Homework 3: QR and Eigenproblems

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2015)
Stanford University
Due Monday, April 27, 2:15pm (**beginning** of class—changed!)

Textbook problems: 5.11 (40 points), 6.8 (30 points), 6.10 (30 points)
Extra credit: 5.8 (5 points), 6.12 (5 points)

Revision of exercise 6.8: We will say $\langle \cdot, \cdot \rangle$ is an *inner product* on $\mathbb{R}^n$ if it satisfies:

1. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle \; \forall \vec{x}, \vec{y} \in \mathbb{R}^n$,
2. $\langle \alpha \vec{x}, \vec{y} \rangle = \alpha \langle \vec{x}, \vec{y} \rangle \; \forall \vec{x}, \vec{y} \in \mathbb{R}^n, \alpha \in \mathbb{R}$,
3. $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle \; \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, and
4. $\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality if and only if $\vec{x} = \vec{0}$.

(a) Given an inner product $\langle \cdot, \cdot \rangle$, show that there exists a matrix $A \in \mathbb{R}^{n \times n}$ such that $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top A \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Also, show that there exists a matrix $M \in \mathbb{R}^{n \times n}$ such that $\langle \vec{x}, \vec{y} \rangle = (M\vec{x}) \cdot (M\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. [This shows that all inner products are dot products after suitable rotation, stretching, and shearing of $\mathbb{R}^n$!]

(b) A *Mahalanobis metric* on $\mathbb{R}^n$ is a distance function of the form $d(\vec{x}, \vec{y}) = \sqrt{\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle}$ for a fixed inner product $\langle \cdot, \cdot \rangle$. Use Exercise 6.8a to write Mahalanobis metrics in terms of matrices M, and show that substituting any invertible matrix $M \in \mathbb{R}^{n \times n}$ into your formula *defines* a Mahalanobis metric.

(c) Suppose we are given several pairs $(\vec{x}_i, \vec{y}_i) \in \mathbb{R}^n \times \mathbb{R}^n$. A typical "metric learning" problem involves finding a nontrivial Mahalanobis metric such that each $\vec{x}_i$ is close to each $\vec{y}_i$ with respect to that metric. Propose an optimization problem for this task that can be solved using eigenvector computation.
Note: Make sure that your optimal Mahalanobis distance is not identically zero, but it is acceptable if your optimization allows *pseudometrics*; that is, there can exist some $\vec{x} \neq \vec{y}$ with $d(\vec{x}, \vec{y}) = 0$.