Ordinary Differential Equations II

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Almost Done!

No class next week!

- **Homework 6:** 11/18 (2 days late!)
- **Homework 7:** 12/2
- **Homework 8?** (optional)
- **Section:** 11/22, 12/6
- **Final exam:** 12/12, 12:15pm

Go to office hours!

Initial Value Problems

$$\text{Find } f(t) : \mathbb{R} \to \mathbb{R}^n$$

$$\text{Satisfying } F[t, f(t), f'(t), f''(t), \ldots, f^{(k)}(t)] = 0$$

$$\text{Given } f(0), f'(0), f''(0), \ldots, f^{(k-1)}(0)$$

Most Famous Example

$$F = ma$$

Newton's second law

Reasonable Assumption

Explicit ODE

An ODE is *explicit* if can be written in the form

$$f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \ldots, f^{(k-1)}(t)].$$

After Reduction

$$\frac{d\vec{y}}{dt} = F[\vec{y}]$$

Forward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_k]$$

- Explicit method
- $O(h^2)$ localized truncation error
- $O(h)$ global truncation error; "first order accurate"

Backward Euler

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Trapezoid Method

$$\vec{y}_{k+1} = \vec{y}_k + h\frac{F[\vec{y}_k] + F[\vec{y}_{k+1}]}{2}$$

- Implicit method
- $O(h^3)$ localized truncation error
- $O(h^2)$ global truncation error; "second order accurate"

Model Equation

$$y' = ay, a < 0 \longrightarrow y_{k+1} = \frac{1}{2}ha(y_{k+1} + y_k)$$

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But this has nothing to do with accuracy.

Is Stability Useful Here?

$$R = \frac{y_{k+1}}{y_k}$$

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Oscillatory behavior!

Pattern

Convergence as $h \to 0$ is not the whole story nor is stability. Often should consider behavior for fixed $h > 0$.

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Qualitative and quantitative analysis needed!

Observation

$$\vec{y}_{k+1} = \vec{y}_k + \int_{t_k}^{t_{k+1}} F[\vec{y}(t)]\, dt$$

Alternative Derivation of Trapezoid Method

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}(F[\vec{y}_k] + F[\vec{y}_{k+1}]) + O(h^3)$$

Implicit method

Heun's Method

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}(F[\vec{y}_k] + F[\vec{y}_k + hF[\vec{y}_k]]) + O(h^3)$$

Replace implicit part with lower-order integrator!

Stability of Heun's Method

$$y' = ay, a < 0$$

$$y_{k+1} = \left(1 + ha + \frac{1}{2}h^2 a\right) y_k$$

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$$\implies h < \sqrt{1 - \frac{4}{|a|}} - 1$$

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Forward Euler: $h < \frac{2}{|a|}$

Runge-Kutta Methods

Apply quadrature to:

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Use low-order integrators

RK4: Simpson's Rule

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$$

$$\text{where } \vec{k}_1 = F\left[\vec{y}_k\right]$$

$$\vec{k}_2 = F\left[\vec{y}_k + \frac{1}{2}h\vec{k}_1\right]$$

$$\vec{k}_3 = F\left[\vec{y}_k + \frac{1}{2}h\vec{k}_2\right]$$

$$\vec{k}_4 = F\left[\vec{y}_k + h\vec{k}_3\right]$$

Using the Model Equation

$$\vec{y}' \approx A\vec{y}$$
$$\implies \vec{y}(t) \approx e^{At}\vec{y}_0$$

Exponential Integrators

$$\vec{y}' = A\vec{y} + G[\vec{y}]$$

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First-order exponential integrator:

$$\vec{y}_{k+1} = e^{Ah}\vec{y}_k - A^{-1}(1 - e^{Ah})G[\vec{y}_k]$$

Returning to Our Reduction

$$\vec{y}'(t) = \vec{v}(t)$$
$$\vec{v}'(t) = \vec{a}(t)$$
$$\vec{a}(t) = F[t, \vec{y}(t), \vec{v}(t)]$$

Returning to Our Reduction

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$$\vec{a}(t) = F[t, \vec{y}(t), \vec{v}(t)]$$

$$\vec{y}(t_k + h) = \vec{y}(t_k) + h\vec{y}'(t_k) + \frac{h^2}{2}\vec{y}''(t_k) + O(h^3)$$

Don't need high-order estimators for derivatives.

