CS164: Computing with Physical Objects Handout # 3
Algorithms for Shape and Motion
Stanford University Wednesday, 7 April 2010

Homework #1: Transformations,parametric curves and surfaces, basic differential geometry [80 points + 5 bonus]
Due Date: Wednesday, 21 April 2010

Doing problems is a very important part of this course. Although you have two weeks to do this assignment, do not delay starting to work on it — several of these problems are not routine exercises. If you cannot solve the problem fully, please write up whatever you can do and document any partial results you have obtained in the process. We intend to be generous with partial credit.

You are encouraged to collaborate in study groups of up to three students on the solution of the homeworks. If you do collaborate on theory problems, you must write up solutions on your own and acknowledge your collaborators by name in the write-up for each problem. If you obtain a solution with outside help (e.g., through literature search, another student not in the class, etc.), acknowledge your source, and write up the solution on your own. For programming problems a single write-up per group is acceptable.

It is very important in this course that every homework be turned in on time. We recognize that occasionally there are circumstances beyond one's control that prevent an assignment from being completed before it is due. You will be allowed two classes of grace during the quarter. This means that you can either hand-in two assignments late by one class, or one assignment late by two classes. Any other assignment handed in late will be penalized by 20% for each class that it is late, unless special arrangements have been made previously with the instructor.

Problem 1. [20 points]

Explore the commutativity of transformations. For two 3D transformations $\mathbf{A}$ and $\mathbf{B}$, we say $\mathbf{A}$ and $\mathbf{B}$ are commutative, or commute, if $\mathbf{AB} = \mathbf{BA}$. Now define the following three 3D transformations:

i. Rotation $\mathbf{R}$ which transforms $+x$ axis to $+y$, $+y$ to $+z$, and $+z$ to $+x$;

ii. Scale $\mathbf{S}$ which scales along x, y and z axes by factors 2, 3 and 4, respectively;

iii. Translation $\mathbf{T}$ which moves along x, y and z axes by 3, 2 and 1 units, respectively.

Questions:

(a) (5 points). Write $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ as 4×4 homogeneous matrices;

(b) (5 points). Compute the compositions $\mathbf{RS}, \mathbf{SR}, \mathbf{ST}, \mathbf{TS}, \mathbf{TR}$ and $\mathbf{R}$. Are $\mathbf{R}$ and $\mathbf{S}$ commutative? How about $\mathbf{S}$ and $\mathbf{T}$, $\mathbf{T}$ and $\mathbf{R}$?

(c) (5 points). Fill in the following table with commutativity of transformations in general. For instance, if any rotation is commutative with any translation, fill row 1 col 3 with Y, otherwise with N.

(d) (5 points). For all N cells in the table, state sufficient and necessary conditions under which the corresponding transformations become commutative.

	Rotation	Scale	Translation
Rotation			
Scale	$\times$		
Translation	$\times$	$\times$	

Problem 2. [10 points]

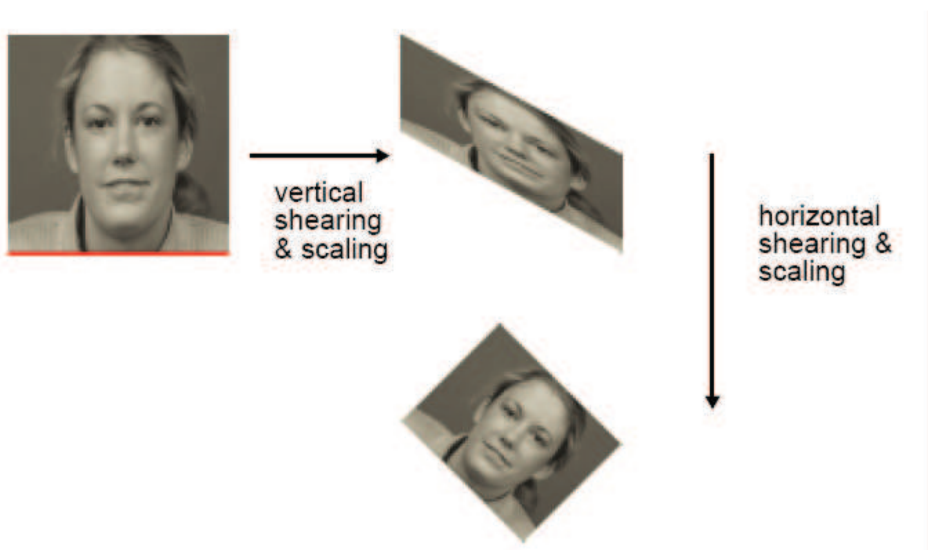

Figure 1: Rotation by shearing.

In this problem we explore the connection between rotation an shear transforms in 2D. This has been exploited in image processing, as shearing maps allow image transforms to be performed in scan-line order. For brevity, we write matrices in the non-homogeneous form, as 2×2 arrays. Let **R** be the general 2-D rotation around the origin by an angle θ:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(a) (5 points). Show that each rotation **R** can be decomposed into the product of three shearing maps (say an x-shear, followed by an y-shear, and then another x-shear).

