Homework 3: QR and Eigenproblems

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Fall 2013), Stanford University

Due Monday, October 21, at midnight

Problem 1 (30 points). *Despite his best intentions, your instructor occasionally makes mistakes during lecture. The first two parts have solutions close to what you can find in the course notes and are intended to fix these mistakes.*

(a) *Suppose we apply the Cholesky factorization to compute $C = LL^\top$. We can isolate element ℓ_{kk} of L by writing L blockwise as follows:*

$$L = \begin{pmatrix} L_{11} & \vec{0} & 0 \\ \vec{\ell}_k^\top & \ell_{kk} & \vec{0}^\top \\ L_{31} & \vec{\ell}_k' & L_{33} \end{pmatrix}$$

Derive the following two relationships:

$$\ell_{kk} = \sqrt{c_{kk} - \|\vec{\ell}_k\|_2^2}$$
$$L_{11}\vec{\ell}_k = \vec{c}_k$$

where $\vec{c}_k$ contains the elements of C in the same position as $\vec{\ell}_k$. How does this suggest an algorithm for computing the Cholesky factorization?

(b) *Draw a picture to illustrate why $2\text{proj}_{\vec{v}}\vec{b} - \vec{b}$ is the reflection of $\vec{b}$ over $\vec{v}$. Derive a matrix $H_{\vec{v}}$ such that $-H_{\vec{v}}$ performs this reflection (notice the negative sign), and prove that $H_{\vec{v}}$ is orthogonal.*

(c) *Show that for any two vectors $\vec{x}$ and $\vec{y}$ with $\vec{x} \neq \vec{y}$ and $\|\vec{x}\|_2 = \|\vec{y}\|_2$, there exists some $H_{\vec{v}}$ with $H_{\vec{v}}\vec{x} = \vec{y}$.*

Problem 2 (40 points). *Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.*

(a) *Define a matrix $\sqrt{A} \in \mathbb{R}^{n\times n}$ and show that $(\sqrt{A})^2 = A$. Notice that generally speaking $\sqrt{A}$ is not the same as L in the Cholesky factorization $A = LL^\top$.*

(b) *Do most matrices have unique square roots? Why or why not?*

(c) *We can define the* exponential *of A as $e^A \equiv \sum_{k=0}^{\infty} \frac{1}{k!}A^k$; this sum is unconditionally convergent (you do not have to prove this!). Write an alternative expression for e^A in terms of the eigenvectors and eigenvalues of A.*

(d) *If $AB = BA$, show $e^{A+B} = e^A e^B$.*

(e) *Show that the ordinary differential equation* $\vec{y}'(t) = -A\vec{y}$ *with* $\vec{y}(0) = \vec{y}_0$ *for some* $\vec{y}_0 \in \mathbb{R}^n$ *is solved by* $\vec{y}(t) = e^{-At}\vec{y}_0$. *What happens as* $t \to \infty$*?*

Problem 3 (30 points; adapted from CS 205A 2012)**.** *Some QR factorization review:*

(a) *Suppose* $A \in \mathbb{R}^{m\times n}$ *is factored* $A = QR$. *Show that* $P_0 = I_{m\times m} - QQ^\top$ *is the projection matrix onto the null space of* $A^\top$.

(b) *Suppose we consider* $\vec{a} \in \mathbb{R}^n$ *as an* $n \times 1$ *matrix. Write out its "reduced" QR factorization explicitly.*

(c) *Can a matrix* $A \in \mathbb{R}^{m\times n}$ *be factored into* $A = RQ$ *where* R *is upper triangular and* Q *is orthogonal? How?*

(d) *Can a matrix* $A \in \mathbb{R}^{m\times n}$ *be factored into* $A = QL$ *where* L *is lower triangular?*