial topology is that counting alone enables us to make statements about the connectivity of a space. We shall experience a first instance of this marvelous theory in the *Euler characteristic*. This topological invariant gives a simple algorithm for classifying 2-manifolds, turning our existential classification from the last lecture into a computational method.

3.1 Geometric Definition

We begin with a definition of simplicial complexes that seems to mix geometry and topology. Combinations allow us to represent regions of space with very few points. In other words, allow us to describe simple cells which become our building blocks later.

Definition 3.1 (combinations) Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^d$. A *linear combination* is $x = \sum_{i=0}^{k} \lambda_i p_i$, for some $\lambda_i \in \mathbb{R}$. An *affine combination* is a linear combination with $\sum_{i=0}^{k} \lambda_i = 1$. A *convex combination* is a an affine combination with $\lambda_i \geq 0$, for all i. The set of all convex combinations is the *convex hull*.

You may have seen the concept of *independence* in studying linear algebra.

Definition 3.2 (independence) A set S is *linearly (affinely) independent* if no point in S is a linear (affine) combination of the other points in S.

Figure 1 shows the linear, affine, and convex combinations of three affinely independent points in $\mathbb{R}^3$. We may now define our basic building block.

Definition 3.3 (k-simplex) A k*-simplex* is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the *vertices* of the simplex.

A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$. We show low-dimensional simplices with their names in Figure 2. Since all the points defining a simplex are affinely independent, so is any subset of them. This causes the simplex to have an interesting structure: it is composed of simplices of lower-dimension, or its *faces*.

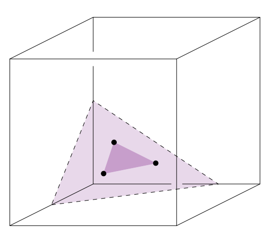


Figure 1. Combinations. The linear combinations of three affinely independent points in $\mathbb{R}^3$ covers the whole space. The affine combinations fill the plane defined by the three points. The convex hull is the triangle defined by the three points.

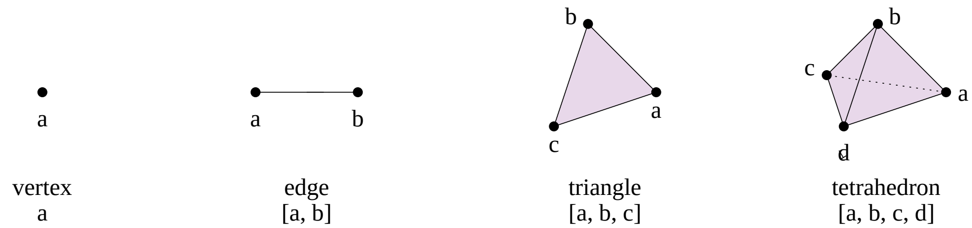


Figure 2. k-simplices, for each $0 \leq k \leq 3$.

Definition 3.4 (face, coface) Let σ be a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$. A simplex τ defined by $T \subseteq S$ is a *face* of σ and has σ as a *coface*. The relationship is denoted with $\sigma \geq \tau$ and $\tau \leq \sigma$. Note that $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

Note that a simplex is always a face of itself by this definition.

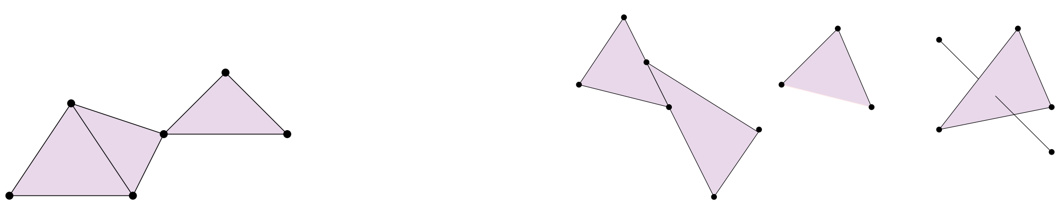
We attach simplices together to represent spaces. This attaching is very much like using Lego blocks to build castles: we can only attach Lego blocks on the special interfaces. Similarly, we may only attach simplices along their special interfaces: their faces. The following definition formally defines our structures, which we call *complexes*.

Definition 3.5 (simplicial complex) A *simplicial complex* K is a finite set of simplices such that

1. every face of a simplex in K is in K, and
2. the non-empty intersection of any two simplices of K is a face of each of them.

The *dimension* of K is $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$. The *vertices* of K are the zero-simplices in K. A simplex is *principal* if it has no proper coface in K.

Here, *proper* has the same definition as for sets. So, a simplicial complex is a collection of simplices that fit together nicely, as shown in Figure 3(a), as opposed to simplices in 3(b).



(a) The middle triangle shares an edge with the triangle on the left, and a vertex with the triangle on the right.

(b) In the middle, the triangle is missing an edge. The simplices on the left and right intersect, but not along shared simplices.

Figure 3. A simplicial complex (a) and disallowed collections of simplices (b).

Recall that each k-simplex is a k-dimensional subspace of $\mathbb{R}^d$. By putting them together nicely in a simplicial complex, we have made sure that the resulting complex is also a subspace of $\mathbb{R}^d$. In other words, we now have a simplicial complex is a combinatorial representation of a topological space.

Definition 3.6 (underlying space) The *underlying space* $|K|$ of a simplicial complex K is $|K| = \cup_{\sigma \in K} \sigma$.

$|K|$ is a topological space as we can induce a topology on it from $\mathbb{R}^d$. We defined two topological spaces to be equivalent if there was a homeomorphism mapping one to the other. We may now define two simplicial complexes to be equivalent if their corresponding underlying spaces are homeomorphic. We begin at the level of simplices and show that any two k-simplices are homeomorphic. While this is intuitively clear, it's nice to define an explicit homeomorphism which is useful in practice.

Definition 3.7 (barycentric coordinates) Let σ be the k-simplex defined on $S = \{v_0, v_1, \ldots, v_k\}$. Then we may write an point $x \in \sigma$ as a linear sum

$$x = \sum_{i=0}^{k} \lambda_i v_i,$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. The λ_i are the *barycentric coordinates* of x.

