# Ordinary Differential Equations I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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# Theme of Last Three Weeks

The unknown is an entire function $f$.

# New Twist

**So far:**

$f$ (or its derivative/integral) known at isolated points

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**Instead:**

Optimize *properties* of $f$

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- Simulate a particle system obeying a physical law
- Approximate $f_0$ with $f$ but transfer properties of $g_0$

# Today: Initial Value Problems

$$\text{Find } f(t):\mathbb{R}\to\mathbb{R}^n$$
$$\text{Satisfying } F[t, f(t), f'(t), f''(t), \ldots, f^{(k)}(t)] = 0$$
$$\text{Given } f(0), f'(0), f''(0), \ldots, f^{(k-1)}(0)$$

# Most Famous Example

$$F = ma$$

*Newton's second law*

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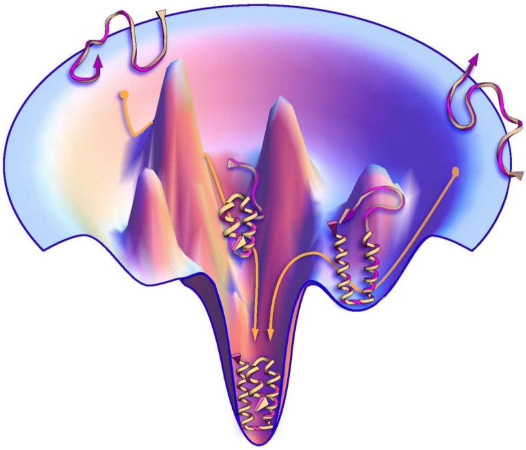
$n$ particles $\Longrightarrow$ simulation in $\mathbb{R}^{3n}$

# Protein Folding



http://www.sciencedaily.com/releases/2012/11/121122152910.htm

# Gradient Descent

$$\min_{\vec{x}} E(\vec{x})$$

$$\implies \frac{d\vec{x}}{dt} = -\nabla E(\vec{x})$$

# Crowd Simulation


http://video.wired.com/watch/building-a-better-zombie-wwz-exclusive

http://gamma.cs.unc.edu/DenseCrowds/

# Examples of ODEs

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- $y' = ay$: linear in $y$
- $y' = ay + e^t$: time and position-dependent
- $y'' + 3y' - y = t$: multiple derivatives of $y$
- $y'' \sin y = e^{ty'}$: nonlinear in $y$ and $t$.

# Reasonable Assumption

## Explicit ODE

An ODE is *explicit* if can be written in the form

$$f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \ldots, f^{(k-1)}(t)].$$

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*Otherwise need to do root-finding!*

# Reduction to First Order

$$\frac{d}{dt}\begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_{k-1}(t) \\ g_k(t) \end{pmatrix} = \begin{pmatrix} g_2(t) \\ g_3(t) \\ \vdots \\ g_k(t) \\ F[t, g_1(t), g_2(t), \ldots, g_{k-1}(t)] \end{pmatrix}$$

# Example

$$y''' = 3y'' - 2y' + y$$

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$$\frac{d}{dt}\begin{pmatrix} y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \\ w \end{pmatrix}$$

# Autonomous ODE

$$\vec{y}' = F[\vec{y}]$$

No dependence of $F$ on $t$

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$$g'(t) = \begin{pmatrix} f'(t) \\ \bar{g}'(t) \end{pmatrix} = \begin{pmatrix} F[f(t), \bar{g}(t)] \\ 1 \end{pmatrix}$$

# Two Visualizations

- Slope field
- Phase space

# Existence and Uniqueness

$$\frac{dy}{dt} = 2y/t$$

**Two cases:**

$$y(0) = 0,\ y(0) \neq 0$$

# Existence and Uniqueness

**Theorem: Local existence and uniqueness**

Suppose $F$ is continuous and Lipschitz, that is, $\|F[\vec{y}] - F[\vec{x}]\|_2 \leq L\|\vec{y} - \vec{x}\|_2$ for some $L$. Then, the ODE $f'(t) = F[f(t)]$ admits exactly one solution for all $t \geq 0$ regardless of initial conditions.

# Linearization of 1D ODEs

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# Model Equation

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$$\implies y(t) = Ce^{at}$$

# Three Cases

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$$y' = ay, y(t) = Ce^{at}$$

1. $a = 0$: Stable
2. $a < 0$: Stable; solutions get closer
3. $a > 0$: Unstable; mistakes in initial data amplified

# Multidimensional Case

$$\vec{y}' = A\vec{y}, A\vec{y}_i = \lambda_i \vec{y}_i$$

$$\vec{y}(0) = \sum_i c_i \vec{y}_i$$

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Stability depends on $\max_i |\lambda_i|$.

# Integration Strategies

Given $\vec{y}_k$ at time $t_k$, generate $\vec{y}_{k+1}$ assuming

$$\vec{y}\,' = F[\vec{y}].$$

# Forward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_k]$$

- Explicit method
- $O(h^2)$ localized truncation error
- $O(h)$ global truncation error; "first order accurate"

# Model Equation

$$y' = ay \longrightarrow y_{k+1} = (1 + ah)y_k$$

For $a < 0$, stable when $h < \frac{2}{|a|}$.

# Backward Euler

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But this has nothing to do with accuracy.

Good for *stiff* equations.

# Forward and Backward Euler on Linear ODE

$$\vec{y}\,' = A\vec{y}$$

- Forward Euler: $\vec{y}_{k+1} = (I + hA)\vec{y}_k$
- Backward Euler: $\vec{y}_{k+1} = (I - hA)^{-1}\vec{y}_k$

Next