CS233, CME251: Geometric and Topological Data Analysis

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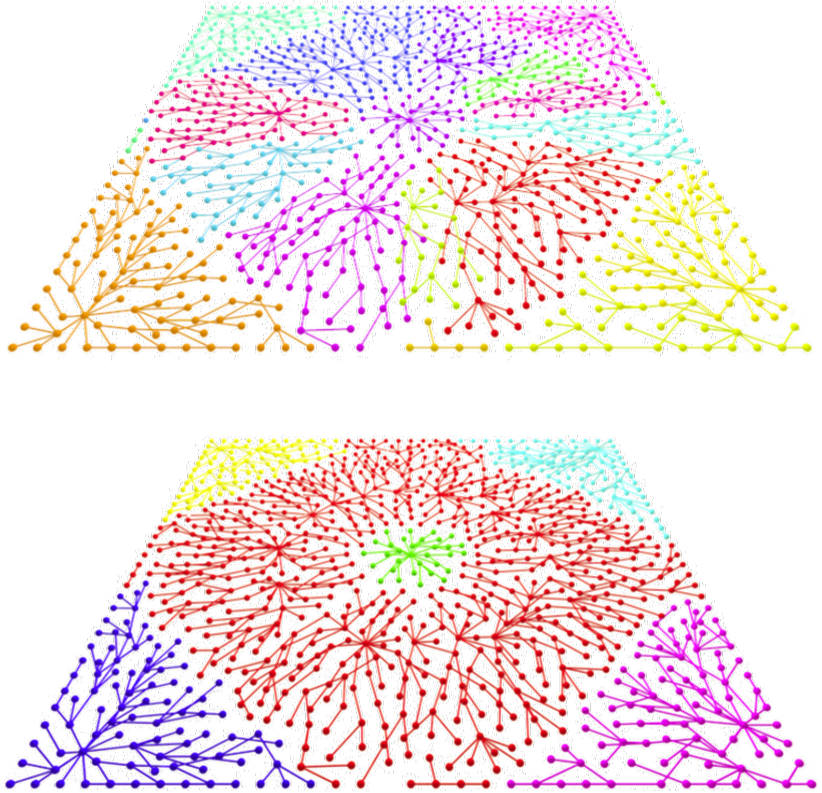
1891

