CS 205a Spring 2012-2013 Midterm 1

Please write your name on the top right of this page. The exam is closed book/closed notes and no calculators are allowed. You have 1 hour and 15 minutes to complete the exam.

Multiple Choice (4 x 1 pt each)

For each of the following questions, circle all answers which are correct. You must circle **ALL** of the answers for a given question correctly to receive credit.

1. Which of the following statements about the solution to linear least-squares problem are true?
   (a) It minimizes $||Ax - b||_2$
   (b) Its associated residual lies in the nullspace of A^T
   (c) It can only be computed using the normal equations
   (d) It can be found by solving $A^T Ax = A^T x$
2. Which of the following statements about condition numbers are true?
   (a) The condition number for solving $Ax = b$ is $||A||||A^{-1}||$ for nonsingular matrix A
   (b) The choice of b affects the condition number of solving $Ax = b$
   (c) A problem is better conditioned if it has a larger condition number
   (d) The condition number can be negative
3. Consider the multi-variate optimization problem $min_{\vec{x}} f(\vec{x})$
   (a) When the Hessian is positive-definite, we are at a local maximizer
   (b) Solving this minimization problem is exactly equivalent to finding some $\vec{x}$ such that $\nabla f(\vec{x}) = 0$
   (c) If f is continuous and differentiable, then the Hessian is guaranteed not to be singular
   (d) Steepest Descent performs poorly when the Hessian is poorly conditioned
4. Which of the following are properties of symmetric matrices?
   (a) All the eigenvalues are positive numbers
   (b) All the eigenvectors are orthogonal to each other
   (c) All projection matrices are symmetric matrices
   (d) The condition number is equal to the largest eigenvalue

QR factorization and least squares problems (10 pts)

Suppose you are computing the QR factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}:$$

1. How many Householder transformations are required? (2 pts)

2. What is the first Householder transformation matrix? (4 pts)

3. What does the first column of A become as a result of applying the first Householder transformation? (2 pt)

4. Show how you find the least-squares solution to $Ax = b$ using QR factorization. (2 pts)

Optimization and Nonlinear Equations (7 pts)

1. Let $f : \mathbb{R}^n \Rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{2}x^T Ax - x^T b + c$ where A is an $n \times n$ symmetric negative definite matrix, b and x are vectors in $\mathbb{R}^n$, and c is a scalar. Does $f(x)$ have a maximizer or minimizer? Show that Newton's method for minimizing (or maximizing) this function converges in one iteration from any starting point x_0. (3 pts)

2. If the steepest descent method is used on this problem, what happens if the starting value x_0 is an eigenvector of A? (2 pt)

3. Consider any minimization problem, draw a graph to show a case that the steepest descent method can be very slow to converge. (2pt)

Eigenvalues (9 pts)

1. Prove that the eigenvalues of a positive definite matrix are positive. (2 pts)

2. Prove that the eigenvalues of an orthogonal matrix all have magnitudes of one. (2 pts)

3. Compute all the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. (2 pt)

4. State how you can use the Power Method to find the eigenvalues and eigenvectors. (3 pts)

Singular Value Decomposition (10 pts)

1. Write down the SVD for the following matrices (no proof is necessary): (4 pts)

   $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

   $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$:

   $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 10 & 0 \end{pmatrix}$:

   (-5):

2. In PCA what is done to the singular values of A? How is this justified? (2 pt)

3. State one method you know that can be used to compute the SVD of a matrix. (2 pts)

4. What is the relation between singular values and the 2-norm of a matrix? (2 pts)