Multiple Choice (4 x 1 pt each)

For each of the following questions, circle ALL answers that are correct.

1. Let $A$ be a symmetric positive definite $n \times n$ matrix that you attempt to solve using the Conjugate Gradient method. If $A$ has 3 distinct eigenvalues, what is the maximum number of steps (in theory) that the CG solver needs to converge?

   (a) $n$
   (b) 3
   (c) $n - 3$
   (d) None of the above is correct.

2. Recall the Newmark method:

   $x^{n+1} = x^n + \Delta t v^n + \frac{\Delta t^2}{2} \left[ (1 - 2\beta)\alpha^n + 2\beta\alpha^{n+1} \right]$
   
   $v^{n+1} = v^n + \Delta t \left[ (1 - \gamma)\alpha^n + \gamma\alpha^{n+1} \right]$

   Which of the following methods is not equivalent for some choice of $\alpha$ and $\beta$ parameters?

   (a) Constant Acceleration
   (b) 3rd-order accurate Runge-Kutta.
   (c) Central Differencing
   (d) Trapezoidal Rule

3. In general, which of the following methods are the most suitable for solving heat equations?

   (a) 2nd-order Runge Kutta.
   (b) Forward Euler.
   (c) Backward Euler.
   (d) 4th-order Runge Kutta.

4. The Monte Carlo method

   (a) is widely used for problems in lower dimensions
   (b) has an error proportional to $1/\sqrt{n}$
   (c) is widely used for problems in higher dimensions
   (d) is deterministic
Conjugate Gradient (10 pts)

1. Apply the Conjugate Gradient method to find the solution of the following linear system
\[
\begin{pmatrix}
3 & 1 \\
1 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
x
\end{pmatrix}
=
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]
Use the initial guess \( \bar{x}_0 = [0, 0]^T \). (5 pts)
2. Let the CG iteration be applied to solve a linear system $AX = b$ with $A$ as a symmetric positive definite matrix. $(s_0, s_1, \cdots, s_k, \cdots, s_n)$ is the sequence of search directions generated in the iteration. Assume the initial error can be expressed as $e_0 = \sum_{j=0}^{n} c_j s_j$. First derive the relations between step size of CG at $k^{th}$ step, i.e. $\alpha_k$, and the coefficient $c_k$; Second, derive the expression of the error after $k$ iterations $e_k$. (5 pts)
Optimization (6 pts)

Minimize \( f(x) = 5x_1 + 3x_2^2 + x_3^2 + 2x_4 \) subject to the constraints \( x_1 + x_2 = x_3 \) and \( x_2 - x_4 = 2 \) using Lagrange multipliers. Show your work, including a problem of the form \( Ax = b \).
Ordinary Differential Equations (10 pts)

The following scheme has been proposed for solving \( y' = f(y) \):

\[
\begin{align*}
  y^* &= y_n + \frac{h}{2} f(y_n) \\
  y_{n+1} &= y_n + hf(y^*)
\end{align*}
\]

with \( h \) being the time step.

1. Applying this method to the model problem \( y' = \lambda y \), what is the maximum step size \( h \) for \( \lambda \) being negative real in order to have a stable solution? (4 pts)

2. Still applying this method to the model problem \( y' = \lambda y \), what is the order of global accuracy of this method? (4 pts)
3. Convert the sixth-order non-linear ODE \( y^{(6)} = 3y^{(4)} + 4y''' \cdot y'' - (y' + y) \cdot y' \) into a system of first-order equations. Show your variable assignments. (2 pts)
Interpolation (10 pts)

1. State the main disadvantage of the monomial basis functions. (1 pt)

2. Given the set of sample points (-2,12), (-1,3), (0,10) and (3,7), construct an interpolating polynomial using monomial basis functions. (4 pts)
3. Given the same set of sample points (-2,12), (-1,3), (0,10) and (3,7), construct an interpolating polynomial using Newton interpolation. (3 pts)

4. Suppose the function that these points are sampled from is \( O(x^3) \). What does that imply about your answers in parts 2 and 3? (1 pt)

5. What are the advantages and disadvantages of Lagrange interpolation? (1 pt)