

Nonlinear Systems

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Part III: Nonlinear Problems

Not all numerical problems
can be solved with `\` in
Matlab.

Question

Have we already seen a
nonlinear problem?

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minimize $\|A\vec{x}\|_2$
such that $\|\vec{x}\|_2 = 1 \leftarrow$ nonlinear!

Root-Finding Problem

Given: $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Find: \vec{x}^* with $f(\vec{x}^*) = \vec{0}$

Issue: Regularizing Assumptions

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$$f(x) = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

Typical Regularizing Assumptions

Continuous

$$f(\vec{x}) \rightarrow f(\vec{y}) \text{ as } \vec{x} \rightarrow \vec{y}$$

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$$C^k$$

k derivatives exist and are continuous

Today

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Property of Continuous Functions

Intermediate Value Theorem

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose $f(x) < u < f(y)$. Then, there exists z between x and y such that $f(z) = u$.

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Used in Homework 1!

Reasonable Starting Point

$$f(\ell) \cdot f(r) < 0$$

Bisection Algorithm

1. Compute $c = \ell + r/2$.
2. If $f(c) = 0$, return $x^* = c$.
3. If $f(\ell) \cdot f(c) < 0$, take $r \leftarrow c$. Otherwise take $\ell \leftarrow c$.
4. Return to step ?? until $|r - \ell| < \varepsilon$; then return c .

Two Important Questions

1. Does it converge?

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Yes! Unconditionally.

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Yes! Unconditionally.

2. How quickly?

Convergence Analysis

Examine E_k with
 $|x_k - x^*| < E_k.$

Bisection: Linear Convergence

$$E_{k+1} \leq \frac{1}{2}E_k$$

for $E_k \equiv |r_k - \ell_k|$

Fixed Points

$$g(x^*) = x^*$$

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Question:
Same as root-finding?

Simple Strategy

$$x_{k+1} = g(x_k)$$

Convergence Criterion

$$\begin{aligned} E_k &\equiv |x_k - x^*| \\ &= |g(x_{k-1}) - g(x^*)| \end{aligned}$$

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$$\begin{aligned} E_k &\equiv |x_k - x^*| \\ &= |g(x_{k-1}) - g(x^*)| \\ &\leq C|x_{k-1} - x^*| \\ &\quad \text{if } g \text{ is Lipschitz} \\ &= CE_{k-1} \end{aligned}$$

Alternative Criterion

Lipschitz *near* x^* with good starting point.

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e.g. C^1 with $|g'(x^*)| < 1$

Convergence Rate of Fixed Point

When it converges...
Always linear (why?)

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Often quadratic!

Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

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Fixed point iteration on

$$g(x) \equiv x - \frac{f(x)}{f'(x)}$$

Convergence of Newton

Simple Root

A root x^* with $f'(x^*) \neq 0$.

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Quadratic convergence in this case!

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Quadratic convergence in this case!

Higher-order approximations?

Issue

Differentiation is hard!

Secant Method

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

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Trivia:

Converges at rate $\frac{1+\sqrt{5}}{2} \approx 1.6180339887\dots$

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Trivia:

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("Golden Ratio")

Hybrid Methods

Want: Convergence rate of secant/Newton with convergence guarantees of bisection

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e.g. **Dekker's Method:** Take secant step if it is in the bracket, bisection step otherwise

Single-Variable Conclusion

- ▶ Unlikely to solve exactly, so we settle for iterative methods
- ▶ Must check that method converges at all
- ▶ Convergence rates:
 - ▶ Linear: $E_{k+1} \leq CE_k$ for some $0 \leq C < 1$
 - ▶ Superlinear: $E_{k+1} \leq CE_k^r$ for some $r > 1$
 - ▶ Quadratic: $r = 2$
 - ▶ Cubic: $r = 3$
- ▶ Time *per* iteration also important