New Formulae

$$\vec{v}_{k+1} = \vec{v}_k + \int_{t_k}^{t_{k+1}} \vec{a}(t)\, dt$$

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$$\vec{a}(\tau) = (1-\gamma)\vec{a}_k + \gamma\vec{a}_{k+1} + \vec{a}'(\tau)(\tau - h\gamma - t_k) + O(h^2)$$
$$\text{for } \tau \in [a, b]$$

Newmark Class

$$
\begin{aligned}
\vec{y}_{k+1} &= \vec{y}_k + h\vec{v}_k + \left(\frac{1}{2} - \beta\right) h^2\vec{a}_k + \beta h^2\vec{a}_{k+1} \\
\vec{v}_{k+1} &= \vec{v}_k + (1-\gamma)h\vec{a}_k + \gamma h\vec{a}_{k+1} \\
\vec{a}_k &= F[t_k, \vec{y}_k, \vec{v}_k]
\end{aligned}
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\vec{v}_{k+1} &= \vec{v}_k + (1-\gamma) h \vec{a}_k + \gamma h \vec{a}_{k+1} \\
\vec{a}_k &= F[t_k, \vec{y}_k, \vec{v}_k]
\end{aligned}
$$

Parameters: β, γ (can be implicit or explicit!)

Constant Acceleration: $\beta = \gamma = 0$

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_k$$
$$\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_k$$

Explicit, exact for constant acceleration, linear accuracy

Constant Implicit Acceleration: $\beta = 1/2, \gamma = 1$

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_{k+1}$$
$$\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_{k+1}$$

Backward Euler for velocity, midpoint for position; globally first-order accurate

Trapezoid: $\beta = 1/4, \gamma = 1/2$

$$\vec{x}_{k+1} = \vec{x}_k + \frac{1}{2}h(\vec{v}_k + \vec{v}_{k+1})$$
$$\vec{v}_{k+1} = \vec{v}_k + \frac{1}{2}h(\vec{a}_k + \vec{a}_{k+1})$$

Trapezoid for position and velocity; globally second-order accurate

Central Differencing: $\beta = 0, \gamma = 1/2$

$$\vec{v}_{k+1} = \frac{\vec{y}_{k+2} - \vec{y}_k}{2h}$$
$$\vec{a}_{k+1} = \frac{\vec{y}_{k+1} - 2\vec{y}_{k+1} + \vec{y}_k}{h^2}$$

Central difference for position and velocity; globally second-order accurate

Newmark Schemes

- Generalizes large class of integrators
- Second-order accuracy (exactly) when $\gamma = 1/2$
- Unconditional stability when $4\beta > 2\gamma > 1$ (otherwise radius is function of β, γ)

Staggered Grid

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_{k+1/2}$$

$$\vec{v}_{k+3/2} = \vec{v}_{k+1/2} + h\vec{a}_{k+1}$$

$$\vec{a}_{k+1} = F\left[t_{k+1}, \vec{x}_{k+1}, \frac{1}{2}(\vec{v}_{k+1/2} + \vec{v}_{k+3/2})\right]$$

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Leapfrog: $\vec{a}$ not a function of $\vec{v}$; explicit

Staggered Grid

$$\begin{aligned}\vec{y}_{k+1} &= \vec{y}_k + h\vec{v}_{k+1/2}\\ \vec{v}_{k+3/2} &= \vec{v}_{k+1/2} + h\vec{a}_{k+1}\\ \vec{a}_{k+1} &= F\left[t_{k+1}, \vec{x}_{k+1}, \frac{1}{2}(\vec{v}_{k+1/2} + \vec{v}_{k+3/2})\right]\end{aligned}$$

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Otherwise: Symmetry in $\vec{v}$ dependence

Next