Norms, Sensitivity and Conditioning

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Announcements

- Reminder: HW0 due Thursday 11:59pm.
- `http://cs205a.stanford.edu` now works
- JuliaBox fine for online. Try JUNO offline.

Questions

Gaussian elimination works in theory, but what about floating point precision?

How much can we trust $\vec{x}_0$ if $0 < \|A\vec{x}_0 - \vec{b}\| \ll 1$?

Recall: Backward Error

Backward Error

The amount a problem statement would have to change to realize a given approximation of its solution

Example 1: $\sqrt{x}$

Example 2: $A\vec{x} = \vec{b}$

Perturbation Analysis

How does $\vec{x}$ change if we solve
$$(A + \delta A)\vec{x} = \vec{b} + \delta\vec{b}?$$

Two viewpoints:

- Thanks to floating point precision, A and $\vec{b}$ are approximate
- If $\vec{x}_0$ isn't the exact solution, what is the backward error?

What is "Small?"

What does it mean for a statement to hold for *small* $\delta\vec{x}$?

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Vector norm

A function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ satisfying:

1. $\|\vec{x}\| = 0$ iff $\vec{x} = 0$
2. $\|c\vec{x}\| = |c|\|\vec{x}\| \; \forall c \in R, \vec{x} \in \mathbb{R}^n$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \; \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

Our Favorite Norm

$$\|\vec{x}\|_2 \equiv \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

p-Norms

For $p \geq 1$,

$$\|\vec{x}\|_p \equiv \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

Taxicab norm: $\|\vec{x}\|_1$

∞ Norm

$$\|\vec{x}\|_\infty \equiv \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$$

How are Norms Different?

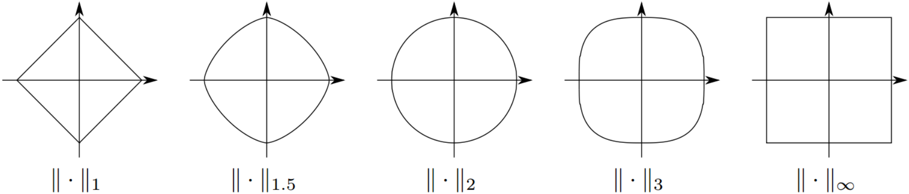

Figure 4.7 The set $\{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\| = 1\}$ for different vector norms $\|\cdot\|$.

How are Norms the Same?

Equivalent norms

Two norms $\|\cdot\|$ and $\|\cdot\|'$ are *equivalent* if there exist constants c_{low} and c_{high} such that
$c_{low}\|\vec{x}\| \leq \|\vec{x}\|' \leq c_{high}\|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.

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Theorem

All norms on $\mathbb{R}^n$ are equivalent.

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All norms on $\mathbb{R}^n$ are equivalent.

$(10000, 1000, 1000)$ vs. $(10000, 0, 0)$?

Matrix Norms: "Unrolled" Construction

Convert to vector, and use vector p-norm:

$$A \in \mathbb{R}^{m \times n} \leftrightarrow \texttt{a[:]} \in \mathbb{R}^{mn}$$

- Achieved by `vecnorm(A, p)` in Julia.

Special Case: Frobenius norm (p=2):

$$\|A\|_{\text{Fro}} \equiv \sqrt{\sum_{ij} a_{ij}^2}$$

Matrix Norms: "Induced" Construction

Maximum stretching of a unit vector by A:

$$\|A\| \equiv \max\{\|A\vec{x}\| : \|\vec{x}\| = 1\}$$

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Different matrix norms induced by different vector p-norms.

Case p=2: What is the norm induced by $\|\cdot\|_2$?

