Homework 1: Numerics & Linear Systems (LU)

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2017)
Stanford University
Due Thursday, April 20, 11:59pm

This homework is one of the more difficult assignments in CS 205A. Please make ample use of office hours, Piazza, and other resources, and get started early.

Note: As mentioned in lecture, although it is not worth course credit you may wish as an exercise on your own to implement algorithms we discuss in class. We are happy to help you debug and experiment with your implementations in office hours and on Piazza. From last week's lectures, the obvious choices for implementation are Gaussian elimination (with or without pivoting) and LU factorization.

Textbook problems: 2.3 (20 points), 2.7 (20 points; see revised 2.7(b) below), 3.12(a-e) (25 points)

Revision of exercise 2.7(b): Suppose we are given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and compute their dot product as s_n via the recurrence:

$$s_0 \equiv 0$$
$$s_k \equiv s_{k-1} + x_k y_k.$$

In practice, both the addition and multiplication steps of computing s_k from s_{k-1} induce numerical error. Use $\hat{s}_k$ to denote the actual value computed incorporating numerical error, and denote $e_k \equiv |\hat{s}_k - s_k|$. Show that

$$|e_n| \leq n\varepsilon_{\max}\bar{s}_n + O(n\varepsilon_{\max}^2\bar{s}_n),$$

where $\bar{s}_n \equiv \sum_{k=1}^{n} |x_k||y_k|$. You can assume that adding $x_1 y_1$ to zero incurs no error, so $\hat{s}_1 = (1 + \varepsilon^{\times})x_1 y_1$, where $\varepsilon^{\times}$ encodes the error induced by multiplying x_1 and y_1. You also can assume that $n\varepsilon_{\max} < 1$.

Julia Programming Assignment (35 points): In this question you will use the normal equations with Tikhonov regularization to fit a 1D polynomial to a set of points. To make things numerically interesting, you will use the a monomial basis which results in (cue the drumroll) the Vandermonde matrix–a famous ill-conditioned matrix. In this question, you will interpolate samples of the analytical function

$$f(x) = sin(\pi x).$$

for $x \in [0, 1]$. Your monomial interpolant (with $N = n + 1$ terms) is given by

$$g(x) = \sum_{j=0}^{n} c_j x^j.$$

Use $M = m + 1$ uniformly sampled positions

$$x_i = ih, \qquad i = 0 \ldots m,$$

with spacing $h = 1/m$, to generate M samples of the test function f,

$$f_i = f(x_i), \qquad i = 0 \ldots m.$$

To estimate the polynomial coefficients $\vec{c}$, you will assemble and solve the linear system

$$V\vec{c} = \vec{f}$$

where V is the M-by-N Vandermonde matrix with entries, $V_{ij} = (x_i)^j$.

For simplicity, we will use the first $N = 16$ monomial basis functions, and fit to M evenly spaced function samples.

(a) Implement methods to estimate the monomial coefficients using the following methods:

- Exactly determined case with $M = N$. Use an LU solve.
- Over-determined case with $M = 4N$. Construct the normal equations, and use a Cholesky solve.
- Over-determined case with $M = 4N$. Normal equations with Tikhonov regularization (α of your choosing). Use a Cholesky solve.

(b) In each case, state a representative matrix condition number of the linear system solved in each of the three cases. You may use Julia's `cond()` method.

(c) To objectively estimate the fitting error, numerically approximate the L1 fitting error integral $E = \int_0^1 |g(x) - f(x)|dx$ using 2000 function samples, i.e., use the equispaced midpoint integration method,

$$E_{2000} \equiv \frac{1}{2000} \sum_{k=1}^{2000} |g(z_k) - f(z_k)|$$

where z_k is the midpoint of the k^{th} interval. Which technique in (a) provides the lowest E_{2000} error?

(d) What range of α values (or specific value) is most effective?

(e) Plot your best polynomial fits, along with $f(x)$ and the first four nonzero terms of the Taylor/Maclaurin series for $f(x)$,

$$sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + O(x^9)$$