Motivation	Parametric Regression	Least Squares	Cholesky Factorization	Sparsity	Special Structure	Preview
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# Designing and Analyzing Linear Systems

#### CS 205A: Mathematical Methods for Robotics, Vision, and Graphics

Justin Solomon

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#### Preview

# Where Are We?

- Reliable methods for factoring/solving linear systems
- Strategies for understanding when our methods will succeed

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# **Obvious Question**

# So what?

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Least Squares

**Cholesky Factorization** 

$$f(\vec{x}) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$
  
Find  $\{a_1, \dots, a_n\}.$ 

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# n **Experiments**

$$\vec{x}^{(k)} \mapsto y^{(k)} \equiv f(\vec{x}^{(k)})$$

$$y^{(1)} = f(\vec{x}^{(1)}) = a_1 x_1^{(1)} + a_2 x_2^{(1)} + \dots + a_n x_n^{(1)}$$
$$y^{(2)} = f(\vec{x}^{(2)}) = a_1 x_1^{(2)} + a_2 x_2^{(2)} + \dots + a_n x_n^{(2)}$$

#### **Note:** The *x*'s may have different units

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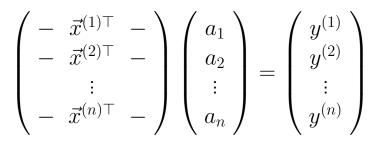
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# Linear System for $\vec{a}$

**Cholesky Factorization** 

Least Squares



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### **General Case**

$$f(\vec{x}) = a_1 f_1(\vec{x}) + a_2 f_2(\vec{x}) + \dots + a_m f_m(\vec{x})$$

$$\begin{pmatrix} f_1(\vec{x}^{(1)}) & f_2(\vec{x}^{(1)}) & \cdots & f_m(\vec{x}^{(1)}) \\ f_1(\vec{x}^{(2)}) & f_2(\vec{x}^{(2)}) & \cdots & f_m(\vec{x}^{(2)}) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(\vec{x}^{(m)}) & f_2(\vec{x}^{(m)}) & \cdots & f_m(\vec{x}^{(m)}) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{pmatrix}$$
  
*f* can be *nonlinear*!

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# **Two Important Cases**

Least Squares

**Cholesky Factorization** 

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$
  
"Vandermonde system"

$$f(x) = a\cos(x + \phi)$$
  
Mini-Fourier

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# Something Fishy

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# Why should you have to do exactly *n* experiments?

# What if $y^{(k)}$ is measured with error?

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## Overfitting

#### Overfitting

# Representing noise rather than the underlying relationships in a statistical dataset

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# **Interpretation of Linear Systems**

**Cholesky Factorization** 

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$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} - \vec{r}_1^\top & - \\ - \vec{r}_2^\top & - \\ \vdots & \cdots & \vdots \\ - \vec{r}_n^\top & - \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_n \cdot \vec{x} \end{pmatrix}$$

"Guess  $\vec{x}$  by observing its dot products with  $\vec{r_i}$ 's."

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Parametric Regression



# The observations are likely to be incompatible.

 $A\vec{x} \approx \vec{b}$ 

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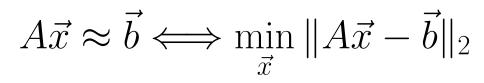
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#### Least Squares



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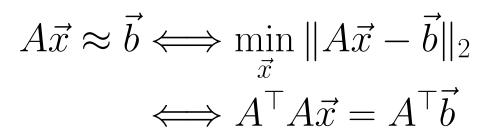
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### Least Squares



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# **Normal Equations**

# $A^{\top}A\vec{x} = A^{\top}\vec{b} \xleftarrow{?} \vec{x} = (A^{\top}A)^{-1}A^{\top}\vec{b}$

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# Regularization

Tikhonov regularization ("Ridge Regression;" Gaussian prior):  $\min \|A\vec{x} - \vec{b}\|_2^2 + \alpha \|\vec{x}\|_2^2$ Lasso (Laplace prior):  $\min \|A\vec{x} - \vec{b}\|_2^2 + \alpha \|\vec{x}\|_1$ Flastic Net:  $\min_{\vec{x}} \|A\vec{x} - \vec{b}\|_2^2 + \alpha \|\vec{x}\|_2^2 + \beta \|\vec{x}\|_1$ 



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# Regularization

Tikhonov regularization ("Ridge Regression;" Gaussian prior):  $\min_{\vec{x}} \|A\vec{x} - \vec{b}\|_2^2 + \alpha \|\vec{x}\|_2^2$ 

Least Squares "In the Wild"

Least Squares

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# Example: Image alignment

 $\vec{y}_k \approx A\vec{x}_k + b$  $A \in \mathbb{R}^{2 \times 2}$  $\vec{h} \in \mathbb{R}^2$ 

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# **A Ridiculously Important Matrix**

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# **Symmetric** *B* is *symmetric* if $B^{\top} = B$ .

