Ordinary Differential Equations II


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Almost Done!

No class next week!

- **Homework 6:** 11/18 (2 days late!)
  - **Homework 7:** 12/2
  - **Homework 8?** (optional)
  - **Section:** 11/22, 12/6
- **Final exam:** 12/12, 12:15pm

Go to office hours!
Initial Value Problems

Find $f(t) : \mathbb{R} \rightarrow \mathbb{R}^n$

Satisfying $F[t, f(t), f'(t), f''(t), \ldots, f^{(k)}(t)] = 0$

Given $f(0), f'(0), f''(0), \ldots, f^{(k-1)}(0)$
Most Famous Example

\[ F = ma \]

*Newton’s second law*
Reasonable Assumption

Explicit ODE

An ODE is *explicit* if can be written in the form

\[ f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \ldots, f^{(k-1)}(t)]. \]
After Reduction

\[
\frac{d\vec{y}}{dt} = F[\vec{y}]
\]
Forward Euler

\[ \vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_k] \]

- Explicit method
- \(O(h^2)\) localized truncation error
- \(O(h)\) global truncation error;
  “first order accurate”
Backward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_{k+1}]$$

- Implicit method
- $O(h^2)$ localized truncation error
- $O(h)$ global truncation error; “first order accurate”
Trapezoid Method

\[ \vec{y}_{k+1} = \vec{y}_k + h \frac{F[\vec{y}_k] + F[\vec{y}_{k+1}]}{2} \]

- Implicit method
- \(O(h^3)\) localized truncation error
- \(O(h^2)\) global truncation error; “second order accurate”
Model Equation

\[ y' = ay, \ a < 0 \quad \rightarrow \quad y_{k+1} = \frac{1}{2}ha(y_{k+1} + y_k) \]
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Unconditionally stable!
Model Equation

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*Unconditionally* stable!

But this has nothing to do with accuracy.
Is Stability Useful Here?

\[ R = \frac{y_{k+1}}{y_k} \]
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Oscillatory behavior!
Convergence as $h \to 0$ is not the whole story nor is stability. Often should consider behavior for fixed $h > 0$. 
Pattern

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Qualitative and quantitative analysis needed!
Observation

\[ \vec{y}_{k+1} = \vec{y}_k + \int_{t_k}^{t_{k+1}} F[\vec{y}(t)] \, dt \]
Alternative Derivation of Trapezoid Method

\[ \vec{y}_{k+1} = \vec{y}_k + \frac{h}{2} \left( F[\vec{y}_k] + F[\vec{y}_{k+1}] \right) + O(h^3) \]

Implicit method
Heun’s Method

\[ \vec{y}_{k+1} = \vec{y}_k + \frac{h}{2} \left( F[\vec{y}_k] + F[\vec{y}_k + hF[\vec{y}_k]] \right) + O(h^3) \]

Replace implicit part with lower-order integrator!
Stability of Heun’s Method

\[ y' = ay, \quad a < 0 \]

\[ y_{k+1} = \left(1 + ha + \frac{1}{2}h^2a\right)y_k \]
Stability of Heun’s Method

\[ y' = ay, \quad a < 0 \]

\[ y_{k+1} = \left(1 + ha + \frac{1}{2}h^2a\right)y_k \]

\[ \implies h < \sqrt{1 - \frac{4}{|a|}} - 1 \]
Stability of Heun’s Method

\[ y' = ay, \ a < 0 \]

\[ y_{k+1} = \left(1 + ha + \frac{1}{2}h^2a\right) y_k \]

\[ \implies h < \sqrt{1 - \frac{4}{|a|}} - 1 \]

Forward Euler: \( h < \frac{2}{|a|} \)
Runge-Kutta Methods

Apply quadrature to:

\[
\mathbf{y}_{k+1} = \mathbf{y}_k + \int_{t_k}^{t_{k+1}} F[\mathbf{y}(t)] \, dt
\]
Runge-Kutta Methods

Apply quadrature to:

\[ \vec{y}_{k+1} = \vec{y}_k + \int_{t_k}^{t_{k+1}} F[\vec{y}(t)] \, dt \]

Use low-order integrators
RK4: Simpson’s Rule

\[ \vec{y}_{k+1} = \vec{y}_k + \frac{h}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4) \]

where \( \vec{k}_1 = F[\vec{y}_k] \)

\[ \vec{k}_2 = F \left[ \vec{y}_k + \frac{1}{2}h\vec{k}_1 \right] \]

\[ \vec{k}_3 = F \left[ \vec{y}_k + \frac{1}{2}h\vec{k}_2 \right] \]

\[ \vec{k}_4 = F \left[ \vec{y}_k + h\vec{k}_3 \right] \]
Using the Model Equation

\[ \vec{y}' \approx A \vec{y} \]

\[ \Rightarrow \vec{y}(t) \approx e^{At} \vec{y}_0 \]
Exponential Integrators

\[ \vec{y}' = A \vec{y} + G[\vec{y}] \]
Exponential Integrators

\[ \vec{y}' = A\vec{y} + G[\vec{y}] \]

