Partial Differential Equations I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

Justin Solomon

Almost Done!

- **Homework 7:** 12/2 (today!)
- **Last lecture:** 12/4
- **Homework 8:** 12/9 (optional)
- **Section:** 12/6 (final review)
- **Final exam:** 12/12, 12:15pm (**Gates B03**)

Go to office hours!

Course Reviews

On Axess!

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Request for Help

$$\text{CS 205A notes} \xmapsto{\textit{your help!}} \text{Textbook}$$

- Review text
- Write reference implementations
- Solidify your CS205A knowledge

Initial Value Problems

$$\text{Find } f(t) : \mathbb{R} \to \mathbb{R}^n$$
$$\text{Satisfying } F[t, f(t), f'(t), f''(t), \ldots, f^{(k)}(t)] = 0$$
$$\text{Given } f(0), f'(0), f''(0), \ldots, f^{(k-1)}(0)$$

Most Famous Example

$$F = ma$$

Newton's second law

This Week

Couple relationships between derivatives.

- *Pressure gradient* determining fluid flow
- *Image operators* using x and y derivatives

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Partial Differential Equations (PDE)

Useful Operators

$$f : \mathbb{R}^3 \to \mathbb{R}, \vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\text{Gradient: } \nabla f \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$\text{Divergence: } \nabla \cdot \vec{v} \equiv \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$\text{Curl: } \nabla \times \vec{v} \equiv \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)$$

$$\text{Laplacian: } \nabla^2 f \equiv \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Gradient Operator Notation

$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

Physics Convention

For $f(t; x, y, z)$,

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

No t!

Incompressible Navier-Stokes

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \vec{f}$$

- $t \in [0, \infty)$: Time
- $\vec{v}(t) : \Omega \to \mathbb{R}^3$: Velocity
- $\rho(t) : \Omega \to \mathbb{R}$: Density
- $p(t) : \Omega \to \mathbb{R}$: Pressure
- $\vec{f}(t) : \Omega \to \mathbb{R}^3$: External forces (e.g. gravity)

Homework 9

Prove or give a counter-example of the following statement:
In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.

- Millennium Prize Problems in Mathematics
($1,000,000 prize!)

Maxwell's Equations

$$\text{Gauss's law: } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\text{Gauss's (other) law: } \nabla \cdot \vec{B} = 0$$

$$\text{Faraday's law: } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{Ampère's law: } \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Laplace's Equation

$$\text{minimize}_f \int_{\Omega} \|\nabla f(\vec{x})\|_2^2 \, d\vec{x}$$

$$\text{such that } f(\vec{x}) = g(\vec{x}) \, \forall x \in \partial\Omega$$

Laplace's Equation

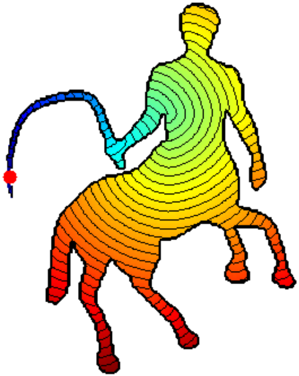
$$\text{minimize}_f \int_\Omega \|\nabla f(\vec{x})\|_2^2 \, d\vec{x}$$
$$\text{such that } f(\vec{x}) = g(\vec{x}) \, \forall x \in \partial\Omega$$

$$\implies \nabla^2 f(\vec{x}) \equiv 0$$

Eikonal Equation

Satisfied by distance functions d:

$$\|\nabla d\|_2 = 1$$

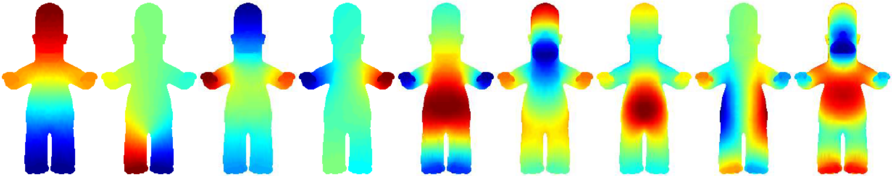


https://www.ceremade.dauphine.fr/~peyre/numerical-tour/tours/shapes_5_geodesic_descriptors/