Lecture 7
27 April 2020

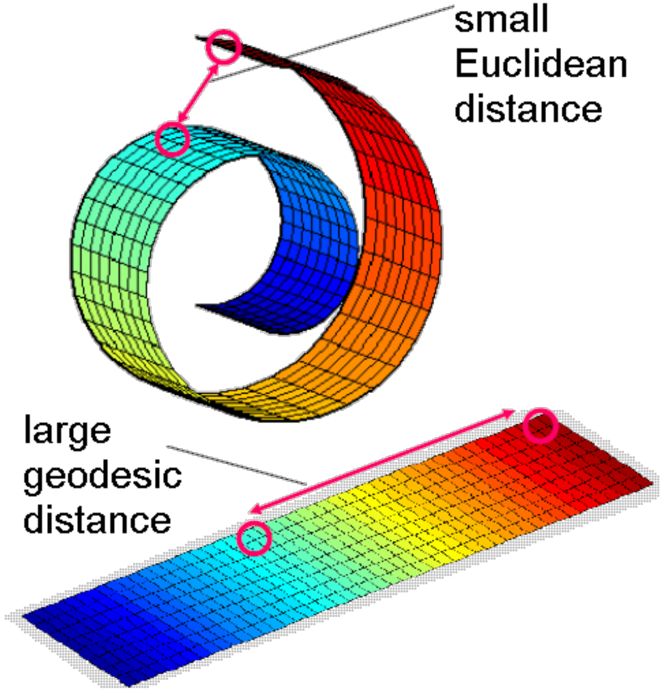


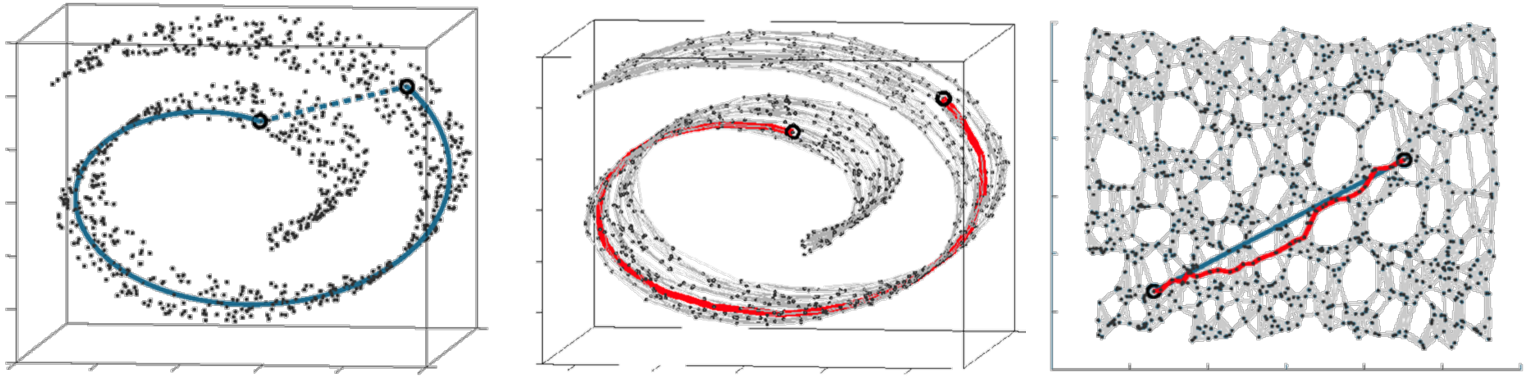
Last Time: Non-Linear Dimensionality Reduction

Isomap, Laplacian Eigenmaps, Locally Linear Embeddings, t-SNE

ISOMAP (J. B. Tenenbaum, V. de Silva, and J. C. Langford)

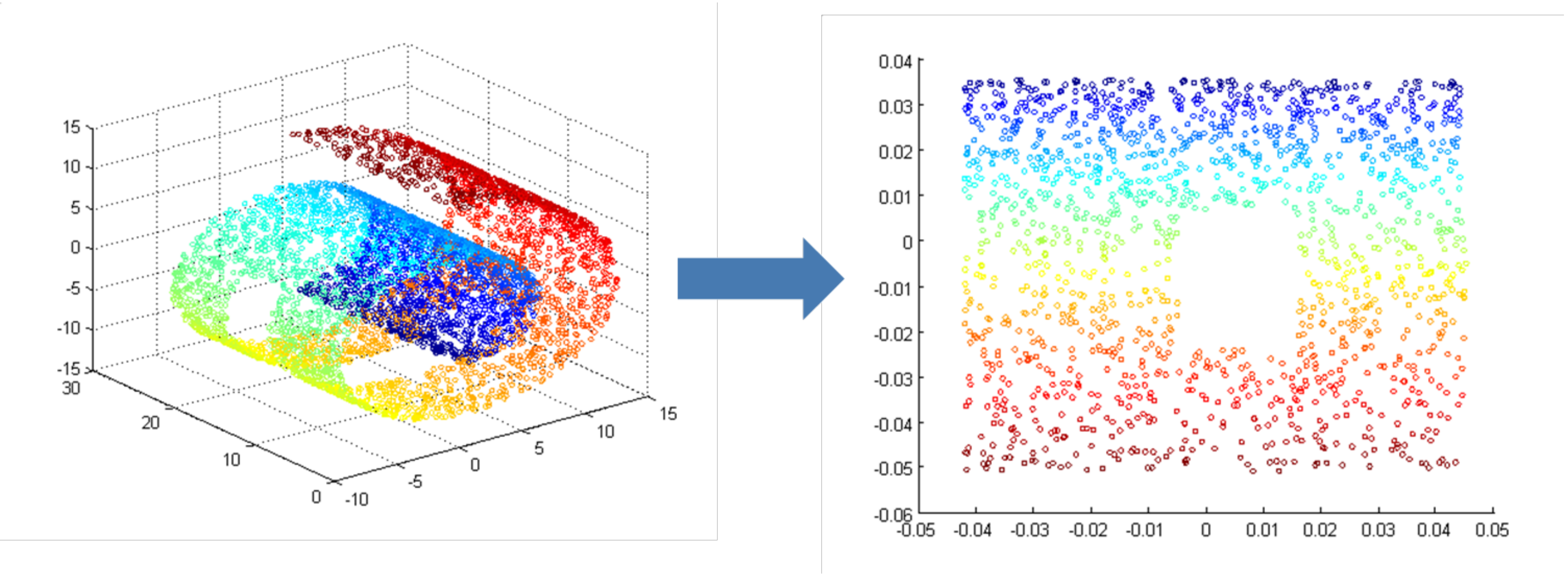
- Example of non-linear structure (Swiss roll)
  - Only the geodesic distances reflect the true low-dimensional geometry of the manifold
- ISOMAP (Isometric Feature Mapping)
  - Uses the geodesic manifold distances between all pairs
  - Preserves the intrinsic geometry of the data -- Preserves the geodesic distances
  - How to estimate geodesic distances between point pairs?





