Eigenproblems I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate
Setup

**Given:** Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

**Find:** Correlations between different dimensions
Simplest Model

One-dimensional subspace

\[ \vec{x}_i \approx c_i \hat{v} \]
Simplest Model

One-dimensional subspace

\[ \vec{x}_i \approx c_i \vec{v} \]

Equivalently:

\[ \vec{x}_i \approx c_i \hat{\vec{v}} \]
Review

What is $c_i$?
Review

What is $c_i$?

$$c_i = \vec{x}_i \cdot \hat{U}$$
Variational Idea

\[ \text{minimize } \sum_i \| \vec{x}_i - \text{proj}_{\hat{v}} \vec{x}_i \|^2 \]

such that \( \| \hat{v} \| = 1 \)
Equivalent Optimization

\[
\begin{align*}
\text{maximize} & \quad ||X^T \hat{v}||^2 \\
\text{such that} & \quad ||\hat{v}||^2 = 1
\end{align*}
\]
End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.
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Eigenvector of $XX^\top$ with largest eigenvalue.

“First principal component”
Physics (in one slide)

Newton:

\[ \vec{F} = m \frac{d^2 \vec{x}}{dt^2} \]
Physics (in one slide)

Newton:

$$\vec{F} = m \frac{d^2 \vec{x}}{dt^2}$$

Hooke:

$$\vec{F}_s = k(\vec{x} - \vec{y})$$
First-Order System

\[
\frac{d}{dt} \begin{pmatrix} \vec{X} \\
\vec{V} \end{pmatrix} = \begin{pmatrix} 0 & I \\
M^{-1}K & 0 \end{pmatrix} \begin{pmatrix} \vec{X} \\
\vec{V} \end{pmatrix}
\]
General ODE

\[ \dot{X} = AX \]
Eigenvector Solution

\[ \vec{x}' = A\vec{x} \]

\[ A\vec{x}_i = \lambda_i \vec{x}_i \]

\[ \vec{x}(0) = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]
Eigenvector Solution

\[ \ddot{\vec{x}} = A \vec{x} \]

\[ A \vec{x}_i = \lambda_i \vec{x}_i \]

\[ \vec{x}(0) = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ \vec{x}(t) = c_1 e^{\lambda_1 t} \vec{x}_1 + \cdots + c_k e^{\lambda_k t} \vec{x}_k \]
Aside: Matrix Inverse

\[ \vec{b} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ A \vec{x} = \vec{b} \]
Aside: Matrix Inverse

\[ \vec{b} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ A \vec{x} = \vec{b} \]

\[ \implies \vec{x} = \frac{c_1}{\lambda_1} \vec{x}_1 + \cdots + \frac{c_n}{\lambda_n} \vec{x}_n \]
Setup

**Have:** $n$ items in a dataset

$w_{ij} \geq 0$ similarity of items $i$ and $j$

$w_{ij} = w_{ji}$

**Want:** $x_i$ embedding on $\mathbb{R}$
**Quadratic Energy**

\[
E(\vec{x}) = \sum_{ij} w_{ij} (x_i - x_j)^2
\]
Optimization

minimize $E(\vec{x})$
Optimization

minimize $E(\vec{x})$

such that $\|\vec{x}\|^2 = 1$
Optimization

minimize $E(\vec{x})$

such that $\|\vec{x}\|^2 = 1$

$\vec{1} \cdot \vec{x} = 0$
Simplification

\[ E(\vec{x}) = \vec{x}^\top (2A - 2W)\vec{x} \]
Second smallest eigenvector of $2A - 2W$. 
Definitions

Eigenvalue and eigenvector

An eigenvector \( \vec{x} \neq \vec{0} \) of a matrix \( A \in \mathbb{R}^{n \times n} \) is any vector satisfying \( A\vec{x} = \lambda \vec{x} \) for some \( \lambda \in \mathbb{R} \); the corresponding \( \lambda \) is known as an eigenvalue. Complex eigenvalues and eigenvectors satisfy the same relationships with \( \lambda \in \mathbb{C} \) and \( \vec{x} \in \mathbb{C}^n \).
Definitions

Eigenvalue and eigenvector

An eigenvalue $\vec{x} \neq \vec{0}$ of a matrix $A \in \mathbb{R}^{n \times n}$ is any vector satisfying $A\vec{x} = \lambda \vec{x}$ for some $\lambda \in \mathbb{R}$; the corresponding $\lambda$ is known as an eigenvalue. Complex eigenvalues and eigenvectors satisfy the same relationships with $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

Scale doesn’t matter!

$\rightarrow \|\vec{x}\| \equiv 1$
Definitions

**Spectrum and spectral radius**

The *spectrum* of $A$ is the set of eigenvalues of $A$. The *spectral radius* $\rho(A)$ is the eigenvalue $\lambda$ maximizing $|\lambda|$. 
Eigenproblems in the Wild

- ODE/PDE problems
- Minimize/maximize $\| A\vec{x} \|$ such that $\| \vec{x} \| = 1$
- Rayleigh quotient:

$$\frac{\vec{x}^\top A\vec{x}}{\| \vec{x} \|^2}$$
Two Basic Properties

Proved in notes

**Lemma**

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.
Two Basic Properties

*Proved in notes*

**Lemma**

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

**Lemma**

Eigenvectors corresponding to distinct eigenvalues must be linearly independent.
Two Basic Properties

Proved in notes

Lemma
Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

Lemma
Eigenvectors corresponding to distinct eigenvalues must be linearly independent.

$\rightarrow$ at most $n$ eigenvalues
Nondefective

\( A \in \mathbb{R}^{n \times n} \) is nondefective or diagonalizable if its eigenvectors span \( \mathbb{R}^n \).
Diagonalizability

**Nondefective**

A $\in \mathbb{R}^{n \times n}$ is *nondefective* or *diagonalizable* if its eigenvectors span $\mathbb{R}^n$.

\[
D = X^{-1}AX
\]
Extending to $\mathbb{C}^{n \times n}$

Complex conjugate

The complex conjugate of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$. 
Extending to $\mathbb{C}^{n \times n}$

**Complex conjugate**

The *complex conjugate* of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

**Conjugate transpose**

The *conjugate transpose* of $A \in \mathbb{C}^{m \times n}$ is $A^H \equiv \bar{A}^\top$. 
Hermitian Matrix

\[ A = A^H \]
Properties

Lemma

All eigenvalues of Hermitian matrices are real.
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All eigenvalues of Hermitian matrices are real.

Lemma
Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.
Spectral Theorem

Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian (if $A \in \mathbb{R}^{n \times n}$, suppose it is symmetric). Then, $A$ has exactly $n$ orthonormal eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$. 
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Full set: $D = X^\top AX$