Final Examination

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2016), Stanford University

- The exam runs for 3 hours.
- The exam contains seven problems. You must complete the first problem and five of problems 2-7. **Circle the problems you want graded on the chart below; otherwise we will grade the first five questions on which you have provided any written answer.**
- The exam is closed-book. You may use two double-sided $8^1/_2'' \times 11''$ sheets of notes.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet and indicate that you have done so. Additional pages are provided at the back; please indicate on the original problem if you continue your answer there.
- Do not spend too much time on any problem. Read them all before beginning.
- Show your work, as partial credit will be awarded.

Circle the five additional problems you want graded.

Problem	(1)	2	3	4	5	6	7	Total
Score								

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

    (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

    (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Signature

Name

SUID

You must complete this problem.

Problem 1 (Short answer).

(a) Given a quadrature rule of the form $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$ what is a quadrature rule for $\int_a^b f(x)dx$ expressed in terms of the w_i and x_i values? [2 points]

(b) (Heath 4.29) Consider the roots of the function $f(\vec{x}, \lambda) = \begin{pmatrix} A\vec{x} - \lambda\vec{x} \\ \vec{x}^\top\vec{x} - 1 \end{pmatrix}$ mapping $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Derive an expression for the Newton update $\begin{pmatrix} \delta\vec{x}_k \\ \delta\lambda_k \end{pmatrix}$ where $\begin{pmatrix} \vec{x}_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} \vec{x}_k \\ \lambda_k \end{pmatrix} + \begin{pmatrix} \delta\vec{x}_k \\ \delta\lambda_k \end{pmatrix}$.
(Hint: First derive the Jacobian $J(\vec{x}, \lambda) \in \mathbb{R}^{(n+1)\times(n+1)}$.) [3 points]

You must complete this problem.

(c) Consider the degree-n polynomial $p(x) = \sum_{k=0}^{n} C_k x^k$ where $C_n = 1$. Its *companion matrix* is the n-by-n sparse matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -C_0 & -C_1 & -C_2 & \ldots & -C_{n-1} \end{bmatrix}.$$

Show that if λ_j is a root of p, i.e., $p(\lambda_j) = 0$, then $\vec{v} = [1 \quad \lambda_j \quad \lambda_j^2 \quad \ldots \quad \lambda_j^{n-1}]^\top$ is an eigenvector of M corresponding to eigenvalue λ_j. [2 points]

(d) Give **three reasons** why it may be preferrable to compute the roots of a degree-n polynomial $p(x)$ by computing the eigenvalues of M instead of using root-finding techniques. [3 points]

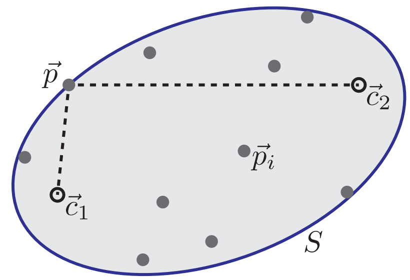
(1)

(2)

(3)

Circle which five of problems 2-7 you want graded on page 1.

Problem 2 (Optimization).

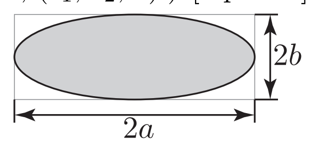


Consider the problem of finding the smallest ellipse enclosing n points $\vec{p}_i \in \mathbb{R}^2,\ i = 1 \ldots n$. The ellipse can be parameterized by $(\vec{c}_1,\, \vec{c}_2,\, s)$ where the location of its two focii are $\vec{c}_1$ and $\vec{c}_2 \in \mathbb{R}^2$, and s is the length of an inextensible virtual string (*dashed line*) which connects the two focii to any point $\vec{p}$ on the ellipse's surface, S. If we define a string-like length function as

$$\ell(\vec{p}, \vec{c}_1, \vec{c}_2) \equiv \|\vec{p} - \vec{c}_1\|_2 + \|\vec{p} - \vec{c}_2\|_2,$$

then all points $\vec{p} \in S$ satisfy $s = \ell(\vec{p}, \vec{c}_1, \vec{c}_2)$. Furthermore, all points inside the ellipse will have ℓ values smaller than s.

(a) Write down an optimization problem to determine the parameters of **the smallest-area ellipse that encloses all n points**. Note that an ellipse has major axis length $2a = s$, minor axis length $2b$, and area πab. (Hint: Express b in terms of ellipse parameters, $(\vec{c}_1,\, \vec{c}_2,\, s)$.) [5 points]



(b) Write down the KKT conditions for your optimization problem. [5 points]

Circle which five of problems 2-7 you want graded on page 1.

Problem 3 (ODE).

(Solomon 15.11(a)) "Fehlberg's method:" Suppose we carry out a single time step of $\vec{y}\,' = F[\vec{y}]$ with size h starting from $\vec{y}(0) = \vec{y}_0$. Make the following definitions:

$$\begin{aligned}\vec{v}_1 &\equiv F[\vec{y}_0]\\ \vec{v}_2 &\equiv F[\vec{y}_0 + h\vec{v}_1]\\ \vec{v}_3 &\equiv F[\vec{y}_0 + \frac{h}{4}(\vec{v}_1 + \vec{v}_2)].\end{aligned}$$

We can write two estimates of $\vec{y}(h)$:

$$\begin{aligned}\vec{y}^{\,(1)} &\equiv \vec{y}_0 + \frac{h}{2}(\vec{v}_1 + \vec{v}_2)\\ \vec{y}^{\,(2)} &\equiv \vec{y}_0 + \frac{h}{6}(\vec{v}_1 + \vec{v}_2 + 4\vec{v}_3).\end{aligned}$$

(a) Show that there is some constant $K \in \mathbb{R}$ such that $\vec{y}^{\,(1)} = \vec{y}(h) + Kh^3 + O(h^4)$. [5 points]

(b) Show that $\vec{y}^{(2)} = \vec{y}(h) + O(h^4)$. [5 points]

Circle which five of problems 2-7 you want graded on page 1.

