Eigenproblems I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Announcements

- Homework 1: Due tonight
- Homework 2: Out today. Tomography example.
- More office hours!
- Today's class: Eigenproblem defns and examples.
- Next class: Computing eigenvalue decompositions.

Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

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Find: Correlations between different dimensions

Simplest Model

One-dimensional subspace

$$\vec{x}_i \approx c_i \vec{v}, \quad \vec{v} \text{ unknown}$$

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Equivalently:

$$\vec{x}_i \approx c_i \hat{v}$$

$$\hat{v} \text{ unknown with } \|\hat{v}\|_2 = 1$$

Variational Idea

$$\text{minimize}_{\hat{v}} \sum_i \|\vec{x}_i - \text{proj}_{\hat{v}}\, \vec{x}_i\|_2^2$$

$$\text{such that } \|\hat{v}\|_2 = 1$$

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What does the constraint do?

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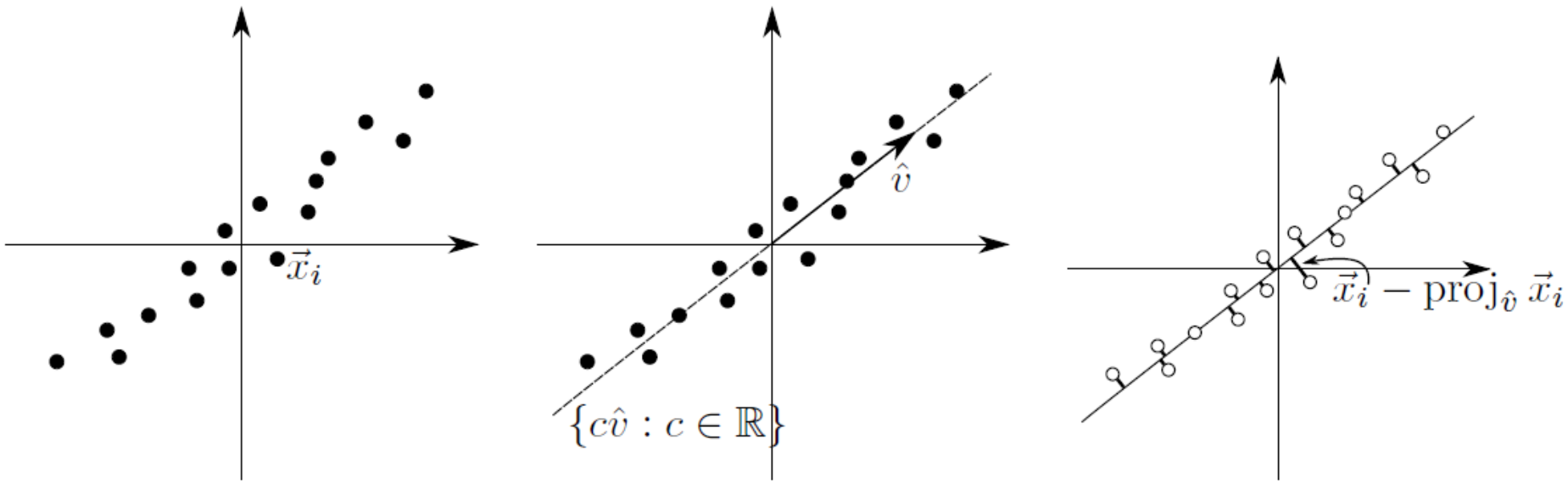
What does the constraint do?

- Does not affect optimal $\hat{v}$
- Removes scaling ambiguity

Geometric Interpretation



(a) Input data (b) Principal axis (c) Projection error

Review from Last Lecture

$$\min_{c_i} \|\vec{x}_i - c_i \hat{v}\|_2$$

What is c_i?

Review from Last Lecture

$$\min_{c_i} \|\vec{x}_i - c_i\hat{v}\|_2$$

What is c_i?

$$\boxed{c_i = \vec{x}_i \cdot \hat{v}}$$

Equivalent Optimization

$$\begin{aligned} &\text{maximize } \|X^\top \hat{v}\|_2^2 \\ &\text{such that } \|\hat{v}\|_2^2 = 1 \end{aligned}$$

End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.

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"First principal component"

More after SVD!

