Eigenproblems I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

Doug James (and Justin Solomon)
Announcements

- Homework 1: Due tonight
- Homework 2: Out today. Tomography example.
- More office hours!
- Today’s class: Eigenproblem defns and examples.
- Next class: Computing eigenvalue decompositions.
Setup

**Given:** Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate
**Setup**

**Given:** Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

**Find:** Correlations between different dimensions
Simplest Model

One-dimensional subspace

$$\tilde{x}_i \approx c_i \tilde{v}, \quad \tilde{v} \text{ unknown}$$
Simplest Model

One-dimensional subspace

\[ \vec{x}_i \approx c_i \vec{v}, \quad \vec{v} \text{ unknown} \]

Equivalently:

\[ \hat{\vec{x}}_i \approx c_i \hat{\vec{v}} \]

\( \hat{\vec{v}} \) unknown with \( \| \hat{\vec{v}} \|_2 = 1 \)
Variational Idea

\[
\text{minimize} \hat{v} \sum_{i} \| \vec{x}_i - \text{proj}_\hat{v} \vec{x}_i \|^2_2
\]

such that \( \| \hat{v} \|_2 = 1 \)
Variational Idea

\[
\text{minimize } \hat{\nu} \sum_i \| \vec{x}_i - \text{proj}_{\hat{\nu}} \vec{x}_i \|^2 \\
\text{such that } \| \hat{\nu} \|_2 = 1
\]

What does the constraint do?
**Variational Idea**

\[
\text{minimize } \hat{\nu} \sum_i \| \vec{x}_i - \text{proj}_{\hat{\nu}} \vec{x}_i \|^2_2 \\
\text{such that } \| \hat{\nu} \|^2_2 = 1
\]

What does the constraint do?

- Does not affect optimal \( \hat{\nu} \)
- Removes scaling ambiguity
Geometric Interpretation

(a) Input data
(b) Principal axis
(c) Projection error

\{c\hat{v} : c \in \mathbb{R}\}
Review from Last Lecture

\[ \min_{c_i} \| \vec{x}_i - c_i \hat{v} \|_2 \]

What is \( c_i \)?
Review from Last Lecture

\[
\min_{c_i} \| \vec{x}_i - c_i \hat{v} \|_2
\]

What is \( c_i \)?

\[
c_i = \vec{x}_i \cdot \hat{v}
\]
Equivalent Optimization

\[
\text{maximize} \; \| X^\top \hat{v} \|_2^2 \\
\text{such that} \; \| \hat{v} \|_2^2 = 1
\]
End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.
End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.

“First principal component”

More after SVD!
Physics (in one slide)

Newton:

\[ \vec{F} = m \frac{d^2 \vec{x}}{dt^2} \]
Physics (in one slide)

Newton:

\[ \vec{F} = m \frac{d^2 \vec{x}}{dt^2} \]

Hooke:

\[ \vec{F}_s = k(\vec{x} - \vec{y}) \]
First-Order System

\[ M \dddot{\mathbf{X}} = K \mathbf{X} \]

\[ \vec{M} \dddot{\vec{X}} = \vec{K} \vec{X} \]

\[ \left( \begin{array}{c} \vec{X} \\ \vec{V} \end{array} \right) = \left( \begin{array}{cc} 0 & I_{3n \times 3n} \\ M^{-1}K & 0 \end{array} \right) \left( \begin{array}{c} \vec{X} \\ \vec{V} \end{array} \right) \]
General ODE

\[ \vec{Y}' = B \vec{Y} \]
Eigenvector Solution

\[ \vec{y}' = B\vec{y} \]

\[ B\vec{y}_i = \lambda_i \vec{y}_i \]

\[ \vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k \]
Eigenvector Solution

\[ \vec{y}' = B\vec{y} \]

\[ B\vec{y}_i = \lambda_i \vec{y}_i \]

\[ \vec{y}(0) = c_1 \vec{y}_1 + \cdots + c_k \vec{y}_k \]

\[ \rightarrow \vec{y}(t) = c_1 e^{\lambda_1 t} \vec{y}_1 + \cdots + c_k e^{\lambda_k t} \vec{y}_k \]
Application: Modal Sound Synthesis

Major role in physics-based sound synthesis

https://www.youtube.com/watch?v=dMUHp8i6E5E
Organizing a Collection

(a) Database of photos

(b) Spectral embedding
Setup

Have: \( n \) items in a dataset

\[
\begin{align*}
  w_{ij} & \geq 0 \text{ similarity of items } i \text{ and } j \\
  w_{ij} & = w_{ji}
\end{align*}
\]

Want: \( x_i \) embedding on \( \mathbb{R} \)
Quadratic Energy

\[ E(\vec{x}) = \sum_{ij} w_{ij} (x_i - x_j)^2 \]
minimize $E(\vec{x})$
Optimization

\[
\begin{align*}
\text{minimize} & \quad E(\vec{x}) \\
\text{such that} & \quad ||\vec{x}||_2^2 = 1
\end{align*}
\]
minimize $E(\vec{x})$

such that $\|\vec{x}\|_2^2 = 1$

$\vec{1} \cdot \vec{x} = 0$
Simplification

\[ E(\vec{x}) = 2\vec{x}^\top (A - W)\vec{x} \]
Desired

Eigenvector of $A - W$ with second smallest eigenvalue.
**Definitions**

**Eigenvalue and eigenvector**

An *eigenvector* \( \vec{x} \neq \vec{0} \) of \( A \in \mathbb{R}^{n \times n} \) satisfies

\[
A\vec{x} = \lambda \vec{x}
\]

for some \( \lambda \in \mathbb{R} \); \( \lambda \) is an *eigenvalue*. *Complex* eigenvalues and eigenvectors instead have \( \lambda \in \mathbb{C} \) and \( \vec{x} \in \mathbb{C}^n \).
Definitions

