Introduction	Initial Value Problems	Theory	Model Equations	Simple Integration
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Ordinary Differential Equations I

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics

Justin Solomon

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Theme of Last Three Weeks

The unknown is an entire function f.

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New Twist

So far: f (or its derivative/integral) known at isolated points

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New Twist

So far: f (or its derivative/integral) known at isolated points

Instead: Optimize *properties* of f

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Example Problems

Approximate f₀ with f but make it smoother (or sharper!)

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Example Problems

- Approximate f₀ with f but make it smoother (or sharper!)
- Simulate a particle system obeying a physical law

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Example Problems

- Approximate f₀ with f but make it smoother (or sharper!)
- Simulate a particle system obeying a physical law
- ► Approximate f₀ with f but transfer properties of g₀

Initial Value Problems

Introduction

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Simple Integration

Today: Initial Value Problems

Find $f(t) : \mathbb{R} \to \mathbb{R}^n$ Satisfying $F[t, f(t), f'(t), f''(t), \dots, f^{(k)}(t)] = 0$ Given $f(0), f'(0), f''(0), \dots, f^{(k-1)}(0)$

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Initial Value Problems

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Simple Integration

Most Famous Example

F = ma

Newton's second law

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Initial Value Problems

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Most Famous Example

F = ma

Newton's second law n particles \implies simulation in \mathbb{R}^{3n}

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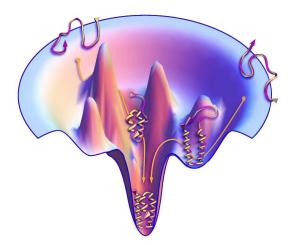
Initial Value Problems

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Protein Folding



http://www.sciencedaily.com/releases/2012/11/121122152910.htm

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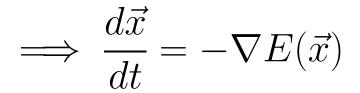
Initial Value Problems

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Gradient Descent

 $\min_{\vec{x}} E(\vec{x})$



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Crowd Simulation



http://video.wired.com/watch/building-a-better-zombie-wwz-exclusive

http://gamma.cs.unc.edu/DenseCrowds/

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Initial Value Problems

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Simple Integration

Examples of ODEs

• $y' = 1 + \cos t$: solved by integrating both sides

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Initial Value Problems

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Examples of ODEs

y' = 1 + cos t: solved by integrating both sides
y' = ay: linear in y

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Initial Value Problems

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Examples of ODEs

- y' = 1 + cos t: solved by integrating both sides
 y' = ay: linear in y
- $y' = ay + e^t$: time and position-dependent

Initial Value Problems

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Simple Integration

Examples of ODEs

- y' = 1 + cos t: solved by integrating both sides
 y' = ay: linear in y
- $y' = ay + e^t$: time and position-dependent
- y'' + 3y' y = t: multiple derivatives of y

Initial Value Problems

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Examples of ODEs

- y' = 1 + cos t: solved by integrating both sides
 y' = ay: linear in y
- $y' = ay + e^t$: time and position-dependent
- y'' + 3y' y = t: multiple derivatives of y
- $y'' \sin y = e^{ty'}$: nonlinear in y and t.

Initial Value Problems

Theory

Model Equations

Simple Integration

Reasonable Assumption

Explicit ODE An ODE is *explicit* if can be written in the form $f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \dots, f^{(k-1)}(t)].$

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Reasonable Assumption

Explicit ODE An ODE is *explicit* if can be written in the form $f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \dots, f^{(k-1)}(t)].$

Otherwise need to do root-finding!

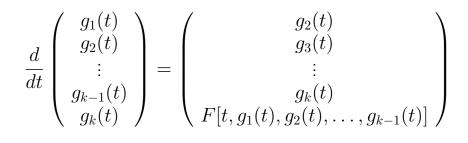
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Initial Value Problems

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Reduction to First Order



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Example

$$y''' = 3y'' - 2y' + y$$

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Example

$$y''' = 3y'' - 2y' + y$$

$$\frac{d}{dt} \begin{pmatrix} y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix}$$

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Initial Value Problems

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Simple Integration

Autonomous ODE

$$\vec{y'} = F[\vec{y}]$$

No dependence of ${\cal F}$ on t

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Initial Value Problems

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Simple Integration

Autonomous ODE

$$\vec{y'} = F[\vec{y}]$$

No dependence of ${\cal F}$ on t

$$g'(t) = \begin{pmatrix} f'(t) \\ \bar{g}'(t) \end{pmatrix} = \begin{pmatrix} F[f(t), \bar{g}(t)] \\ 1 \end{pmatrix}$$

