Problem 1 (50 points). We start by revisiting a midterm problem to help motivate our discussion of differential equations. Throughout this problem, assume $f, g : [0, 1] \to \mathbb{R}$ are differentiable functions with $g(0) = g(1) = 0$. We will derive continuous and discrete versions of the screened Poisson equation, used for smoothing functions.

(a) So far our optimization problems have been to find points $\vec{x}^* \in \mathbb{R}^n$, but sometimes our unknown is an entire function. Thankfully, our “variational” approach still is valid in this case. Explain in a few words what the following energies, which take a function $f$ as input, measure about $f$:

(i) $E_1[f] \equiv \int_0^1 (f(t) - f_0(t))^2 \, dt$ for some given function $f_0 : [0, 1] \to \mathbb{R}$

(ii) $E_2[f] \equiv \int_0^1 (f'(t))^2 \, dt$

(b) For an energy functional $E[\cdot]$ like the two above, explain how the following expression for $dE(f; g)$ (the Gateaux derivative of $E$) can be thought of as the “directional derivative of $E$ at $f$ in the $g$ direction:”

$$dE(f; g) = \left. \frac{d}{d\epsilon} E[f + \epsilon g] \right|_{\epsilon = 0}$$

(c) Again assuming $g(0) = g(1) = 0$, derive the following formulae:

(i) $dE_1(f, g) = \int_0^1 2(f(t) - f_0(t))g(t) \, dt$

(ii) $dE_2(f, g) = \int_0^1 -2f''(t)g(t) \, dt$ [Hint: Apply integration by parts to get rid of $g'(t)$.]

(d) Suppose we wish to approximate $f_0$ with a smoother function $f$. One reasonable model for doing so is to minimize $E[f] \equiv E_1[f] + \alpha E_2[f]$ for some $\alpha > 0$. Using the result of (c), argue informally that an $f$ minimizing this energy should satisfy the differential equation $f(t) - f_0(t) = \alpha f''(t)$ for $t \in (0, 1)$.

(e) [open-ended] Now, suppose we discretize $f$ on $[0, 1]$ using $n$ evenly-spaced samples $f_1^1, f_2^2, \ldots, f_n^n \in \mathbb{R}$ and $f_0$ using samples $f_0^1, f_2^2, \ldots, f_0^n$. Devise a discrete analog of $E[f]$ as a quadratic energy in the $f_k$’s. For $k \notin \{1, n\}$, does differentiating $E$ with respect to $f_k$ yield a result analogous to (d)?

EC. Explain how the finite elements method (FEM) relates to this construction.
Problem 2 (15 points). Many interpolation algorithms we discussed require evaluation of the polynomial
\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k. \]
This formula takes \( O(k^2) \) time to evaluate. Give an \( O(k) \) algorithm for evaluating \( f(x) \).

Problem 3 (25 points; adapted from CS 205A 2012). Suppose \( f \in \mathcal{C}^1([a,b]) \).

(a) Find a quadratic polynomial \( g(x) \) such that \( f'(a) = g'(a), f'(b) = g'(b), \) and \( f\left(\frac{a+b}{2}\right) = g\left(\frac{a+b}{2}\right) \).
[Hint: Write \( g(x) \) as a polynomial in \( (x-a+b)/2 \).]

(b) Define a quadrature rule for finding \( \int_a^b f(x) \, dx \) by integrating the interpolant \( g(x) \) on \([a,b]\).

(c) Prove that this scheme has degree-three accuracy. [In fact, it does not have degree four accuracy.]

(d) Define the corresponding composite quadrature rule for \( \int_a^b f(x) \, dx \) obtained by subdividing \([a,b]\) into \( n \) subintervals of equal size.

Problem 4 (10 points). Write code to fit a degree \( k - 1 \) polynomial to \( k \) evenly-spaced points in \([-1,1]\) sampling \( f(x) = |x| \). Plot the resulting polynomials for \( k = 3, 5, 7, 9, 11 \). What does this plot illustrate about high-degree polynomial interpolation? [Please submit your code via email in a .zip file.]