Problem 4 (Gauss-Newton Matrix Inversion).

Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular and $X_0 \in \mathbb{R}^{n\times n}$ is given. The iteration defined by

$$X_{k+1} = X_k(2I - AX_k)$$

is the matrix analog of Newton's method for the function $f(x) = a - (1/x)$. Define the error at the k^{th} step by $E_k \equiv A^{-1} - X_k$, and the (easier to compute) residual by $R_k \equiv I - AX_k$.

(a) Derive an expression for R_{k+1} in terms of R_k. [3 points]

(b) Derive an expression for E_{k+1} in terms of E_k. [3 points]

(c) What is a good choice for X_0? Why? [4 points]

Circle which five of problems 2-7 you want graded on page 1.

Problem 5 (Extrapolating to Zero). Recall that Richardson extrapolation and Romberg integration exploit the specific polynomial structure of the error versus h to reduce truncation error. Consider the case where you can numerically evaluate an expensive function $f(h)$, but only know that the leading order error is of the form

$$f(h) \approx f_0 + Ch^\alpha$$

where $f_0 = f(0)$, but C and $\alpha > 0$ are unknown. Given three function evaluations

$$f_1 = f(h_1), \qquad f_2 = f(h_2), \qquad f_3 = f(h_3),$$

at

$$h_1 = h, \qquad h_2 = 2h, \qquad h_3 = 4h,$$

derive a method to estimate f_0. (Hint: First solve for α, then C, and finally f_0.) [10 points]

Circle which five of problems 2-7 you want graded on page 1.

Problem 6 (Integration and Differentiation).

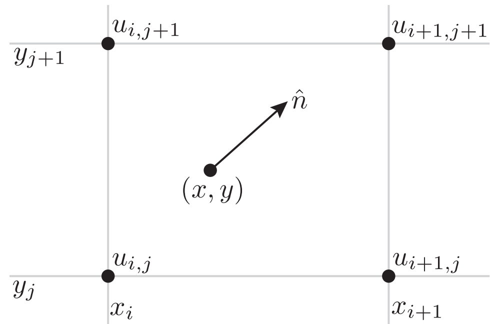
(a) (C&K 11.2.9) The problem

$$\begin{aligned} x''(t) &= x + y - 2x' + 3y' + logt \\ y''(t) &= 2x - 3y + 5x' + ty' - sint \\ x(0) &= 1, \qquad x'(0) = 2 \\ y(0) &= 3, \qquad y'(0) = 4 \end{aligned}$$

is to be put into the form of an autonomous system of five first-order equations, i.e., without explicit t dependence. Give the resulting system and the appropriate initial values. [4 points]

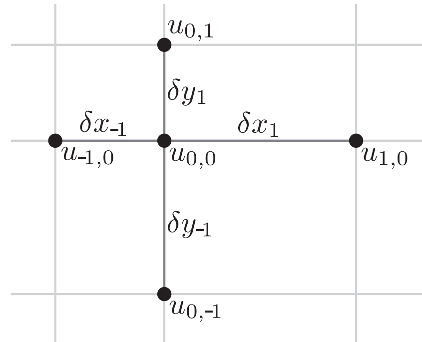
(b) Derive a numerical integration formula of the form $\int_{-h}^{h} f(x)dx \approx Af(0) + Bf'(-h) + Cf''(h)$, that is exact for polynomials of as high a degree as possible. [3 points]

(c) Consider a lattice of square cells with vertices $(x_i, y_j) = (ih, jh)$, and a function $u(x, y)$ discretized at uniformly spaced vertices, $u_{i,j} = u(x_i, y_j)$. Provide an estimate for the directional derivative, $\frac{\partial u}{\partial n} = \hat{n} \cdot \nabla u$ (where $\hat{n}$ is a unit vector) at a location (x, y), where $x \in [x_i, x_{i+1}]$ and $y \in [y_j, y_{j+1}]$ and so (x, y) is in the cell with corner values $u_{i,j}$, $u_{i+1,j}$, $u_{i,j+1}$ and $u_{i+1,j+1}$. (Hint: Compute the gradient of a bilinear interpolant.) [3 points]



Circle which five of problems 2-7 you want graded on page 1.

Problem 7 (PDEs).



In this problem you will discretize the 2D Laplacian on a graded mesh, i.e., a nonuniform 2D rectilinear grid (see figure). Specifically you will determine the 5-point finite-difference approximation to the Laplacian $\nabla^2 u$ on nonuniform grids with varying δx and δy spacing.

(a) First, consider the simpler 1D problem of interpolating $u(x)$ at three nonuniform locations,

$$(x, u): \quad (-\delta x_{-1},\, u_{-1}),\;\; (0,\, u_0),\;\; (\delta x_1,\, u_1).$$

Give an expression for a quadratic 1D interpolant, $f(x)$, passing through these values. [4 points]

(b) Derive an expression for the interpolant's second derivative at zero, $f''(0)$. [4 points]

(c) Use this result to write an expression for the Laplacian $\nabla^2 u = u_{xx} + u_{yy}$. (Hint: Evaluate it at the origin $(0, 0)$, and use neighboring values at $(\pm\delta x_{\pm 1}, 0)$ and $(0, \pm\delta y_{\pm 1})$.) [2 points]

This page is for additional work.

Problem number:

Problem number: