

Ordinary Differential Equations II

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

Doug James (and Justin Solomon)

Almost Done!

Today is last lecture!

- ▶ **Homework 7:** Due tonight by midnight
 - ▶ No extensions!
- ▶ **Office hours:** Same for me this week (Tu/Th)
- ▶ **Final exam:** Sat June 10, 3:30-6:30pm
(Room: NVIDIA Auditorium)
 - ▶ Covers all material—emphasis on 2nd half
 - ▶ Two cheatsheets—can reuse one from midterm
 - ▶ Similar format to midterm
 - ▶ OAE: Tentatively: 2:00pm start (pls confirm time)
 - ▶ SCPD: Arrange similar time (or slightly later date)

Study Tips

- ▶ Practice
Especially on modeling problems.
- ▶ Do sanity checks
Come up with a function and minimize it. Factor a matrix.
- ▶ Implement and experiment
Try it out in Julia.
- ▶ Iterate
Look at solution, put it away, and try again.
- ▶ Time your practices
- ▶ Ask questions

Initial Value Problems

Find $f(t) : \mathbb{R} \rightarrow \mathbb{R}^n$

Satisfying $F[t, f(t), f'(t), f''(t), \dots, f^{(k)}(t)] = 0$

Given $f(0), f'(0), f''(0), \dots, f^{(k-1)}(0)$

Most Famous Example

$$\vec{F} = m\vec{a}$$

Newton's second law

Reasonable Assumption

Explicit ODE

An ODE is *explicit* if can be written in the form

$$f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \dots, f^{(k-1)}(t)].$$

After Reduction

$$\frac{d\vec{y}}{dt} = F[\vec{y}]$$

Forward Euler

$$\vec{y}_{k+1} = \vec{y}_k + hF[\vec{y}_k]$$

- ▶ Explicit method
- ▶ $O(h^2)$ localized truncation error
- ▶ $O(h)$ global truncation error;
“first-order accurate”

Backward Euler

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Trapezoid Method

$$\vec{y}_{k+1} = \vec{y}_k + h \frac{F[\vec{y}_k] + F[\vec{y}_{k+1}]}{2}$$

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- ▶ $O(h^3)$ localized truncation error
- ▶ $O(h^2)$ global truncation error;
“second-order accurate”

Example

$$\vec{y}' = A\vec{y}$$

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$$\longrightarrow \vec{y}_{k+1} = \left(I_{n \times n} - \frac{hA}{2} \right)^{-1} \left(I_{n \times n} + \frac{hA}{2} \right) \vec{y}_k$$

Model Equation

$$y' = ay, a < 0 \longrightarrow y_{k+1} = y_k + h \frac{ay_k + ay_{k+1}}{2}$$

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Unconditionally stable!

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But this has nothing to do with accuracy.

Is Stability Useful Here?

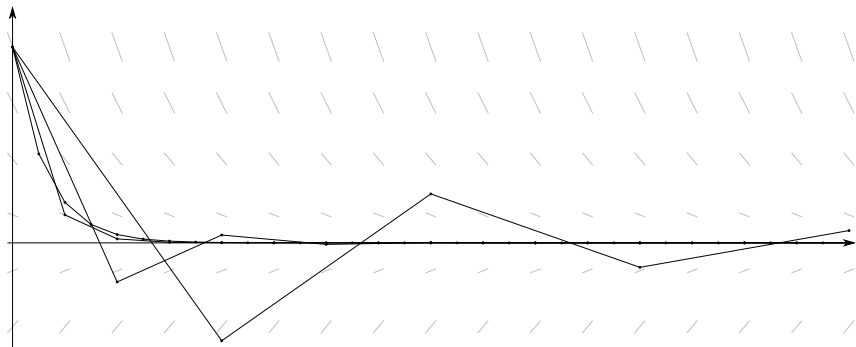
$$R = \frac{y_{k+1}}{y_k}$$

Is Stability Useful Here?

$$R = \frac{y_{k+1}}{y_k}$$

Oscillatory behavior!