Laplacian Eigenmaps (M. Belkin and P. Niyogi)

- Start same as Isomap, but use a spectral embedding in lieu of MDS



Hole distorts long geodesic distances, but affects less diffusion distances

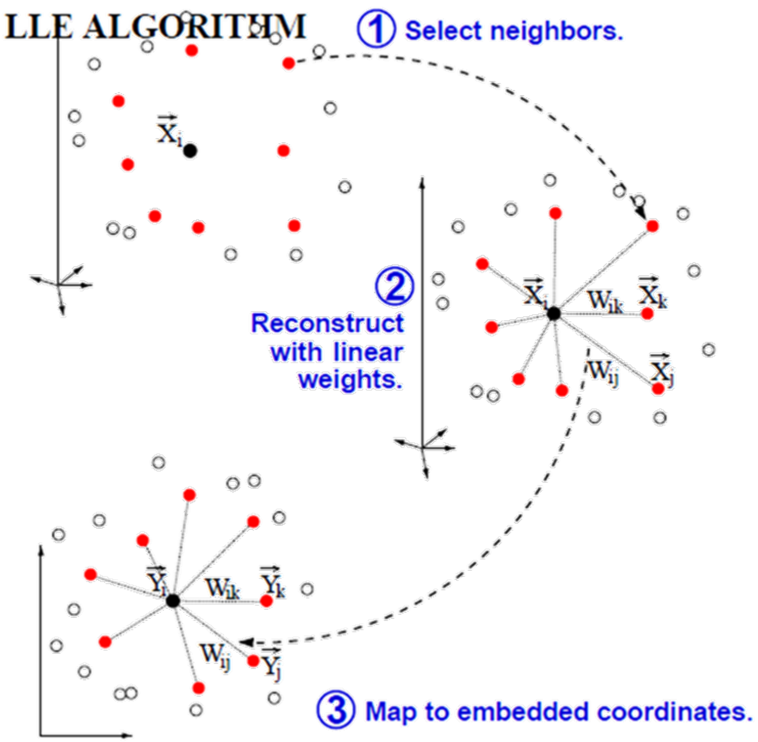
Locally Linear Embeddings (LLE) (S. T. Roweis and L. K. Saul)

- Define neighborhood relations between points (build NN graph)
  - *k* nearest neighbors
  - ε-balls
- Find weights that reconstruct each data point from its neighbors:

$$\min_{\sum_j w_{ij}=1} \left\| \mathbf{x_i} - \sum_{j\in N(i)} w_{ij}\mathbf{x_j} \right\|^2$$

- Find low-dimensional coordinates so that the same weights hold: $\mathbf{x_1}', \ldots, \mathbf{x_n}' \in R^d$