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Two Visualizations

Slope fieldPhase space

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Simple Integration

Existence and Uniqueness

 $\frac{dy}{dt} = 2y/t$

Two cases: y(0) = 0, $y(0) \neq 0$

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Existence and Uniqueness

Theorem: Local existence and uniqueness

Suppose F is continuous and Lipschitz, that is, $\|F[\vec{y}] - F[\vec{x}]\|_2 \le L \|\vec{y} - \vec{x}\|_2$ for some L. Then, the ODE f'(t) = F[f(t)] admits exactly one solution for all $t \ge 0$ regardless of initial conditions.

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Initial Value Problems

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Linearization of 1D ODEs

y' = F[y]

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Simple Integration

Linearization of 1D ODEs

y' = F[y]

$\longrightarrow y' = ay + b$

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Linearization of 1D ODEs

$$y' = F[y]$$

$\longrightarrow y' = ay + b$

$$\longrightarrow \bar{y}' = a\bar{y}$$

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Model Equation

y' = ay

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Initial Value Problems

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Simple Integration

Model Equation

y' = ay

 $\implies y(t) = Ce^{at}$

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Three Cases

$$y' = ay, y(t) = Ce^{at}$$

1. a = 0: Stable

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Initial Value Problems

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Simple Integration

Three Cases

$$y' = ay, y(t) = Ce^{at}$$

a = 0: Stable a < 0: Stable; solutions get closer

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Initial Value Problems

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Simple Integration

Three Cases

$$y' = ay, y(t) = Ce^{at}$$

1. a = 0: Stable

- **2.** a < 0: Stable; solutions get closer
- **3.** *a* > 0: Unstable; mistakes in initial data amplified

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Initial Value Problems

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Simple Integration

Multidimensional Case

$$\vec{y'} = A\vec{y}, A\vec{y_i} = \lambda_i \vec{y_i}$$
$$\vec{y}(0) = \sum_i c_i \vec{y_i}$$

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Initial Value Problems

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Simple Integration

Multidimensional Case

$$\vec{y}' = A\vec{y}, A\vec{y}_i = \lambda_i \vec{y}_i$$
$$\vec{y}(0) = \sum_i c_i \vec{y}_i$$
$$\implies \vec{y}(t) = \sum_i c_i e^{\lambda_i t} \vec{y}_i$$

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Simple Integration

Multidimensional Case

$$\vec{y}' = A\vec{y}, A\vec{y}_i = \lambda_i \vec{y}_i$$
$$\vec{y}(0) = \sum_i c_i \vec{y}_i$$
$$\implies \vec{y}(t) = \sum_i c_i e^{\lambda_i t} \vec{y}_i$$

Stability depends on $\max_i |\lambda_i|$.

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Simple Integration

Integration Strategies

Given \vec{y}_k at time t_k , generate \vec{y}_{k+1} assuming $\vec{y'} = F[\vec{y}]$.

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Initial Value Problems

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Simple Integration

Forward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_k]$$

- Explicit method
- $O(h^2)$ localized truncation error
- O(h) global truncation error;
 "first order accurate"

Initial Value Problems

Theory 00000000 Model Equations

Simple Integration

Model Equation

$$y' = ay \longrightarrow y_{k+1} = (1 + ah)y_k$$

For $a < 0$, stable when $h < \frac{2}{|a|}$.

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Initial Value Problems

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Simple Integration

Backward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_{k+1}]$$

- Implicit method
- $O(h^2)$ localized truncation error
- O(h) global truncation error;
 "first order accurate"

Initial Value Problems

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Simple Integration

Model Equation

$$y' = ay \longrightarrow y_{k+1} = \frac{1}{1 - ah}y_k$$

Unconditionally stable!

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Simple Integration

Model Equation

$$y' = ay \longrightarrow y_{k+1} = \frac{1}{1 - ah}y_k$$

Unconditionally stable! But this has nothing to do with accuracy.

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Model Equation

$$y' = ay \longrightarrow y_{k+1} = \frac{1}{1 - ah}y_k$$

Unconditionally stable! But this has nothing to do with accuracy.

Good for *stiff* equations.

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Initial Value Problems

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Simple Integration

Forward and Backward Euler on Linear ODE

$$\vec{y'} = A\vec{y}$$

Forward Euler: ÿ_{k+1} = (I + hA)ÿ_k
 Backward Euler: ÿ_{k+1} = (I − hA)⁻¹ÿ_k

▶ Next

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