Example



Useful for mid-size time steps

Pattern

Convergence as $h \rightarrow 0$ is not the whole story nor is stability. Often should consider behavior for fixed $h > 0$.

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Qualitative and quantitative analysis needed!

Observation

$$\vec{y}_{k+1} = \vec{y}_k + \int_{t_k}^{t_k+h} F[\vec{y}(t)] dt$$

Alternative Derivation of Trapezoid Method

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}(F[\vec{y}_k] + F[\vec{y}_{k+1}]) + O(h^3)$$

Implicit method

Heun's Method

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{2}(F[\vec{y}_k] + F[\vec{y}_k + hF[\vec{y}_k]]) + O(h^3)$$

Replace implicit part with lower-order integrator!

Stability of Heun's Method

$$y' = ay, a < 0$$
$$y_{k+1} = \left(1 + ha + \frac{1}{2}h^2a^2 \right) y_k$$

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$$\implies h < \frac{2}{|a|}$$

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$$\text{Forward Euler: } h < \frac{2}{|a|}$$

Runge-Kutta Methods

Apply quadrature to:

$$\vec{y}_{k+1} = \vec{y}_k + \int_{t_k}^{t_{k+1}} F[\vec{y}(t)] dt$$

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Use low-order integrators

RK4: Simpson's Rule

$$\vec{y}_{k+1} = \vec{y}_k + \frac{h}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$$

where $\vec{k}_1 = F[\vec{y}_k]$

$$\vec{k}_2 = F\left[\vec{y}_k + \frac{1}{2}h\vec{k}_1\right]$$

$$\vec{k}_3 = F\left[\vec{y}_k + \frac{1}{2}h\vec{k}_2\right]$$

$$\vec{k}_4 = F\left[\vec{y}_k + h\vec{k}_3\right]$$

Fehlberg's method (15.11(a))

$$\vec{y}^{(1)} = \vec{y}_0 + \frac{h}{2}(\vec{v}_1 + \vec{v}_2) = y(h) + O(h^3)$$

$$\vec{y}^{(2)} = \vec{y}_0 + \frac{h}{6}(\vec{v}_1 + \vec{v}_2 + 4\vec{v}_3) = y(h) + O(h^4)$$

where

$$\vec{v}_1 = F[\vec{y}_0]$$

$$\vec{v}_2 = F[\vec{y}_0 + h\vec{v}_1]$$

$$\vec{v}_3 = F\left[\vec{y}_0 + \frac{h}{4}(\vec{v}_1 + \vec{v}_2)\right]$$

Proof by Mathematica.

Using the Model Equation

$$\vec{y}' \approx A\vec{y}$$
$$\implies \vec{y}(t) \approx e^{At}\vec{y}_0$$

Exponential Integrators

$$\vec{y}' = A\vec{y} + G[\vec{y}]$$

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First-order exponential integrator:

$$\vec{y}_{k+1} = e^{Ah}\vec{y}_k + A^{-1}(e^{Ah} - I_{n \times n})G[\vec{y}_k]$$

Returning to Our Reduction

$$\vec{y}'(t) = \vec{v}(t)$$

$$\vec{v}'(t) = \vec{a}(t)$$

$$\vec{a}(t) = F[t, \vec{y}(t), \vec{v}(t)]$$

Returning to Our Reduction

$$\vec{y}'(t) = \vec{v}(t)$$

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$$\vec{a}(t) = F[t, \vec{y}(t), \vec{v}(t)]$$

$$\vec{y}(t_k + h) = \vec{y}(t_k) + h\vec{y}'(t_k) + \frac{h^2}{2}\vec{y}''(t_k) + O(h^3)$$

Don't need high-order estimators for derivatives.