Physics (in one slide)

Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

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Hooke:

$$\vec{F}_s = k(\vec{x} - \vec{y})$$

First-Order System

$$M\vec{X}'' = K\vec{X}$$

$$\longrightarrow \frac{d}{dt}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix} = \begin{pmatrix} 0 & I_{3n\times 3n} \\ M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} \vec{X} \\ \vec{V} \end{pmatrix}$$

General ODE

$$\vec{Y}' = B\vec{Y}$$

Eigenvector Solution

$$\vec{y}' = B\vec{y}$$

$$B\vec{y}_i = \lambda_i \vec{y}_i$$

$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

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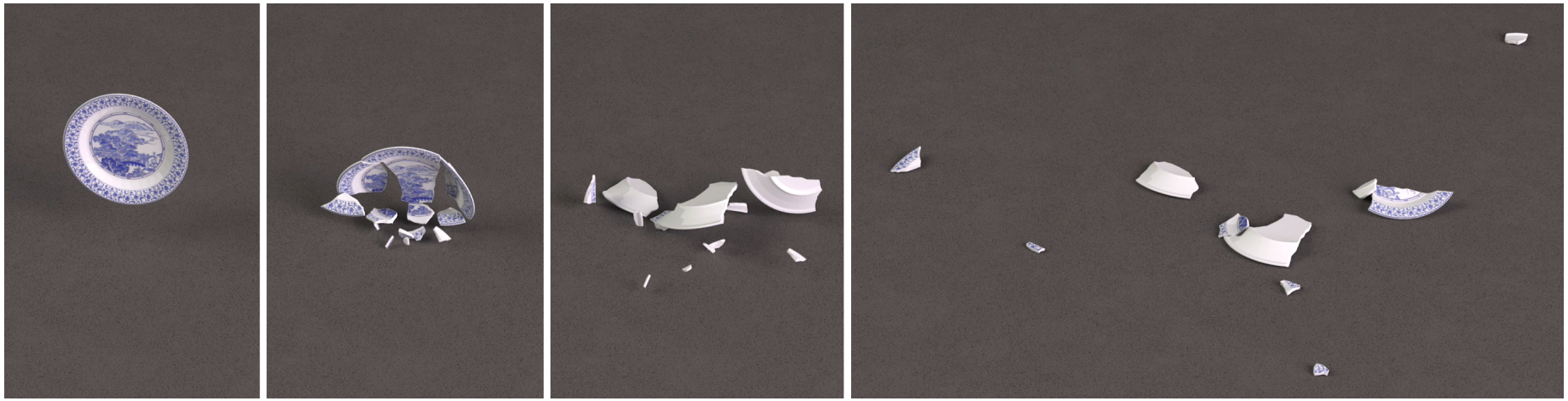
$$\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$$

$$\longrightarrow \vec{y}(t) = c_1 e^{\lambda_1 t}\vec{y}_1 + \cdots + c_k e^{\lambda_k t}\vec{y}_k$$