Harmonic Analysis

To find resonant frequencies of a domain:

$$\nabla^2 f = \lambda f$$



http://graphics.stanford.edu/courses/cs468-13-spring/assets/lecture12.pdf

Boundary Value Problems

- *Dirichlet conditions*: Value of $f(\vec{x})$ on $\partial\Omega$
- *Neumann conditions*: Derivatives of $f(\vec{x})$ on $\partial\Omega$
- *Mixed* or *Robin conditions*: Combination

Second-Order Model Equation

$$\sum_{ij} a_{ij} \frac{\partial f}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial f}{\partial x_i} + cf = 0$$

Second-Order Model Equation

$$\sum_{ij} a_{ij} \frac{\partial f}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial f}{\partial x_i} + cf = 0$$

$$(\nabla^\top A \nabla + \nabla \cdot \vec{b} + c) f = 0$$

Well-Posed PDE

- Solution exists
- Solution is unique
- Continuous dependence on boundary conditions

- Hadamard, 1902

Classification of Second-Order PDE

$$(\nabla^\top A \nabla + \nabla \cdot \vec{b} + c) f = 0$$

- If A is *positive or negative definite*, system is *elliptic*.
- If A is *positive or negative semidefinite*, the system is *parabolic*.
- If A has only one eigenvalue of different sign from the rest, the system is *hyperbolic*.
- If A satisfies none of the criteria, the system is *ultrahyperbolic*.

Elliptic PDE

A is positive (or negative) definite!

- Existence/uniqueness theory
- Elliptic regularity: Solutions are C^∞ under weak conditions
- Model equation: Laplace equation $f_{xx} + f_{yy} = g$
- Boundary conditions

Elliptic PDE

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- Model equation: Laplace equation
  $f_{xx} + f_{yy} = g$
- Boundary conditions

[1D example]

Parabolic PDE

A is positive semi-definite

- Short-term existence/uniqueness
- Model equation: Heat equation $f_t = \alpha \nabla^2 f$
- Boundary conditions: Time and space

Hyperbolic PDE

A is has one eigenvalue of opposite sign

- Model equation: Wave equation
  $f_{tt} - c^2\nabla^2 f = 0$
- Not necessarily dampening over time
- Boundary conditions: Time and space (incl. first derivative)

Derivative as Operator on C^∞

$$\frac{d}{dx}(af(x) + bg(x))$$

$$= a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

Recall: Central Differencing

$$f''(x) = \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] + O(h)$$

Second Derivative Operator

$n + 1$ samples on $[0, 1]$

$$y_k'' \equiv \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

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Draw stencil

Boundary Conditions

- Dirichlet: $y_{-1} = y_{n+1} = 0$
- Neumann: $y_{-1} = y_0$ and $y_{n+1} = y_n$
- Periodic: $y_{-1} = y_n$ and $y_{n+1} = y_0$

Derivative Operator Matrix

$$h^2\vec{w} = L_1\vec{y}$$

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

Dirichlet

Derivative Operator Matrix

$$h^2\vec{w} = L_1\vec{y}$$

$$\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}$$

Neumann (null space!)

Derivative Operator Matrix

$$h^2\vec{w} = L_1\vec{y}$$

$$\begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

Periodic (null space!)

Stencil for 2D Grid

$$(\nabla^2 y)_{k,\ell} \equiv \frac{1}{h^2}\Big(y_{(k-1),\ell} + y_{k,(\ell-1)} + y_{(k+1),\ell} + y_{k,(\ell+1)} - 4y_{k,\ell}\Big)$$

What About First Derivative?

- Potential for asymmetry at notes
- Centered differences: Fencepost problem
- Possible resolution: Imitate leapfrog

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