New Formulae

$$\vec{v}_{k+1} = \vec{v}_k + \int_{t_k}^{t_{k+1}} \vec{a}(t) dt$$

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$$\vec{a}(\tau) = (1 - \gamma)\vec{a}_k + \gamma\vec{a}_{k+1} + \vec{a}'(\tau)(\tau - h\gamma - t_k) + O(h^2)$$

for $\tau \in [a, b]$

Newmark Class

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \left(\frac{1}{2} - \beta\right) h^2 \vec{a}_k + \beta h^2 \vec{a}_{k+1}$$

$$\vec{v}_{k+1} = \vec{v}_k + (1 - \gamma)h\vec{a}_k + \gamma h\vec{a}_{k+1}$$

$$\vec{a}_k = F[t_k, \vec{y}_k, \vec{v}_k]$$

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Parameters: β, γ (can be implicit or explicit!)

Constant Acceleration: $\beta = \gamma = 0$

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_k$$

$$\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_k$$

Explicit, exact for constant acceleration, linear accuracy

Constant Implicit Acceleration:

$$\beta = 1/2, \gamma = 1$$

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_k + \frac{1}{2}h^2\vec{a}_{k+1}$$

$$\vec{v}_{k+1} = \vec{v}_k + h\vec{a}_{k+1}$$

Backward Euler for velocity, midpoint for position; globally first-order accurate

Trapezoid: $\beta = 1/4, \gamma = 1/2$

$$\vec{x}_{k+1} = \vec{x}_k + \frac{1}{2}h(\vec{v}_k + \vec{v}_{k+1})$$

$$\vec{v}_{k+1} = \vec{v}_k + \frac{1}{2}h(\vec{a}_k + \vec{a}_{k+1})$$

Trapezoid for position and velocity; globally second-order accurate

Central Differencing: $\beta = 0, \gamma = 1/2$

$$\vec{v}_{k+1} = \frac{\vec{y}_{k+2} - \vec{y}_k}{2h}$$
$$\vec{a}_{k+1} = \frac{\vec{y}_{k+2} - 2\vec{y}_{k+1} + \vec{y}_k}{h^2}$$

Central difference for position and velocity;
globally second-order accurate

Newmark Schemes

- ▶ Generalizes large class of integrators
- ▶ Second-order accuracy (exactly) when $\gamma = 1/2$
- ▶ Unconditional stability when $4\beta > 2\gamma > 1$ (otherwise radius is function of β, γ)

Staggered Grid

$$\vec{y}_{k+1} = \vec{y}_k + h\vec{v}_{k+1/2}$$

$$\vec{v}_{k+3/2} = \vec{v}_{k+1/2} + h\vec{a}_{k+1}$$

$$\vec{a}_{k+1} = F \left[t_{k+1}, \vec{x}_{k+1}, \frac{1}{2}(\vec{v}_{k+1/2} + \vec{v}_{k+3/2}) \right]$$

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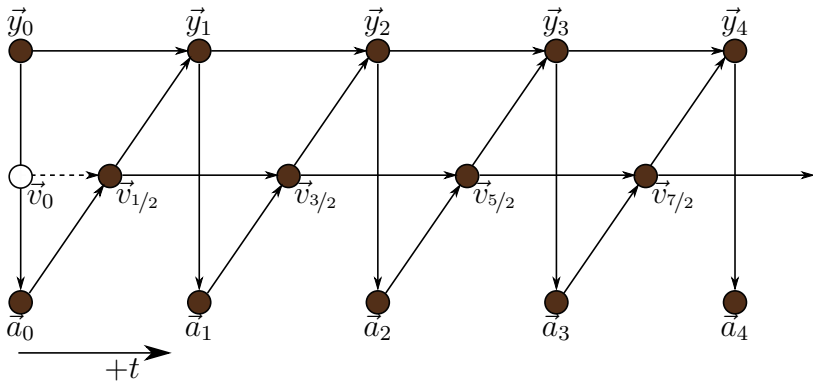
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Otherwise: Symmetry in \vec{v} dependence

Leapfrog Integration

[▶ Next](#)