(b) (5 points). If we consider more general shear-scale transforms (i.e., transforms that combine shearing and scaling in the shearing direction, so that an x-direction shear-scale or y-direction shear-scale look respectively like

$$\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix})$$

then show that only two shear-scale transforms suffice (say an x-scale-shear followed by an y-scale-shear).

The picture above shows visually how this can happen.

Problem 3. [10 points]

Consider the circular quarter arc AD shown below. Our goal is to approximate this circular arc by a cubic Bézier arc defined by control points A, B, C, and D. We say approximate, because the circle requires a rational parametrization and cannot be represented exactly in parametric polynomial form of any degree. Since the entire picture is symmetric around the diagonal $y = x$, symmetry considerations suggest that the middle two Bézier control points B and C should also be symmetric around the $y = x$ line. Where should the points B and C be placed if we want the Bézier arc defined by A, B, C, and D to pass exactly through M, the midpoint of the quarter arc AD, as well as have the correct tangents at A and D? (It turns out that this choice in fact provides the optimal cubic Bézier approximant to the quarter circle, with a relative error of the order of 10^{-4} on the radius.)

Problem 4. [20 points]

This problem is about curvature computation under different parameterizations and representations.

(5 points). Given a 2d curve $p(\theta) = (\cos\theta, 2\sin(\theta))$ where $0 \leq \theta < 2\pi$, compute its curvature at point $p(\frac{\pi}{2}) = (0, 2)$.

(5 points). Given another 2d curve $p(t) = (\frac{1-t^2}{1+t^2}, \frac{4t}{1+t^2})$ where $t \in R$, compute its curvature at point $p(1) = (0, 2)$.

(5 points). Explain why these two curvature values are the same.

(5 points). Derive the curvature formula for a implicit 2d curve defined as $f(x, y) = 0$. We expect the formula to be a function of $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$. Test your formula on $x^2 + \frac{y^2}{4} - 1 = 0$ at point $(0, 2)$.

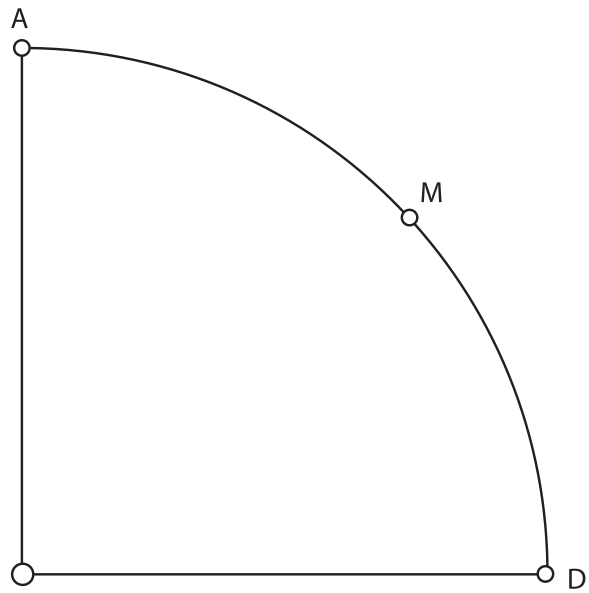

Figure 2: A quarter-circle arc.

Problem 5. [20 points]

(a) (5 points). Consider the curve $c(t) = \left(e^t \sin(t), e^t \cos(t)\right)$.

1) Write down the arc-length parametrization of $c(t)$.
2) Compute the curvature $\kappa(t)$.
3) What can you say about the curvature as $t \to \infty$?

(b) (5 points). Given a planar curve $c(t) = (x(t), y(t))$ describe the effects of the following operations on the curvature $\kappa(t)$ of $c(t)$:

1) Translation
2) Rotation
3) Scaling $x(t) \to rx(t), y(t) \to ry(t)$.

(c) (10 points). Suppose that the curve $c(t)$ is contained in the ball of radius r. In other words $\|c(t)\|_2 \leq r$ for all t. Suppose that $\|c(t_0)\|_2 = r$ for some t_0. Prove that $\kappa(t_0) \geq 1/r$.

Problem 6. [Bonus – 5 points]

The definition of curvature that we gave in class only applies to differentiable curves. However, in practice we often have only an approximation of the curve with a piecewise linear polygonal curve. That is a curve $c(t) = \frac{u_0(t_1-t)+(t-t_0)u_1}{t_1-t_0}$ if $t_0 \leq t <$

t_1, $c(t) = \frac{u_1(t_2-t)+(t-t_1)u_1}{t_2-t_1}$ if $t_1 \leq t < t_2, ...$ Here u_i are points in the plane. Suppose we are given a closed polygonal curve with n points: $u_n = u_0$. Find a definition for curvature that would satisfy the discrete analogue of the total curvature theorem presented in class: $\sum_{i=0}^{n-1} \kappa(u_i) = 2\pi n$ for some integer n. Hint: consider the angles at the vertices of the polygonal curve.