Homework 7: Interpolation, Integration, and Differentiation

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Fall 2013), Stanford University

Due Monday, December 2, at midnight

This is the last required homework of CS 205A. Due to the Thanksgiving holiday you have an extra week to complete it. We may distribute an optional homework during the last week for you to review the end of the course.

Problem 1 (50 points). *We start by revisiting a midterm problem to help motivate our discussion of differential equations. Throughout this problem, assume $f, g : [0,1] \to \mathbb{R}$ are differentiable functions with $g(0) = g(1) = 0$. We will derive continuous and discrete versions of the* screened Poisson equation, *used for smoothing functions.*

(a) *So far our optimization problems have been to find points $\vec{x}^* \in \mathbb{R}^n$, but sometimes our unknown is an entire function. Thankfully, our "variational" approach still is valid in this case. Explain in a few words what the following energies, which take a function f as input, measure about f:*

   (i) $E_1[f] \equiv \int_0^1 (f(t) - f_0(t))^2\, dt$ *for some given function* $f_0 : [0,1] \to \mathbb{R}$

   (ii) $E_2[f] \equiv \int_0^1 (f'(t))^2\, dt$

(b) *For an energy functional $E[\cdot]$ like the two above, explain how the following expression for $dE(f; g)$ (the Gâteaux derivative of E) can be thought of as the "directional derivative of E at f in the g direction:"*

$$dE(f; g) = \frac{d}{d\varepsilon} E[f + \varepsilon g]\big|_{\varepsilon=0}$$

(c) *Again assuming $g(0) = g(1) = 0$, derive the following formulae:*

   (i) $dE_1(f, g) = \int_0^1 2(f(t) - f_0(t))g(t)\, dt$

   (ii) $dE_2(f, g) = \int_0^1 -2f''(t)g(t)\, dt$ *[Hint: Apply integration by parts to get rid of $g'(t)$.]*

(d) *Suppose we wish to approximate f_0 with a smoother function f. One reasonable model for doing so is to minimize $E[f] \equiv E_1[f] + \alpha E_2[f]$ for some $\alpha > 0$. Using the result of (c), argue informally that an f minimizing this energy should satisfy the differential equation $f(t) - f_0(t) = \alpha f''(t)$ for $t \in (0, 1)$.*

(e) *[open-ended] Now, suppose we discretize f on $[0,1]$ using n evenly-spaced samples $f^1, f^2, \ldots, f^n \in \mathbb{R}$ and f_0 using samples $f_0^1, f_0^2, \ldots, f_0^n$. Devise a discrete analog of $E[f]$ as a quadratic energy in the f^k's. For $k \notin \{1, n\}$, does differentiating E with respect to f_k yield a result analogous to (d)?*

EC. *Explain how the finite elements method (FEM) relates to this construction.*

Problem 2 (15 points). *Many interpolation algorithms we discussed require evaluation of the polynomial*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k.$$

This formula takes $O(k^2)$ time to evaluate. Give an $O(k)$ algorithm for evaluating $f(x)$.

Problem 3 (25 points; adapted from CS 205A 2012). *Suppose $f \in C^1([a,b])$.*

- *(a) Find a quadratic polynomial $g(x)$ such that $f'(a) = g'(a)$, $f'(b) = g'(b)$, and $f({}^{a+b}/_2) = g({}^{a+b}/_2)$. [Hint: Write $g(x)$ as a polynomial in $(x - {}^{a+b}/_2)$.]*
- *(b) Define a quadrature rule for finding $\int_a^b f(x)\,dx$ by integrating the interpolant $g(x)$ on $[a,b]$.*
- *(c) Prove that this scheme has degree-three accuracy. [In fact, it does* not *have degree four accuracy.]*
- *(d) Define the corresponding composite quadrature rule for $\int_a^b f(x)\,dx$ obtained by subdividing $[a,b]$ into n subintervals of equal size.*

Problem 4 (10 points). *Write code to fit a degree $k-1$ polynomial to k evenly-spaced points in $[-1,1]$ sampling $f(x) = |x|$. Plot the resulting polynomials for $k = 3, 5, 7, 9, 11$. What does this plot illustrate about high-degree polynomial interpolation? [Please submit your code via email in a* `.zip` *file.]*