Matrix Norms: $\|A\|_2$ Visualization

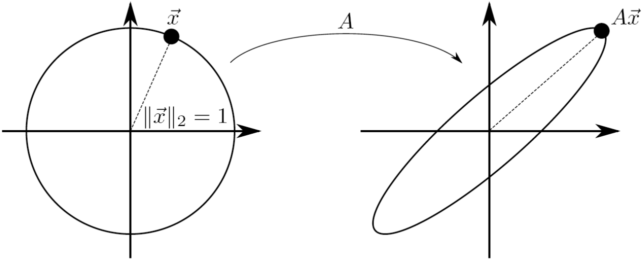

Figure 4.8 The norm $\|\cdot\|_2$ induces a matrix norm measuring the largest distortion of any point on the unit circle after applying A.

Induced two-norm, or *spectral norm*, of $A \in \mathbb{R}^{n\times n}$ is the square root of the largest eigenvalue of A^TA:

$$\|A\|_2^2 = \max\{\lambda : \text{there exists } \vec{x} \in \mathbb{R}^n \text{ with } A^TA\vec{x} = \lambda\vec{x}\}$$

Other Induced Norms

$$\|A\|_1 \equiv \max_j \sum_i |a_{ij}|$$

$$\|A\|_\infty \equiv \max_i \sum_j |a_{ij}|$$

Question

Are all matrix norms equivalent?

Recall: Condition Number

Condition number

Ratio of forward to backward error

Root-finding example:

$$\frac{1}{f'(x^*)}$$

Model Problem

$$(A + \varepsilon\, \delta A)\, \vec{x}(\varepsilon) = \vec{b} + \varepsilon\, \delta\vec{b}$$

Simplification (on the board!)

$$\left.\frac{d\vec{x}}{d\varepsilon}\right|_{\varepsilon=0} = A^{-1}(\delta\vec{b} - \delta A\ \vec{x}(0))$$

$$\frac{\|\vec{x}(\varepsilon) - \vec{x}(0)\|}{\|\vec{x}(0)\|} \leq |\varepsilon|\|A^{-1}\|\|A\|\left(\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} + \frac{\|\delta A\|}{\|A\|}\right) + O(\varepsilon^2)$$

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Relative change: $D \equiv \frac{\delta\vec{b}}{\|\vec{b}\|} + \frac{\|\delta A\|}{\|A\|}$

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Invariant to scaling (unlike determinant!); equals one for the identity.

Condition Number of Induced Norm

$$\operatorname{cond} A = \frac{\max_{\vec{x}\neq\vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}}{\min_{\vec{y}\neq\vec{0}} \frac{\|A\vec{y}\|}{\|\vec{y}\|}} = \frac{\max_{\|\vec{x}\|=1} \|A\vec{x}\|}{\min_{\|\vec{y}\|=1} \|A\vec{y}\|}$$

Condition Number: Visualization

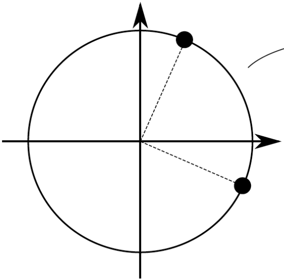

Figure 4.9 The condition number of A measures the ratio of the largest to smallest distortion of any two points on the unit circle mapped under A.

Experiments with an ill-conditioned Vandermonde matrix

Chicken $\Longleftrightarrow$ Egg

$$\operatorname{cond} A \equiv \|A\| \boxed{\|A^{-1}\|}$$

Computing $\|A^{-1}\|$ typically requires solving $A\vec{x} = \vec{b}$, but how do we know the reliability of $\vec{x}$?

To Avoid...

What is the condition number of computing the condition number of A?

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$\vdots$

Instead

Bound the condition number.

- Below: Problem is at *least* this hard
- Above: Problem is at *most* this hard

Potential for Approximation

$$\|A^{-1}\vec{x}\| \leq \|A^{-1}\| \|\vec{x}\|$$

$$\Downarrow$$

$$\operatorname{cond} A = \|A\| \|A^{-1}\| \geq \frac{\|A\| \|A^{-1}\vec{x}\|}{\|\vec{x}\|}$$

Next