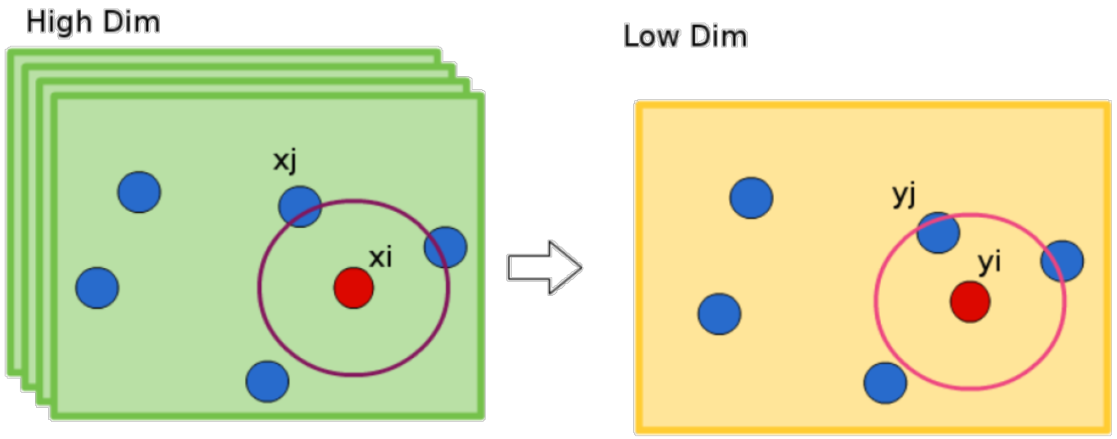
$$\min_{\mathbf{x_1}',\ldots,\mathbf{x_n}'} \sum_i \left\| \mathbf{x_i}' - \sum_{j\in N(i)} w_{ij}\mathbf{x_j}' \right\|^2$$

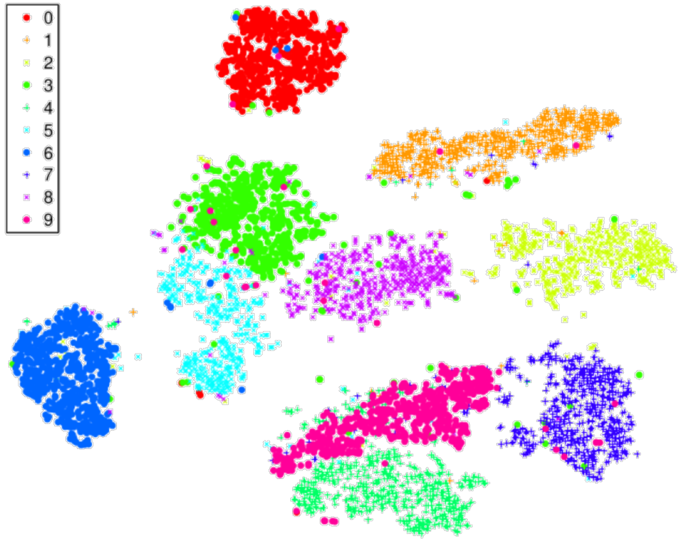


$$x_i' = Y_i$$

t-Distributed Stochastic Neighbor Embedding

Measure pairwise similarities between high-dimensional and low-dimensonal objects





- Similarity of datapoints in High Dimension

$$p_{ij} = \frac{exp(-||x_i - x_j||^2/2\sigma^2)}{\sum_{k \neq l} exp(-||x_l - x_k||^2/2\sigma^2)}$$

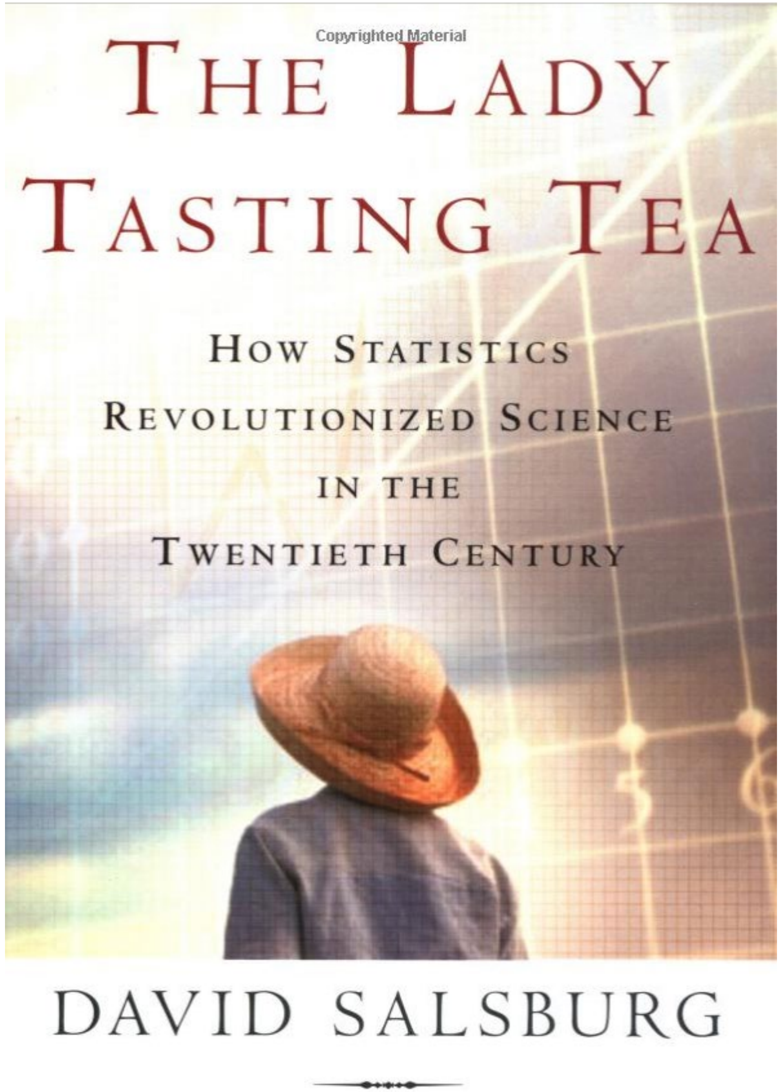
- Similarity of datapoints in Low Dimension