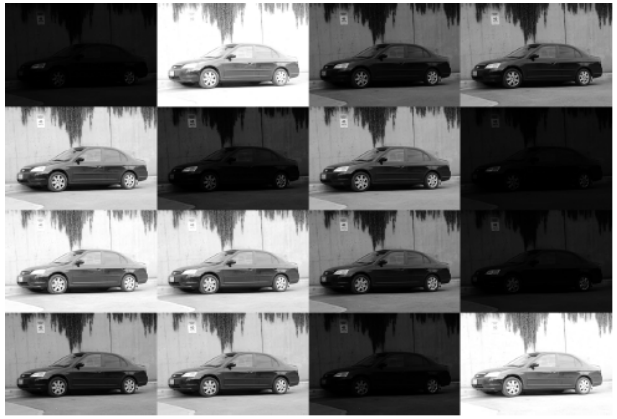
Application: Modal Sound Synthesis



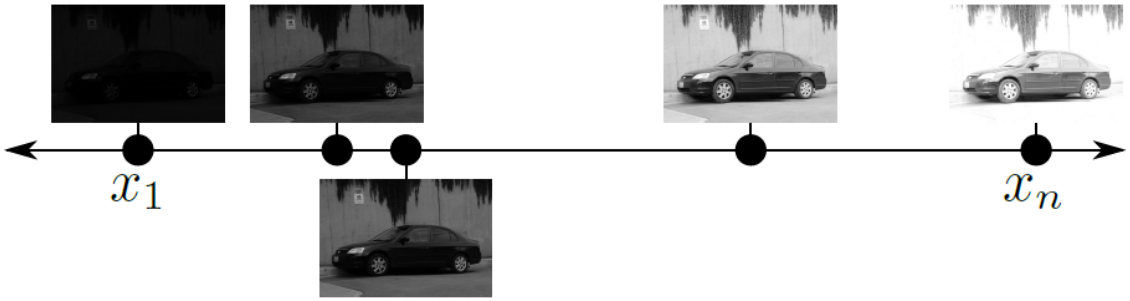
Major role in physics-based sound synthesis

https://www.youtube.com/watch?v=dMUHp8i6E5E

Organizing a Collection



(a) Database of photos



(b) Spectral embedding

Setup

Have: n items in a dataset

$w_{ij} \geq 0$ similarity of items i and j

$w_{ij} = w_{ji}$

Want: x_i embedding on $\mathbb{R}$

Quadratic Energy

$$E(\vec{x}) = \sum_{ij} w_{ij}(x_i - x_j)^2$$

Optimization

$$\text{minimize } E(\vec{x})$$

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$$\text{such that } \|\vec{x}\|_2^2 = 1$$

Optimization

$$\begin{aligned} \text{minimize } & E(\vec{x}) \\ \text{such that } & \|\vec{x}\|_2^2 = 1 \\ & \vec{1} \cdot \vec{x} = 0 \end{aligned}$$

Simplification

$$E(\vec{x}) = 2\vec{x}^\top (A - W)\vec{x}$$

Desired

Eigenvector of $A - W$ with **second** smallest eigenvalue.

Definitions

Eigenvalue and eigenvector

An *eigenvector* $\vec{x} \neq \vec{0}$ of $A \in \mathbb{R}^{n \times n}$ satisfies $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$; λ is an *eigenvalue*. *Complex* eigenvalues and eigenvectors instead have $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

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Scale doesn't matter!

$\longrightarrow$ can constrain $\|\vec{x}\|_2 \equiv 1$

Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$

  (important!)

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- Critical points of Rayleigh quotient:

$$\frac{\vec{x}^\top A\vec{x}}{\|\vec{x}\|_2^2}$$

Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

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$\longrightarrow$ at most n eigenvalues

Diagonalizability

Nondefective

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$$D = X^{-1} A X$$

Definitions

Spectrum and spectral radius

The *spectrum* of A is the set of eigenvalues of A. The *spectral radius* $\rho(A)$ is the eigenvalue λ maximizing $|\lambda|$.

Extending to $\mathbb{C}^{n\times n}$

Complex conjugate

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Conjugate transpose

The *conjugate transpose* of $A \in \mathbb{C}^{m\times n}$ is $A^H \equiv \bar{A}^\top$.

Hermitian Matrix

$$A = A^H$$

Properties

Lemma

All eigenvalues of Hermitian matrices are real.

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Lemma

Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.

Spectral Theorem

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Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian (if $A \in \mathbb{R}^{n \times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$.

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Full set: $D = X^\top A X$

Matrix Inverse

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$$A = XDX^{-1} \quad \implies \quad A^{-1} = XD^{-1}X^{-1}$$

Matrix Square Root

- Given symmetric positive semi-definite (PSD) matrix, U
- Can compute matrix square root, $U^{1/2}$

Application: Polar decomposition

- Given real n-by-n matrix, A
- There exists a unique factorization called the Polar Decomposition

$$A = RU$$

  where R is an n-by-n orthogonal matrix, and U is an n-by-n symmetric PSD right "stretch" matrix.
- Also a left stretch matrix, W, such that $A = WR$.
- Geometric interpretation.

Application: Shape Matching

- *Fast Lattice Shape Matching* (Fast LSM)
- SIGGRAPH 2007 [Rivers and James 2007]
- `http://www.alecrivers.com/fastlsm`
- Need to compute orientation, R, of local particle groups
- Millions of polar decompositions (and eigenvalue decomps) per second