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# **Symmetric** B is symmetric if $B^{\top} = B$ .

## Positive (Semi-)Definite

B is positive semidefinite if for all  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x}^{\top}B\vec{x} \geq 0$ . B is positive definite if  $\vec{x}^{\top}B\vec{x} > 0$ whenever  $\vec{x} \neq \vec{0}$ .

Motivation



# **Pivoting for SPD** C

# **Goal:** Solve $C\vec{x} = \vec{d}$ for symmetric positive definite C.

$$C = \begin{pmatrix} c_{11} \ \vec{v}^{\top} \\ \vec{v} \ \tilde{C} \end{pmatrix}$$

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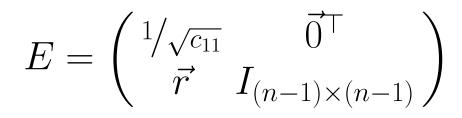
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# **Forward Substitution**





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# Symmetry Experiment

# Try post-multiplication:



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# • Positive definite $\implies$ existence of $\sqrt{c_{11}}$

• Symmetry  $\implies$  apply E to both sides

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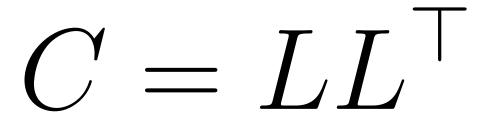
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# **Cholesky Factorization**



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# **Observation about Cholesky**

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# **Observation about Cholesky**

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$$\ell_{kk} = \sqrt{c_{kk} - \|\ell_k\|_2^2}$$

$$L_{11}\vec{\ell_k} = \vec{c}_k$$

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# Interpretation of Cholesky

# <philosophy>

# What is $\vec{x}^{\top}C\vec{x}$ ?

# </philosophy>

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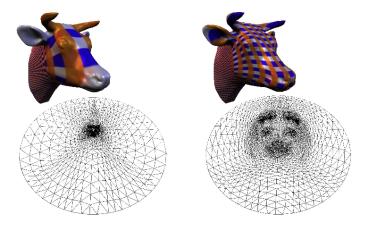
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# Harmonic Parameterization



http://www.mpi-inf.mpg.de/departments/d4/areas/meshproc/rz\_Kowz.jpg

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# **Storing Sparse Matrices**

**Cholesky Factorization** 

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Want O(n) storage if we have O(n) nonzeros!

Examples:

Parametric Regression

Motivation

List of triplets (r,c,val)

Least Squares

For each row r, matrix[r] holds a dictionary c→A[r][c]

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Common strategy: Permute rows/columns

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Mostly heuristic constructions
 Minimizing fill in Cholesky is NP-complete!

 Alternative strategy: Avoid Gaussian elimination altogether Lots more in a few weeks!

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### **Banded Matrices**

$$\left(\begin{array}{cccc}
\times & \times & & \\
\times & \times & \times & \\
& \times & \times & \times & \\
& & & \times & \times & \\
& & & & \times & \times & \\
& & & & & \times & \times & \\
\end{array}\right)$$

- Storage?
- Solving?

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# **Cyclic Matrices**

$$\left(\begin{array}{cccc}a&b&c&d\\d&a&b&c\\c&d&a&b\\b&c&d&a\end{array}\right)$$

- Storage?
- Solving?



#### **Preview**

cond 
$$A^{\top}A = ||A^{\top}A|| ||(A^{\top}A)^{-1}||$$
  
 $\approx ||A^{\top}|| ||A|| ||A^{-1}|| ||(A^{\top})^{-1}||$   
 $= ||A||^2 ||A^{-1}||^2$   
 $= (\text{cond } A)^2$   
 $\Im$ 

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