The definition follows easily from the definition of the k-simplex as each point $x \in \sigma$ is a convex combination. In this manner, we can coordinatize all points in a simplex, provided we put a fixed ordering on the vertices of the simplex. We use barycentric coordinates to map one simplex to another.

Lemma 3.1 *All k-simplices are homeomorphic.*

Proof: Let σ be the k-simplex defined on $S = \{v_0, v_1, \ldots, v_k\}$ and τ, the k-simplex defined on $T = \{v'_0, v'_1, \ldots, v'_k\}$. Let $f\colon S \to T$ be any bijection that maps the vertices of σ to those of τ. We need to extend this discrete map to a continuous map g on all of σ. We define $g\colon \sigma \to \tau$ as follows. Take any point $x \in \sigma$ and write it in terms of its barycentric coordinates: $x = \sum_{i=0}^{k} \lambda_i v_i$. Then we simply map $g(x)$ to the point $y = \sum_{i=0}^{k} \lambda_i v'_i$ in τ, that is, the point in τ that has the same coordinates as x. It is now easy to show that this map is a homeomorphism. This map is often called a *linear* or *simplicial* map. $\square$

Using this lemma, we may prove the following.

Theorem 3.1 *Let K and L be simplicial complexes, and f be a bijection from the vertices of K to those of L such that $\{v_0, \ldots, v_k\}$ spans a simplex in K iff $\{f(v_0), \ldots, f(v_k)\}$ spans a simplex of L. Then, $|K| \approx |L|$.*

Proof: We give a sketch of the proof here. We map each simplex using a linear map and the piece-wise linear maps agree on the faces of simplices and are continuous there. Similarly, the inverse is continuous. $\square$

We may now define what we mean by equivalent simplicial complexes.

Definition 3.8 (isomorphism) Two simplicial complexes K and L are *isomorphic* (or *simplicially homeomorphic*) iff $|K| \approx |L|$. We denote this by $K \cong L$.

3.2 Size of a Simplex

As already mentioned, combinatorial topology derives its power from counting. Now that we have a finite description of a space, we can count easily. So, let's use Figure 2 to count the number of faces of a simplex. For example, an edge has two vertices and an edge as its faces (recall that a simplex is a face of itself.) A tetrahedron has four vertices, six edges, four triangles, and a tetrahedron as faces. These counts are summarized in Table 1. What should the numbers be for a 4-simplex? The numbers in the table may look really familiar to you. If we add a 1 to the left of each row, we get *Pascal's triangle*, as shown in Figure 4. Recall that Pascal's triangle encodes the binomial coefficients: the number of different combinations of l objects out of k objects or $\binom{k}{l}$. Here, we have $k+1$ points representing an k-simplex, any $l+1$ of which defines a l-simplex. To make the relationship complete, we define the empty set $\emptyset$ as the (-1)-*simplex*. This simplex is part of every simplex and allows us to add a column of 1's to the left side of Table 1 to get Pascal's triangle. It also allows us to eliminate the word *non-empty* in the second condition of Definition 3.5, as

k/l	0	1	2	3
0	1	0	0	0
1	2	1	0	0
2	3	3	1	0
3	4	6	4	1
4	?	?	?	?

Table 1. Number of l-simplices in each k-simplex.

$$
\begin{array}{ccccccccccc}
 & & & & & \boxed{\mathbf{1}} & & & & & \\
 & & & & \boxed{\mathbf{1}} & & 1 & & & & \\
 & & & \boxed{\mathbf{1}} & & 2 & & 1 & & & \\
 & & \boxed{\mathbf{1}} & & 3 & & 3 & & 1 & & \\
 & \boxed{\mathbf{1}} & & 4 & & 6 & & 4 & & 1 & \\
\boxed{\mathbf{1}} & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Figure 4. If we add a 1 to the left side of each row in Table 1, we get Pascal's triangle.

the empty set is part of both simplices for non-intersecting simplices. To get the total size of a simplex, we sum each row of Pascal's triangle. A k-simplex has $\binom{k+1}{l+1}$ faces of dimension l and

$$\sum_{l=-1}^{k} \binom{k+1}{l+1} = \sum_{l=0}^{k+1} \binom{k}{l} = 2^{k+1},$$

faces in total, according to the binomial theorem. A simplex, therefore, is a very large object. Mathematicians often do not find it appropriate for "computation", when computation is being done by hand. Simplices are very uniform and simple in structure, however, and therefore provide an ideal computational gadget for computers.

3.3 Abstract Definition

Our discussion on the size of a simplex shows that we can view a simplex as a set along and its power set (the collection of all its subsets. This view of a simplex is attractive because it avoids references to geometry in defining a simplicial complex. It also should give you eerie feelings of déjà vu, as it matches the definition of a topology

Definition 3.9 (abstract simplicial complex) An *abstract simplicial complex* is a set $\mathcal{S}$ of finite sets such that if $A \in S$, so is every subset of A. We say $A \in S$ is an *(abstract) k-simplex* of dimension k if $|A| = k + 1$.

The face and co-face definitions follow as before. Note that the definition automatically allows for $\emptyset$ as a (-1)-simplex. We will often abuse notation and refer to $\mathcal{S}$ as the complex. The abstract definition affirms the notion that topology only cares about how the simplices are connected, and not how they are placed within a space. We now relate this abstract set-theoretic definition to the geometric one by extracting the combinatorial structure of a (geometric) simplicial complex.

Definition 3.10 (vertex scheme) Let K be a simplicial complex with vertices V and let $\mathcal{S}$ be the collection of all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K. The collection $\mathcal{S}$ is called the *vertex scheme* of K.

Clearly, the set K and the the collection $\mathcal{S}$ together form an abstract simplicial complex. To compare abstract simplicial complexes, we need a notion of equivalence.

Definition 3.11 (isomorphism) Let K_1, K_2 be abstract simplicial complexes with vertices V_1, V_2 and subset collections $\mathcal{S}_1, \mathcal{S}_2$, respectively. An *isomorphism* between K_1, K_2 is a bijection $\varphi : V_1 \to V_2$, such that the sets in S_1 and S_2 are the same under the renaming of the vertices by φ and its inverse.

Compare this definition with that of isomorphisms defined for simplicial complexes in Definition 3.8. We may now fully define the relationship between geometric and abstract simplicial complexes.