Eigenvalue and eigenvector

An eigenvector \( \vec{x} \neq \vec{0} \) of \( A \in \mathbb{R}^{n \times n} \) satisfies \( A\vec{x} = \lambda \vec{x} \) for some \( \lambda \in \mathbb{R} \); \( \lambda \) is an eigenvalue. Complex eigenvalues and eigenvectors instead have \( \lambda \in \mathbb{C} \) and \( \vec{x} \in \mathbb{C}^n \).

Scale doesn’t matter!

\[ \vec{x} \] can constrain \( \| \vec{x} \|_2 \equiv 1 \)
Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$

(important!)
Eigenproblems in the Wild

- Optimize $\| A \vec{x} \|_2$ such that $\| \vec{x} \|_2 = 1$

  (important!)

- ODE/PDE problems: Closed solutions and approximations for $\vec{y}' = B \vec{y}$
Eigenproblems in the Wild

- Optimize $\|A\vec{x}\|_2$ such that $\|\vec{x}\|_2 = 1$
  
  *(important!)*

- ODE/PDE problems: Closed solutions and approximations for $\vec{y}' = B\vec{y}$

- Critical points of Rayleigh quotient:

\[
\frac{\vec{x}^\top A\vec{x}}{\|\vec{x}\|^2_2}
\]
Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.
Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

Lemma

Eigenvectors corresponding to distinct eigenvalues must be linearly independent.
Two Basic Properties

Proved in textbook

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

Lemma

Eigenvectors corresponding to distinct eigenvalues must be linearly independent.

→ at most $n$ eigenvalues
Diagonalizability

Nondefective

\( A \in \mathbb{R}^{n \times n} \) is nondefective or diagonalizable if its eigenvectors span \( \mathbb{R}^n \).
Diagonalizability

Nondefective

\( A \in \mathbb{R}^{n \times n} \) is nondefective or diagonalizable if its eigenvectors span \( \mathbb{R}^n \).

\[
D = X^{-1}AX
\]
Definitions

Spectrum and spectral radius

The *spectrum* of $A$ is the set of eigenvalues of $A$. The *spectral radius* $\rho(A)$ is the eigenvalue $\lambda$ maximizing $|\lambda|$. 
Extending to \( \mathbb{C}^{n \times n} \)

**Complex conjugate**

The *complex conjugate* of a number \( z = a + bi \in \mathbb{C} \) is \( \bar{z} \equiv a - bi \).
Extending to $\mathbb{C}^{n \times n}$

**Complex conjugate**

The *complex conjugate* of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

**Conjugate transpose**

The *conjugate transpose* of $A \in \mathbb{C}^{m \times n}$ is $A^H \equiv \bar{A}^\top$. 
Hermitian Matrix

\[ A = A^H \]
Lemma

All eigenvalues of Hermitian matrices are real.
Properties

Lemma
All eigenvalues of Hermitian matrices are real.

Lemma
Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.
Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian (if $A \in \mathbb{R}^{n \times n}$, suppose it is symmetric). Then, $A$ has exactly $n$ orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$. 
Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian (if $A \in \mathbb{R}^{n \times n}$, suppose it is symmetric). Then, $A$ has exactly $n$ orthonormal eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$.

**Full set:** $D = X^\top AX$
Matrix Inverse

\[ \vec{b} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ A \vec{x} = \vec{b} \]
Matrix Inverse

\[ \vec{b} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ A\vec{x} = \vec{b} \]

\[ \implies \vec{x} = \frac{c_1}{\lambda_1} \vec{x}_1 + \cdots + \frac{c_n}{\lambda_n} \vec{x}_n \]
Matrix Inverse

\[ \vec{b} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k \]

\[ A \vec{x} = \vec{b} \]

\[ \implies \vec{x} = \frac{c_1}{\lambda_1} \vec{x}_1 + \cdots + \frac{c_n}{\lambda_n} \vec{x}_n \]

\[ A = XD X^{-1} \implies A^{-1} = XD^{-1} X^{-1} \]
Matrix Square Root

- Given symmetric positive semi-definite (PSD) matrix, $U$
- Can compute matrix square root, $U^{1/2}$
Application: Polar decomposition

- Given real $n$-by-$n$ matrix, $A$
- There exists a unique factorization called the Polar Decomposition

$$A = RU$$

where $R$ is an $n$-by-$n$ orthogonal matrix, and $U$ is an $n$-by-$n$ symmetric PSD right “stretch” matrix.
- Also a left stretch matrix, $W$, such that $A = WR$.
- Geometric interpretation.
Application: Shape Matching

- **Fast Lattice Shape Matching** (Fast LSM)
- SIGGRAPH 2007 [Rivers and James 2007]
- [http://www.alecrivers.com/fastlsm](http://www.alecrivers.com/fastlsm)
- Need to compute orientation, $R$, of local particle groups
- Millions of polar decompositions (and eigenvalue decomp) per second