Midterm Examination

CS 205A: Mathematical Methods for Robotics, Vision, and Graphics (Spring 2016), Stanford University

- The exam runs for 75 minutes.
- The exam contains six problems. You must complete the first problem and four of problems 2-6. **CIRCLE THE PROBLEMS YOU WANT GRADED ON THE CHART BELOW; OTHERWISE WE WILL GRADE THE FIRST FOUR QUESTIONS ON WHICH YOU HAVE PROVIDED ANY WRITTEN ANSWER.**
- The exam is closed book/notes. You may use one double-sided $8^1/_2'' \times 11''$ sheet of notes. No calculators may be used.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem and indicate that you have done so.
- Do not spend too much time on any problem. Read them all before beginning.
- Show your work, as partial credit will be awarded.

Circle the four additional problems you want graded.

Problem	(1)	2	3	4	5	6	Total
Score (/10 each)							

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

(a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

Signature

Name

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YOU MUST COMPLETE THIS PROBLEM.

Problem 1 (True or False). [1 point each]

(1) A matrix with non-unique eigenvalues is defective.

(2) Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalue decomposition $A = XDX^\top$, then its singular value decomposition is $A = XDX^\top$ too.

(3) If two matrices A and B commute, so that $AB = BA$, then analytic matrix functions of A and B also commute, $f(A)f(B) = f(B)f(A)$. (You can assume that a convergent Taylor series of f exists everywhere.)

(4) The eigenvalues of an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ are all 1.

(5) The eigenvalues of a Householder reflection matrix $H \in \mathbb{R}^{n\times n}$ are either 1 or -1.

(6) (Heath Rev 3.3) An over-determined linear least-squares problem $A\vec{x} = \vec{b}$ always has a unique solution $\vec{x}$ that minimizes the Euclidean norm of the residual vector $\vec{r} = \vec{b} - A\vec{x}$.

(7) (Heath Rev 3.8) If $Q \in \mathbb{R}^{n\times n}$ is a Householder transformation matrix, and $\vec{x} \in \mathbb{R}^n$ is an arbitrary vector, then the last k components of the vector $Q\vec{x}$ can be made zero for some $0 < k < n$.

(8) $\text{cond}(cA) = |c|\text{cond}(A)$.

(9) The float-pointing machine epsilon, ε_{mach} (the smallest positive number such that $1 + \varepsilon_{mach} = 1$ in floating-point arithmetic) is approximately $\varepsilon_{mach} \approx |3. * (4./3. - 1.) - 1.|$

(10) (Heath Rev 2.14) If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.

CIRCLE WHICH FOUR OF PROBLEMS 2-6 YOU WANT GRADED ON PAGE 1.

Problem 2 (Short Answers).

(a) Consider the harmonic-series partial sum,

$$S_n = \sum_{m=1}^{n} \frac{1}{m}, \qquad n = 1, 2, 3, \ldots,$$

which is evaluated using a simple **for** loop (with increasing m), in a floating-point implementation with machine epsilon ε_{mach}. How large does n need to be taken before S_n stops increasing? What is the largest value that S_n will achieve? [5 points]

(b) (Like Heath 3.23) Given a matrix $A \in \mathbb{R}^{m \times n}$ with linearly independent columns, show that $A(A^\top A)^{-1}A^\top$ is an orthogonal projection onto the column space of A. (Hint: Use the SVD $A = U\Sigma V^\top$ if you are stuck, but there's a shorter way.) [5 points]

CIRCLE WHICH FOUR OF PROBLEMS 2-6 YOU WANT GRADED ON PAGE 1.

Problem 3 (Matrix Factorizations).

(a) Computing the determinant $\det(A)$ of a general square matrix $A \in \mathbb{R}^{n\times n}$ involves $O(n!)$ operations using the naive approach, which is prohibitive for large matrices. Alternately, show how to compute $\det(A)$ much more efficiently by using an LU factorization of A with partial row pivoting, i.e., $PA = LU$. (Hint: Use $\det(AB) = \det(A)\det(B)$.) [5 points]

(b) Different factorizations have different memory costs. Consider the factorizations

Thin SVD   Partial-pivoted LU QR

for a general *non-symmetric* square matrix $A \in \mathbb{R}^{n\times n}$ (assuming they exist). Arrange them from left-to-right in increasing order of memory cost (# numbers stored) and state whether they are roughly equal ($\approx$) or less than ($<$), i.e., "X $<$ Y $\approx$ Z." (Note: Only consider the number of values which are nonzero and *need* to be stored, e.g., square dense matrices have n^2 entries, triangular have $(n^2 \pm n)/2$ entries (fewer if unit diagonal), diagonal matrices have n entries, etc.) [2 points]

(c) Next, consider the case of *symmetric* matrices, and rank the memory costs of storing the following decompositions

Eigenvalue Decomp. $LL^\top$ $\tilde{L}D\tilde{L}^\top$ QR

(assume that the transpose of an already stored matrix is "free"). [3 points]

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Problem 4 (Matrix two-norms).

Given a matrix $A \in \mathbb{R}^{n\times n}$, show that appending an additional column $\vec{a}$ to form the matrix $A' = [A\ \vec{a}] \in \mathbb{R}^{n\times(n+1)}$ can only increase the matrix's two-norm, i.e., show that $\|A'\|_2 \geq \|A\|_2$. (Hint: The two norm is the largest singular value, which is related to the largest eigenvalue of a particular symmetric matrix.) [10 points]

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Problem 5 (QR and Least squares). (From Heath 3.23) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$, where $\varepsilon < \sqrt{\epsilon_{mach}}$, and ϵ_{mach} is the machine precision of your floating-point number system. In this question you will compare the behavior of the normal equations and QR for solving a least-squares problem $A\vec{x} = \vec{b}$.

(a) Normal equations: Show that the matrix $A^\top A$ is singular in floating-point arithmetic. (Hint: You can say that $1 + \varepsilon^2 \approx 1$ to simplify your floating point expressions.) [2 points]

(b) Compute the QR factorization and show that it is numerically nonsingular. Do so by applying two Householder reflections to A to obtain an upper triangular but non-square matrix, $R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \in \mathbb{R}^{3\times 2}$ where $R_1 \in \mathbb{R}^{2\times 2}$ is the square upper triangular block of R corresponding to the reduced QR factorization $A = Q_1 R_1$. Show that R_1 is nonsingular in floating-point arithmetic. (Note: You do not need to construct Q or Q_1.) [8 points]

CIRCLE WHICH FOUR OF PROBLEMS 2-6 YOU WANT GRADED ON PAGE 1.

Problem 6 (Thin SVD and Eigenvalue Problems). In this problem you will calculate the Thin SVD, $A = \bar{U}\bar{\Sigma}\bar{V}^\top$, for the 3-by-2 matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

in three stages:

(a) Using their definition, compute the two singular values (σ_1 and σ_2) of A by computing the eigenvalues of a related 2-by-2 eigenvalue problem. [4 points]

(b) Compute the two right singular vectors $\hat{v}_i \in \mathbb{R}^2$, such that $\bar{V} = [\hat{v}_1\ \hat{v}_2] \in \mathbb{R}^{2\times 2}$. [3 points]

(c) Compute the two left singular vectors $\vec{u}_i$ using their relation to $\hat{v}_i$, and then normalizing them to get $\hat{u}_i$, such that $\bar{U} = [\hat{u}_1\ \hat{u}_2] \in \mathbb{R}^{3\times 2}$. [3 points]