Theorem 3.2 *Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K. Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes.*

The second statement in the theorem follows directly from Theorem 3.1, since the isomorphism at the abstract level may be extended into a simplicial map to show homeomorphism between the underlying spaces. We may easily show the first statement by constructing K as a subcomplex of the convex hull of the standard vectors in $\mathbb{R}^d$, where d is the number of vertices in K.

Definition 3.12 (Geometric Realization) If an abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of a simplicial complex K, we call K a *geometric realization* of $\mathcal{S}$. A realization is uniquely determined up to an linear isomorphism as defined before.

Having constructed a finite simplicial complex, we may divide it into topological and geometric components. The former will be a abstract simplicial complex, a purely combinatorial object, easily stored and manipulated in a computer system. The latter is a map of the vertices of the complex into the space in which the complex is realized. Again, this map is finite, and can be approximately represented in a computer using a floating point representation.

Example 3.1 (Wavefront OBJ format) This representation of a simplicial complex translates word for word into most common file formats for storing surfaces. One standard format is the OBJ format from *Wavefront*. The format first describes the map which places the vertices in $\mathbb{R}^3$. A vertex with location $(x, y, z) \in \mathbb{R}^3$ gets the line "v x y z" in the file. After specifying the map, the format describes an simplicial complex by only listing its triangles, which are the principal simplices (see Definition 3.5.) The vertices are numbered according to their order in the file and numbered from 1. A triangle with vertices v_1, v_2, v_3 is specified with line "f v_1 v_2 v_3". The description in an OBJ file is often called a "triangle soup", as the topology is specified implicitly and must be extracted.

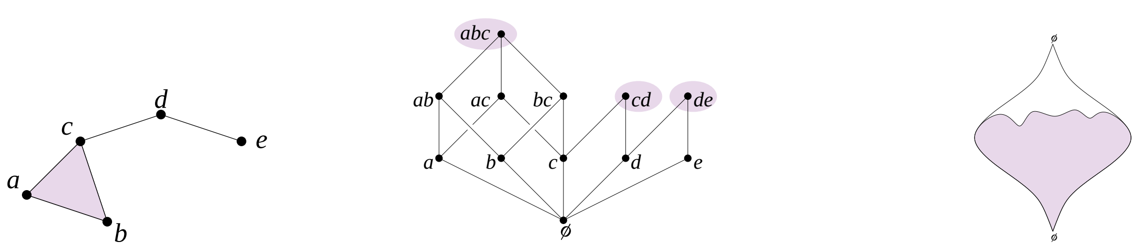
```
v -0.269616 0.228466 0.077226
v -0.358878 0.240631 0.044214
v -0.657287 0.527813 0.497524
v 0.186944 0.256855 0.318011
v -0.074047 0.212217 0.111664
...
f 19670 20463 20464
f 8936 8846 14300
f 4985 12950 15447
f 4985 15447 15448
...
```

Figure 5. Portions of an OBJ file specifying the surface of the Stanford Bunny.

3.4 Subcomplexes

Recall that a simplex is the power set of its simplices. Similarly, a natural view of a simplicial complex is that it is special subset of the power set of all its vertices. The subset is *special* because of the requirements in Definition 3.9. Consider the small complex in Figure 6(a). The diagram 6(b) shows how the simplices connect within the complex: it has a node for each simplex, and an edge indicating a face-coface relationship. The marked principal simplices are the "peaks" of the diagram. This diagram is, in fact, a *poset*.

Definition 3.13 (poset) Let S be a finite set. A *partial order* is a binary relation $\leq$ on S that is reflexive, antisymmetric, and transitive. That is for all $x, y, z \in S$,



(a) A small complex

(b) Poset of the small complex, with principal simplices marked.

(c) An abstract poset: the "water level" of the poset is defined by principal simplices.

Figure 6. Poset view of a simplicial complex

1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ implies $x = y$, and
3. $x \leq y$ and $y \leq z$ implies $x \leq z$.

A set with a partial order is a *partially ordered set* or *poset* for short.

It is clear from the definition that the face relation on simplices is a partial order. Therefore, the set of simplices with the face relation forms a poset. We often abstractly imagine a poset as in Figure 6(c). The set is fat around its waist because the number of possible simplices $\binom{n}{k}$ is maximized for $k \approx n/2$. The principal simplices form a level beneath which all simplices must be included. Therefore, we may recover a simplicial complex by simply storing its principal simplices, as in the case with triangulations in Example 3.1. This view also gives us intuition for extensions of concepts in point set theory to simplicial complexes. A simplicial complex may be viewed as a closed set (it *is* a closed point set, if it is geometrically realized.)

Definition 3.14 (subcomplex, link, star) A *subcomplex* is a simplicial complex $L \subseteq K$. The smallest subcomplex containing a subset $L \subseteq K$ is its closure, $\mathrm{Cl}\, L = \{\tau \in K \mid \tau \leq \sigma \in L\}$. The *star of* L contains all of the cofaces of L, $\mathrm{St}\, L = \{\sigma \in K \mid \sigma \geq \tau \in L\}$. The *link of* L is the boundary of its star, $\mathrm{Lk}\, L = \mathrm{Cl}\,\mathrm{St}\, L - \mathrm{St}\,(\mathrm{Cl}\, L - \{\emptyset\})$.

Figure 7 demonstrates these concepts within the poset for our complex in Figure 6. A subcomplex is the analog of a subset for a simplicial complex. Given a set of simplices, we take all the simplices "below" the set within the poset to get its closure 7(a), and all the simplices "above" the set to get its star 7(b). The face relation is the partial order that defines "above" and "below". Most of the time, the star of a set is an open set (viewed as a point set) and not a simplicial complex. The star corresponds to the notion of a neighborhood for a simplex, and like a neighborhood, it is open. The closure operation completes the boundary of a set as before, making the star a simplicial complex 7(b). The link operation gives us the boundary. In our example, $\mathrm{Cl}\,\{c, e\} - \emptyset = \{c, e\}$, so we remove the simplices from the light regions from those in the dark region in 7(b) to get the link 7(c). Therefore, the link of c and e is the edge ab and the vertex d. Check on Figure 6(a) to see if this matches your intuition of what a boundary should be.

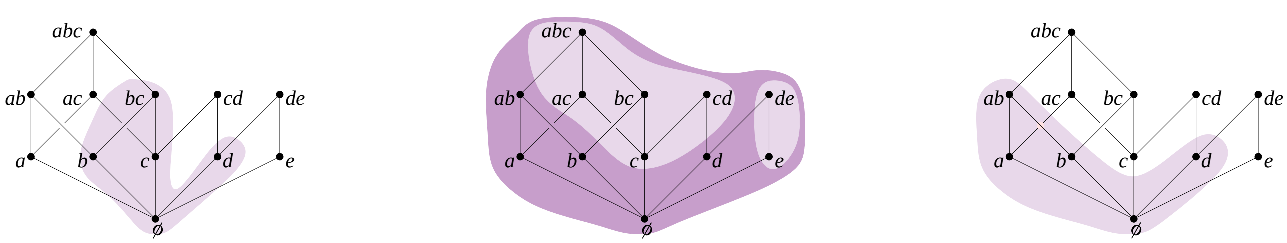
3.5 Triangulations

The primary reason we study simplicial complexes is to represent manifolds.

Definition 3.15 (triangulation) A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.

For example, the boundary of a 3-simplex (tetrahedron) is homeomorphic to a sphere and is a triangulation of the sphere, as shown in Figure 8.

The term "triangulation" is used by different fields with different meanings. For example, in computer graphics, the term most often refers to "triangle soup" descriptions of surfaces. The finite element community often refers to triangle soups as a *mesh*, and may allow other elements, such as quadrangles, as basic building blocks. In areas, three-dimensional meshes composed of tetrahedra are called *tetrahedralizations*. Within topology, a triangulation refers to complexes of *any* dimension, however.



(a) $\mathrm{Cl}\,\{bc, d\}$ (b) $\mathrm{St}\,\{c, e\}$ (light) and its closure $\mathrm{Cl}\,\mathrm{St}\,\{c, e\}$ (dark) (c) $\mathrm{Lk}\,\{c, e\}$

Figure 7. Closure, star, and link of simplices

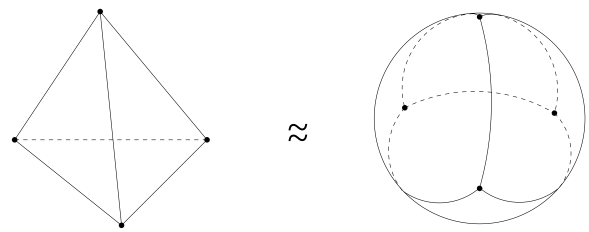


Figure 8. The boundary of a tetrahedron is a triangulation of a sphere, as its underlying space is homeomorphic to the sphere.

3.6 Orientability

We had a definition of orientability in the notes for the first lecture that depended on differentiability. We now extend this definition to simplicial complexes, which are not smooth. This extension further affirms that orientability is a topological property not dependent on smoothness.

Definition 3.16 (orientation) Let K be a simplicial complex. An *orientation* of a k-simplex $\sigma \in K$, $\sigma = \{v_0, v_1, \ldots, v_k\}, v_i \in K$ is an equivalence class of orderings of the vertices of σ, where

$$(v_0, v_1, \ldots, v_k) \sim (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(k)})$$

are equivalent orderings if the parity of the permutation τ is even. We denote an *oriented simplex*, a simplex with an equivalence class of orderings, by $[\sigma]$.

Note that the concept of orientation derives from that fact that permutations may be partitioned into two equivalence classes (if you have forgotten these concepts, you may review *permutations* and *partitions* in the notes from lecture 1 and 2, respectively.) Orientations may be shown graphically using arrows, as shown in Figure 9. We may use oriented simplices to define the concept of orientability to triangulated d-manifolds.

Definition 3.17 (orientability) Two k-simplices sharing a $(k-1)$-face σ are *consistently oriented* if they induce different orientations on σ. A triangulable d-manifold is *orientable* if all d-simplices can be oriented consistently. Otherwise, the d-manifold is *non-orientable*

Last lecture, we saw two basic non-orientable 2-manifolds: the Klein bottle and the projective plane. Our exposition shows that non-orientable manifolds can exist in any dimensions, however.

Example 3.2 (Rendering) The surface of a three-dimensional object is a 2-manifold and may be modeled with a triangulation in a computer. In computer graphics, these triangulations are rendered using light models that assign color to each triangle according to how it is situation with respect to the lights in the scene, and the viewer. To do this, the model needs the normal for each triangle. But each triangle has two normals pointing in opposite directions. To get a correct rendering, we need the normals to be consistently oriented.

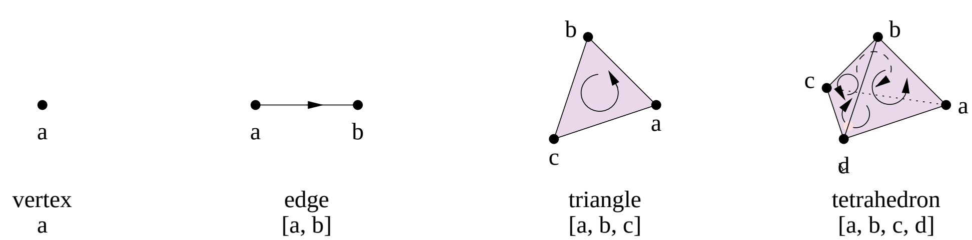


Figure 9. Oriented k-simplices, $0 \leq k \leq 3$. The orientation on the tetrahedron is shown on its faces.

3.7 Euler Characteristic

Having seen orientability for simplicial surfaces, we finish this lecture by looking at our first topological invariant.

Definition 3.18 (invariant) A *(topological) invariant* is a map that assigns the same object to spaces of the same topological type.

Note that an invariant may assign the same object to spaces of different topological type. In other words, an invariant need not be *complete*. All that is required by the definition is that if the spaces have the same type, they are mapped to the same object. Generally, this characteristic of invariants implies their utility in contrapositives: if two spaces are assigned different objects, they have different topological types. On the other hand, if two spaces are assigned the same object, we usually cannot say anything about them. Let us formally state these statements for an invariant f:

$$\begin{array}{rcll} \mathbb{X} \approx \mathbb{Y} & \Longrightarrow & f(\mathbb{X}) = f(\mathbb{Y}) & \\ f(\mathbb{X}) \neq f(\mathbb{Y}) & \Longrightarrow & \mathbb{X} \not\approx \mathbb{Y} & \text{(contrapositive)} \\ f(\mathbb{X}) = f(\mathbb{Y}) & \Longrightarrow & \text{nothing} & \\ \text{If } f(\mathbb{X}) = f(\mathbb{Y}) & \Longrightarrow & \mathbb{X} \approx \mathbb{Y}, & \text{the invariant is complete.} \end{array}$$

The last statement is the converse of the first statement, the definition of an invariant. A good incomplete invariant will have enough differentiating power to be useful through contrapositives. Here, we introduce a famous invariant, the Euler characteristic.

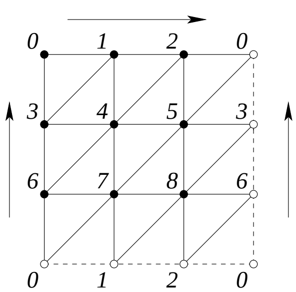
Definition 3.19 (Euler characteristic) Let K be a simplicial complex and $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$. The *Euler characteristic* $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i \tag{1}$$

While it is defined for a simplicial complex, the Euler characteristic is an integer invariant for $|K|$, the underlying space of K. Given *any* triangulation of a space $\mathbb{M}$, we always will get the same integer, which we will call the Euler characteristic of that space $\chi(\mathbb{M})$.

3.8 Algorithm for Classifying 2-Manifolds

Armed with triangulations, orientability, and the Euler characteristic, we return to 2-manifolds to convert our "existential" proof from last lecture to a computational one. We begin with calculating the Euler characteristic for the basic surfaces from the last lecture. We have a triangulation of a sphere $\mathbb{S}^2$ in Figure 8, so $\chi(\mathbb{S}^2) = 4 - 6 + 4 = 2$. To compute the Euler characteristic of the other manifolds, we must build triangulations for them. This is simple, however, by triangulating the diagrams for constructing flat 2-manifolds from the last lecture, as in Figure 10(a). This triangulation gives us $\chi(\mathbb{T}^2) = 9 - 27 + 18 = 0$. We may complete the table in Figure 10(b) in a similar fashion. As $\chi(\mathbb{T}^2) = \chi(\mathbb{K}^2) = 0$, the Euler characteristic by itself is not powerful enough to differentiate between surfaces.



(a) A triangulation for the diagram of the torus $\mathbb{T}^2$

2-Manifold	χ
Sphere $\mathbb{S}^2$	2
Torus $\mathbb{T}^2$	0
Klein bottle $\mathbb{K}^2$	0
Projective plane $\mathbb{R}\mathrm{P}^2$	1

(b) The Euler characteristics of our basic 2-manifolds

Figure 10. A triangulation of the diagram of the torus $\mathbb{T}^2$

Last lecture, we also discussed constructing more complicated surfaces using the connected sum. Suppose we form the connected sum of two surfaces $\mathbb{M}_1, \mathbb{M}_2$ by removing a single triangle from each, and identifying the two boundaries. Clearly, the Euler characteristic should be the sum of the Euler characteristics of the two surfaces, minus 2 for the two missing triangles. In fact, this is true for arbitrary shaped disks.

Theorem 3.3 *For compact surfaces* $\mathbb{M}_1, \mathbb{M}_2$, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

For a compact surface $\mathbb{M}$, let $g\mathbb{M}$ be the connected sum of g copies of $\mathbb{M}$. Combining this theorem with the table in Figure 10(b) we get the following.

Corollary 3.1 $\chi(g\mathbb{T}^2) = 2 - 2g$ *and* $\chi(g\mathbb{R}P^2) = 2 - g$.

These surfaces, along with the sphere, form the equivalence classes of 2-manifolds discussed in the last lecture.

Definition 3.20 (genus) The connected sum of g tori is called a surface with *genus* g.

The genus refers to how many "holes" the multi-donut surface has. We are now ready to give a complete answer to the homeomorphism problem for closed compact 2-manifolds.

Theorem 3.4 (Homeomorphism problem of 2-manifolds) *Closed compact surfaces* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are homeomorphic,* $\mathbb{M}_1 \approx \mathbb{M}_2$ *iff*

1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ *and*
2. *either both surfaces are orientable or both are non-orientable.*

Observe that the theorem is "if and only if". In other words, the Euler characteristic is complete for 2-manifolds. We can easily compute the Euler characteristic of any 2-manifold. Computing orientability is also easy by orienting one triangle and "spreading" the orientation throughout the manifold if it is orientable. Therefore, we have a full computational method for classifying 2-manifolds. As we shall see in the future lectures, the problem is much harder in higher dimensions, forcing us to resort to more elaborate machinery.

Acknowledgments

The material for this lecture is mostly from Munkres [4] and Firby and Gardiner [2], with inspirations from Henle [3] and personal notes. The attributed quote is from Cameron [1]. Thanks to Daniel Russel and Niloy Mitra for proofreading.

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It is by logic we prove, it is by intuition that we invent.
— Henri Poincaré(1854–1912)

6 Homology

In Lecture 3, we learned about a combinatorial method for representing spaces. In Lecture 4, we studied groups and equivalence relations implied by their normal subgroups. In this lecture, we look at a combinatorial and computable functor called *homology* that gives us a finite description of the topology of a space. Homology groups may be regarded as an algebraization of the first layer of geometry in cell structures: how cells of dimension n attach to cells of dimension $n-1$ [1]. Mathematically, the homology groups have a less transparent definition than the fundamental group, and require a lot of machinery to be set up before any calculations. We focus on a weaker form of homology, simplicial homology, that both satisfies our need for a combinatorial functor, and obviates the need for this machinery. Simplicial homology is defined only for simplicial complexes, the spaces we are interested in. Like the Euler characteristic, however, homology is an invariant of the underlying space of the complex. Indeed, the invariance of the Euler characteristic is often derived from the invariance of homology.