First-order exponential integrator:

\[ \vec{y}_{k+1} = e^{Ah}\vec{y}_k - A^{-1}(1 - e^{Ah})G[\vec{y}_k] \]
Returning to Our Reduction

\[ \ddot{y}(t) = \ddot{v}(t) \]
\[ \ddot{v}(t) = \ddot{a}(t) \]
\[ \ddot{a}(t) = F[t, \dot{y}(t), \dot{v}(t)] \]
Returning to Our Reduction

\[ \ddot{y}'(t) = \ddot{v}(t) \]

\[ \ddot{v}'(t) = \ddot{a}(t) \]

\[ \ddot{a}(t) = F[t, \ddot{y}(t), \ddot{v}(t)] \]

\[ \ddot{y}(t_k + h) = \ddot{y}(t_k) + h\ddot{y}'(t_k) + \frac{h^2}{2}\dddot{y}'(t_k) + O(h^3) \]

Don’t need high-order estimators for derivatives.
New Formulae

\[ \vec{v}_{k+1} = \vec{v}_k + \int_{t_k}^{t_{k+1}} \vec{a}(t) \, dt \]
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\[ \vec{v}_{k+1} = \vec{v}_k + \int_{t_k}^{t_{k+1}} \vec{a}(t) \, dt \]

\[ \vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \int_{t_k}^{t_{k+1}} (t_{k+1} - t) \vec{a}(t) \, dt \]
New Formulae

\[ \vec{v}_{k+1} = \vec{v}_k + \int_{t_k}^{t_{k+1}} \vec{a}(t) \, dt \]

\[ \vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \int_{t_k}^{t_{k+1}} (t_{k+1} - t)\vec{a}(t) \, dt \]

\[ \vec{a}(\tau) = (1 - \gamma)\vec{a}_k + \gamma\vec{a}_{k+1} + \vec{a}'(\tau)(\tau - h\gamma - t_k) + O(h^2) \]

for \( \tau \in [a, b] \)
Newmark Class

\[ \ddot{y}_{k+1} = \ddot{y}_k + h\ddot{v}_k + \left(\frac{1}{2} - \beta\right) h^2 \dddot{a}_k + \beta h^2 \dddot{a}_{k+1} \]

\[ \ddot{v}_{k+1} = \ddot{v}_k + (1 - \gamma) h\dddot{a}_k + \gamma h\dddot{a}_{k+1} \]

\[ \dddot{a}_k = F[t_k, \dddot{y}_k, \dddot{v}_k] \]
Newmark Class

\[ \vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \left( \frac{1}{2} - \beta \right) h^2 \vec{a}_k + \beta h^2 \vec{a}_{k+1} \]

\[ \vec{v}_{k+1} = \vec{v}_k + (1 - \gamma)h\vec{a}_k + \gamma h\vec{a}_{k+1} \]

\[ \vec{a}_k = F[t_k, \vec{y}_k, \vec{v}_k] \]

Parameters: \( \beta, \gamma \) (can be implicit or explicit!)
**Constant Acceleration:** $\beta = \gamma = 0$

\[
\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_k
\]

\[
\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_k
\]

Explicit, exact for constant acceleration, linear accuracy
Constant Implicit Acceleration:

\[ \beta = \frac{1}{2}, \gamma = 1 \]

\[
\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_{k+1}
\]

\[
\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_{k+1}
\]

Backward Euler for velocity, midpoint for position; globally first-order accurate
**Trapezoid:** \( \beta = 1/4, \gamma = 1/2 \)

\[
\vec{x}_{k+1} = \vec{x}_k + \frac{1}{2}h(\vec{v}_k + \vec{v}_{k+1})
\]

\[
\vec{v}_{k+1} = \vec{v}_k + \frac{1}{2}h(\vec{a}_k + \vec{a}_{k+1})
\]

Trapezoid for position and velocity; globally second-order accurate
Central Differencing: $\beta = 0, \gamma = \frac{1}{2}$

$$\vec{v}_{k+1} = \frac{\vec{y}_{k+2} - \vec{y}_k}{2h}$$

$$\vec{a}_{k+1} = \frac{\vec{y}_{k+1} - 2\vec{y}_{k+1} + \vec{y}_k}{h^2}$$

Central difference for position and velocity; globally second-order accurate
Newmark Schemes

- Generalizes large class of integrators
- Second-order accuracy (exactly) when $\gamma = 1/2$
- Unconditional stability when $4\beta > 2\gamma > 1$
  (otherwise radius is function of $\beta$, $\gamma$)
Staggered Grid

\[\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_{k+1/2}\]

\[\vec{v}_{k+3/2} = \vec{v}_{k+1/2} + h\vec{a}_{k+1}\]

\[\vec{a}_{k+1} = F\left[ t_{k+1}, \vec{x}_{k+1}, \frac{1}{2}(\vec{v}_{k+1/2} + \vec{v}_{k+3/2}) \right]\]
Staggered Grid

\[ \vec{y}_{k+1} = \vec{y}_k + h\vec{v}_{k+1/2} \]
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\[ \vec{a}_{k+1} = F \left[ t_{k+1}, \vec{x}_{k+1}, \frac{1}{2} (\vec{v}_{k+1/2} + \vec{v}_{k+3/2}) \right] \]

**Leapfrog:** \( \vec{a} \) not a function of \( \vec{v} \); explicit
Staggered Grid

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\[ \vec{a}_{k+1} = F \left[ t_{k+1}, \vec{x}_{k+1}, \frac{1}{2}(\vec{v}_{k+1/2} + \vec{v}_{k+3/2}) \right] \]

Leapfrog: \( \vec{a} \) not a function of \( \vec{v} \); explicit

Otherwise: Symmetry in \( \vec{v} \) dependence