$$q_{ij} = \frac{(1 + ||y_i - y_j||^2)^{-1}}{\sum_{k \neq l}(1 + ||y_k - y_l||^2)^{-1}}$$

Heavy tail Student t-distribution
William Sealy Gosset

$$C = \sum_i KL(P_i||Q_i) = \sum_i \sum_j p_{j|i} log \frac{p_{j|i}}{q_{j|i}}$$

History of Statistics



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THE LADY TASTING TEA

HOW STATISTICS REVOLUTIONIZED SCIENCE IN THE TWENTIETH CENTURY

DAVID SALSBURG

Today:
Topological Data Analysis
(TDA)

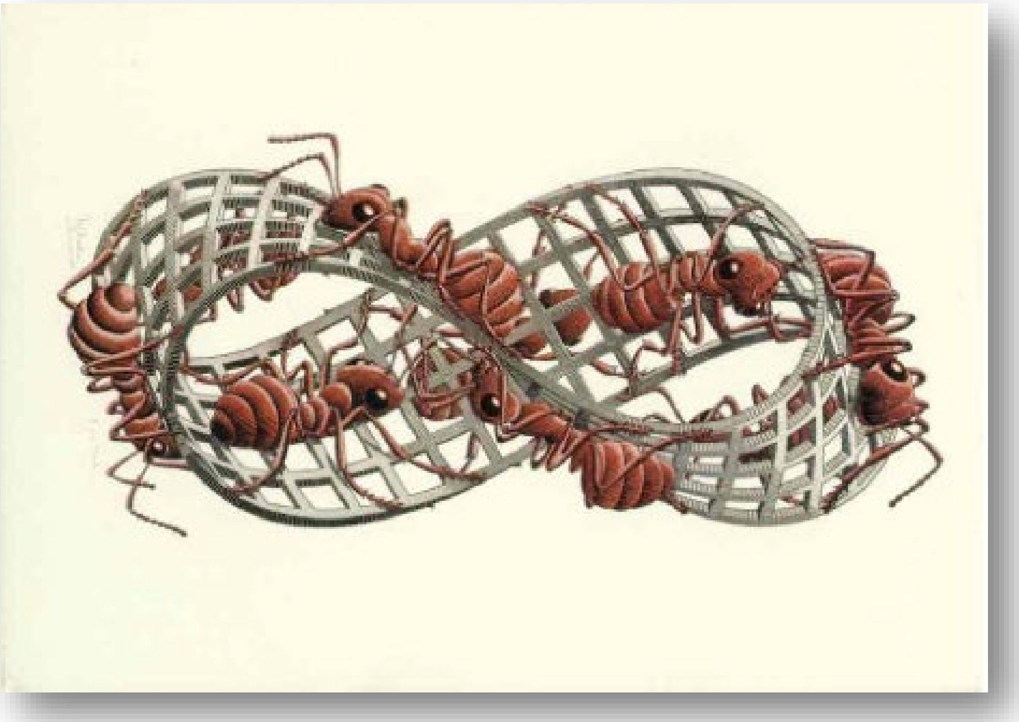
Slides ack: Afra Zomorodian, Ryan Lewis,
Gunnar Carlsson, Samir Chowdhury