Homology groups, unlike the fundamental group, are abelian. In fact, the first homology group is precisely the abelianization of the fundamental group. We pay a price for the generality and computability of homology groups: homology has less differentiating power than homotopy. Once again, however, homology respects homotopy classes, and therefore, classes of homeomorphic spaces.

6.1 Chains and Cycles

To define homology groups, we need simplicial analogs of paths and loops. Let K be a simplicial complex. Recall oriented simplices from Lecture 3. We create the chain group of oriented simplices on the complex.

Definition 6.1 (chain group) The *kth chain group* of a simplicial complex K is $\langle \mathsf{C}_k(K), + \rangle$, the free abelian group on the oriented k-simplices, where $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ have different orientations. An element of $\mathsf{C}_k(K)$ is a *k-chain*, $\sum_q n_q[\sigma_q], n_q \in \mathbb{Z}, \sigma_q \in K$.

We often omit the complex in the notation. A simplicial complex has a chain group in every dimension. As stated earlier, homology examines the connectivity between two immediate dimensions. To do so, we define a structure-relating map between chain groups.

Definition 6.2 (boundary homomorphism) Let K be a simplicial complex and $\sigma \in K$, $\sigma = [v_0, v_1, \ldots, v_k]$. The boundary homomorphism $\partial_k : \mathsf{C}_k(K) \to \mathsf{C}_{k-1}(K)$ is

$$\partial_k \sigma = \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n], \tag{1}$$

where $\hat{v}_i$ indicates that v_i is deleted from the sequence.

It is easy to check that ∂_k is well-defined, that is, ∂_k is the same for every ordering in the same orientation.

Example 6.1 (boundaries) Let us take the boundary of the simplices in Figure 1. .

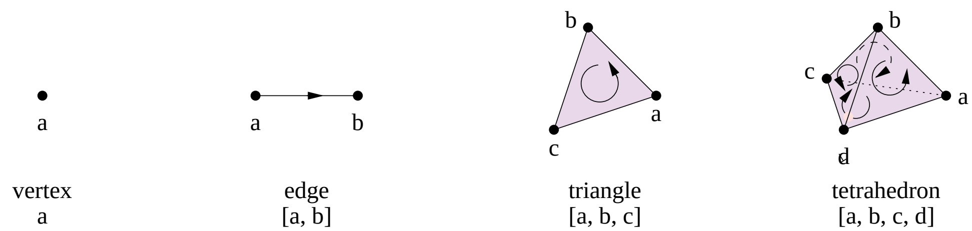


Figure 1. k-simplices, $0 \le k \le 3$. The orientation on the tetrahedron is shown on its faces.

- $\partial_1[a, b] = b - a$.
- $\partial_2[a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b]$.
- $\partial_3[a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c]$.

Note that the boundary operator orients the faces of an oriented simplex. In the case of the triangle, this orientation corresponds to walking around the triangle on the edges, according to the orientation of the triangle.

If we take the boundary of the boundary of the triangle, we get:

$$\partial_1\partial_2[a, b, c] = [c] - [b] - [c] + [a] + [b] - [a] = 0. \tag{2}$$

This is intuitively correct: the boundary of a triangle is a cycle, and a cycle does not have a boundary. In fact, this intuition generalizes to all dimensions.

Theorem 6.1 *$\partial_{k-1}\partial_k = 0$, for all k.*

Proof: The proof is elementary.

$$\begin{aligned}
\partial_{k-1}\partial_k[v_0, v_1, \ldots, v_k] &= \partial_{k-1}\sum_i (-1)^i [v_0, v_1, \ldots, \hat{v_i}, \ldots, v_k] \\
&= \sum_{j<i} (-1)^i(-1)^j [v_0, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_k] \\
&\quad + \sum_{j>i} (-1)^i(-1)^{j-1} [v_0, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_k] \\
&= 0,
\end{aligned}$$

as switching i and j in the second sum negates the first sum. □

The boundary operator connects the chain groups into a *chain complex* C_*:

$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots.$$

with $\partial_k\partial_{k+1} = 0$ for all k. For generality, we often define null boundary operators in dimensions where the domain or codomain of the boundary operator is empty, e.g. $\partial_0 \equiv 0$. A chain complex C_* should be viewed as a single object. Chain complexes are common in homology, but this particular chain complex is one of two we will see in our class.

The boundary operator also allows us to define subgroups of C_k: the group of *cycles* and the group of *boundaries*.

Definition 6.3 (cycle group, boundary group) The *kth cycle group* is

$$\begin{aligned}
\mathsf{Z}_k &= \ker \partial_k \\
&= \{c \in \mathsf{C}_k \mid \partial_k c = \emptyset\}.
\end{aligned}$$

A chain that is an element of Z_k is a *k-cycle*. The *kth boundary group* is

$$\begin{aligned}
\mathsf{B}_k &= \operatorname{im} \partial_{k+1} \\
&= \{c \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1} : c = \partial_{k+1} d\}.
\end{aligned}$$

A chain that is an element of B_k is a *k-boundary*. We also call boundaries *bounding cycles* and cycles not in B_k *non-bounding cycles*.

Both subgroups are normal because our chain groups are abelian. The names match the names we had for loops in the fundamental group, but also extend the notions to other dimensions. Bounding cycles *bound* higher dimensional cycles, as otherwise they would not be in the image of the boundary homomorphism. We can think of them as "filled" cycles, as opposed to "empty" non-bounding cycles. The definitions of the subgroups, along with Theorem 6.1, imply that the subgroups are nested, $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$, as shown in Figure 2.

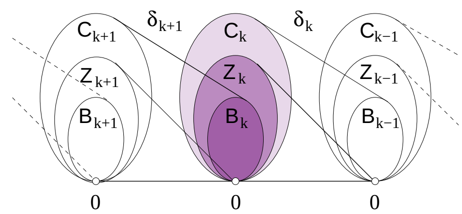


Figure 2. A chain complex with its internals: chain, cycle, and boundary groups, and their images under the boundary operators.

6.2 Simplicial Homology

Chains and cycles are simplicial analogs of the maps called paths and loops in the continuous domain. Following the construction of the fundamental group, we now need a simplicial version of a homotopy to form equivalent classes of cycles. Consider the sum of the non-bounding 1-cycle and a bounding 1-cycle in Figure 3. The two cycles z, b have

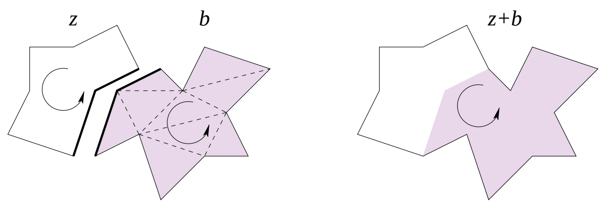


Figure 3. A non-bounding oriented 1-cycle $z \in \mathsf{Z}_k, z \not\in \mathsf{B}_k$ is added to a oriented 1-boundary $b \in \mathsf{B}_k$. The resulting cycle $z + b$ is homotopic to z. The orientation on the cycles is induced by the arrows.

a shared boundary. The edges in the shared boundary appear twice in the sum $z + b$ with opposite signs, so they are eliminated. The resulting cycle $z + b$ is homotopic to z: we may slide the shared portion of the cycles smoothly across the triangles that b bounds. But such homotopies exist for any boundary $b \in \mathsf{B}_1$. Generalizing this argument to all dimensions, we look for equivalent classes of $z + \mathsf{B}_k$ for a k-cycle. But these are precisely the cosets of B_k in Z_k. As B_k is normal in Z_k, the cosets form a group under coset addition.

Definition 6.4 (homology group) The *kth homology group* is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}. \tag{3}$$

If $z_1 = z_2 + \mathsf{B}_k, z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are *homologous* and denote it with $z_1 \sim z_2$.

Homology groups are finitely generated abelian. Therefore, the fundamental theorem of finitely generated abelian groups from Lecture 4 applies. Homology groups describe spaces through their Betti numbers and the torsion subgroups.

Definition 6.5 (*k*th Betti Number) The *kth Betti number* β_k *of a simplicial complex* K is $\beta_k = \beta(\mathsf{H}_k)$, the rank of the free part of H_k.

We can show that $\beta_k = \operatorname{rank} \mathsf{H}_k = \operatorname{rank} \mathsf{Z}_k - \operatorname{rank} \mathsf{B}_k$. The description given by homology is finite, as a n-dimensional simplicial space has at most $n + 1$ nontrivial homology groups.

6.3 Understanding Homology

The description provided by homology groups may not be transparent at first. In this section, we look at a few examples to gain an intuitive understanding of what homology groups capture. Table 1 lists the homology groups of the basic 2-manifolds we first met in Lecture 2. As they are 2-manifolds, the highest non-trivial homology group for any of them is H_2. *Torsion-free* spaces have homology that does not have a torsion subgroup, that is, terms that are finite cyclic groups $\mathbb{Z}_m$. Most of the spaces we are interested are torsion-free. In fact, any space that is a subcomplex of $\mathbb{S}^3$ is torsion-free. We deal with $\mathbb{S}^3$ as it is compact. $\mathbb{R}^3$ is not compact and creates special cases that need to be handled in algorithms. To avoid these difficulties, we add a point at infinity and *compactify* $\mathbb{R}^3$ to get $\mathbb{S}^3$, the three-dimensional

2-manifold	H_0	H_1	H_2
sphere	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$
torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$

Table 1. Homology of basic 2-manifolds.

sphere. This construction mirrors that of the two-dimensional sphere in Lecture 2. Algorithmically, the *one point compactification of* $\mathbb{R}^3$ is easy, as we have a simplicial representation of space.

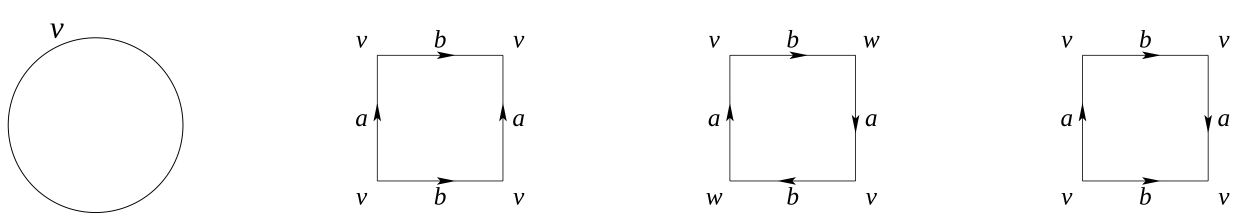
So what does homology capture? For torsion-free spaces in three-dimensions, the Betti numbers (the number of $\mathbb{Z}$ terms in the description) have intuitive meaning as a consequence of the *Alexander Duality*. β_0 measures the number of components of the complex. β_1 is the rank of a basis for the *tunnels*. As H_1 is free, it is a vector-space and β_1 is its rank. β_2 counts the number of *voids* in the complex. Tunnels and voids exist in the complement of the complex in $\mathbb{S}^3$. The distinction might seem tenuous, but this is merely because of our familiarity with the terms. For example, the complex *encloses* a void, and the void is the empty space *enclosed* by the complex.

Using this understanding, we may now examine Table 1. All four spaces have a single component, so $\mathsf{H}_0 = \mathbb{Z}$ and $\beta_0 = 1$. The sphere and the torus enclose a void, so $\mathsf{H}_2 = \mathbb{Z}$ and $\beta_2 = 1$. The non-orientable spaces, on the other hand, are one-sided and cannot enclose any voids, so they have trivial homology in dimension 2. To see what H_1 captures, we look at the diagrams for the 2-manifolds in Figure 4. We may, of course, triangulate these diagrams to obtain abstract simplicial complexes for computing simplicial homology. For now, though, we assume that whatever curve we draw on these manifolds could be "snapped" to some triangulation of the diagrams. To understand 1-cycles and torsion, we need to pay close attention to the boundaries in the diagrams. Recall that a boundary is simply a cycle that *bounds*. In each diagram, we have a boundary, simply, the boundary of the diagram! The manner in which this boundary is labeled determines how the space is connected, and therefore the homology of the space.

It is clear that any simple closed curve drawn on the disk for the sphere is a boundary. Therefore, its homology is trivial in dimension one. The torus has two classes of non-bounding cycles. When we glue the edges marked 'a', edge 'b' becomes a non-bounding 1-cycle and forms a class with all 1-cycles that are homologous to it. We get a different class of cycles when we glue the edges marked 'b'. Each class has a generator, and each generator is free to generate as many *different* classes of homologous 1-cycles as it pleases. Therefore, the homology of a torus in dimension one is $\mathbb{Z} \times \mathbb{Z}$ and $\beta_1 = 2$.

There is a 1-boundary in the diagram, however: the boundary of the disk that we are gluing. Going around this 1-boundary, we get the description $aba^{-1}b^{-1}$. That is, the disk makes the cycle with this description a boundary. Equivalently, the disk adds the relation $aba^{-1}b^{-1} = 1$ to the presentation of the group. But this relation is simply stating that the group is abelian and we already knew that.

Continuing in this manner, we look at the boundary in the diagram for the projective plane. Going around, we get the description $abab$. If we let $c = ab$, the boundary is c^2 and the disk adds the relation $c^2 = 1$ to the group presentation. We need this substitution as an artifact of using this diagram, which we are using for adding some form of uniformity to our treatment. The definition of the cross-cap in Conway's ZIP proof, however, is the one we need here. In other words, we have a cycle c in our manifold that is non-bounding, but becomes bounding when we go around it twice. If we try to generate all the different cycles from this cycle, we just get two classes: the class of cycles homologous to c, and the class of boundaries. But any group with two elements is isomorphic to $\mathbb{Z}_2$, hence the



(a) Sphere $\mathbb{S}^2$ (b) Torus $\mathbb{T}^2$ (c) Projective plane $\mathbb{RP}^2$ (d) Klein bottle $\mathbb{K}^2$

Figure 4. Diagrams for basic 2-manifolds.

description of H_1. You should convince yourself of the verity of the description of H_1 for the Klein bottle in a similar fashion.

6.4 Invariance

Like the Euler characteristic, we define homology using simplicial complexes. From the definition, it seems that homology is capturing extrinsic properties of our representation of a space. We are interested in intrinsic properties of the space, however. We hope that any two different simplicial complexes K and L with homeomorphic underlying spaces $|K| \approx |L|$ have the same homology, the homology of the space itself. Poincaré stated this hope in terms of "the principal conjecture" in 1904.

Conjecture 6.1 (Hauptvermutung) *Any two triangulations of a topological space have a common refinement.*

In other words, the two triangulations can be subdivided until they are the same. This conjecture, like Fermat's last lemma, is deceptively simple. Papakyriakopoulos verified the conjecture for polyhedra of dimension ≤ 2 in 1943 [7], and Moïse proved it for three-dimensional manifolds in 1953 [5]. Unfortunately, the conjecture is false in higher dimensions for general spaces. Milnor obtained a counterexample in 1961 for dimensions six and greater using Lens spaces [4]. Kirby and Siebenmann produced manifold counterexamples in 1969 [2]. The conjecture fails to show the invariance of homology [8].

To settle the question of topological invariance of homology, a more general theory was introduced, that of *singular homology*. This theory is defined using maps on general spaces, thereby eliminating the question of representation. Homology is axiomatized as a sequence of functors with specific properties. Much of the technical machinery required is for proving that singular homology satisfies the axioms of a homology theory, and that simplicial homology is equivalent to singular homology. A result of this theory is the following theorem which states that homology respects homotopy types, and in turn, topological types.

Theorem 6.2 $\mathbb{X} \simeq \mathbb{Y} \Rightarrow \mathsf{H}_*(\mathbb{X}) = \mathsf{H}_*(\mathbb{Y})$

Mathematically speaking, this machinery makes homology less transparent than the fundamental group. Algorithmically, however, simplicial homology is the ideal mechanism to compute topology.

6.5 The Euler-Poincaré Formula

Let's revisit the Euler characteristic now in our new setting. We may redefine the Euler characteristic over a chain complex.

Definition 6.6 (Euler characteristic) $\chi(\mathsf{C}_*) = \sum_i (-1)^i \operatorname{rank}(\mathsf{C}_i)$.

This definition is trivially equivalent to our previous one as the k-simplices are the generators of C_k and $\operatorname{rank}(\mathsf{C}_i) = |\{\sigma \in K \mid \dim \sigma = i\}|$ in that definition. We now denote the sequence of homology functors as H_*. Then, $\mathsf{H}_*(\mathsf{C}_*)$ is a chain complex:

$$\ldots \to \mathsf{H}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{H}_k \xrightarrow{\partial_k} \mathsf{H}_{k-1} \to \ldots.$$

The operators between the homology groups are induced by the boundary operators: we map a homology class to the class of the boundary of one of its members. According to the new definition, the Euler characteristic of our new chain is

$$\chi(\mathsf{H}_*(\mathsf{C}_*)) = \sum_i (-1)^i \operatorname{rank}(\mathsf{H}_i) = \sum_i (-1)^i \beta_i.$$

Surprisingly, the homology functor preserves the Euler characteristic of a chain complex.

Theorem 6.3 (Euler-Poincaré) $\chi(K) = \chi(\mathsf{C}_*) = \chi(\mathsf{H}_*(\mathsf{C}_*))$. *That is,* $\sum_i (-1)^i s_i = \sum_i (-1)^i \beta_i$*, where* $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$ *and* $\beta_i = \operatorname{rank} \mathsf{H}_i$.

The theorem derives the invariance of the Euler characteristic from the invariance of homology.

Example 6.2 We know that the Euler characteristic of a sphere is 2. The Euler-Poincaré relation tells us where this 2 comes from. According to the relation, $\chi(\mathbb{S}^2) = \beta_0 - \beta_1 + \beta_2$. We have $\beta_0 = 1$, as the sphere has one component, $\beta_1 = 0$ as all 1-cycles are contractible, and $\beta_2 = 1$ as the sphere encloses a single void. Similarly, $\chi(\mathbb{T}^2) = 0$, as it has the same Betti numbers as the sphere, except that $\beta_1 = 2$.

Acknowledgments

The actual content of this lecture comes from Munkres [6] and Hatcher [1]. As always, the pedagogical constructions are mine. Massey includes a description of the Alexander Duality [3]. The other citations are referenced within the text.

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