Topology and Data Analysis

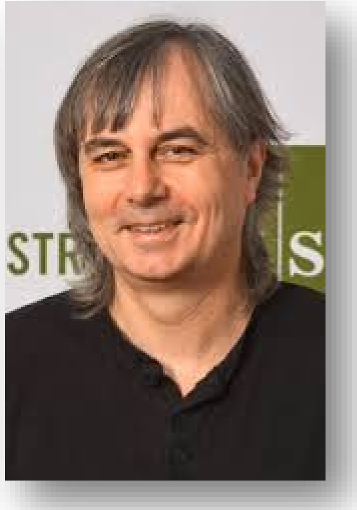
Topology — Computational Topology

Homology — **Persistent Homology**

Topological Data Analysis Applications

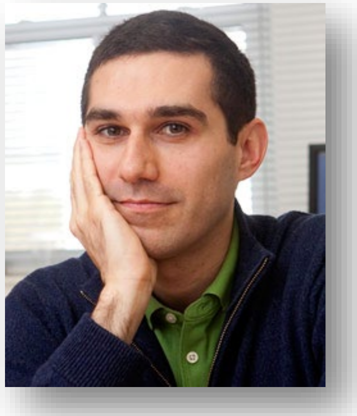


Computational Topology and Data Analysis



Herbert Edelsbrunner

Gunnar Carlsson



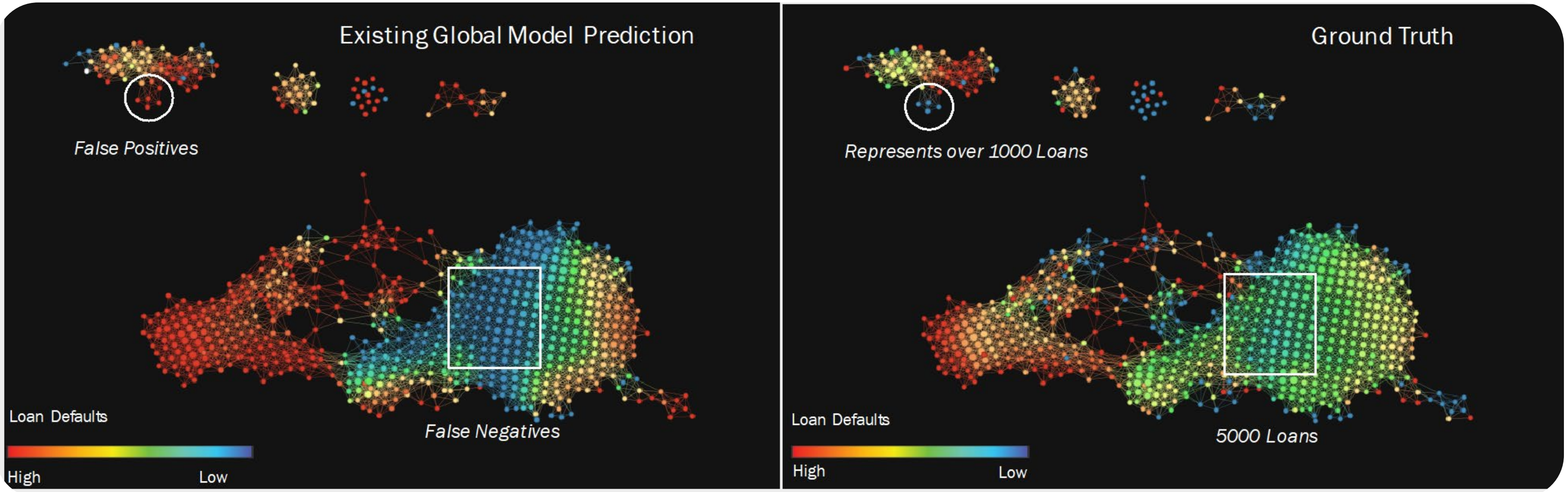
Afra Zomorodian

Frederic Chazal

1990 -- 2020

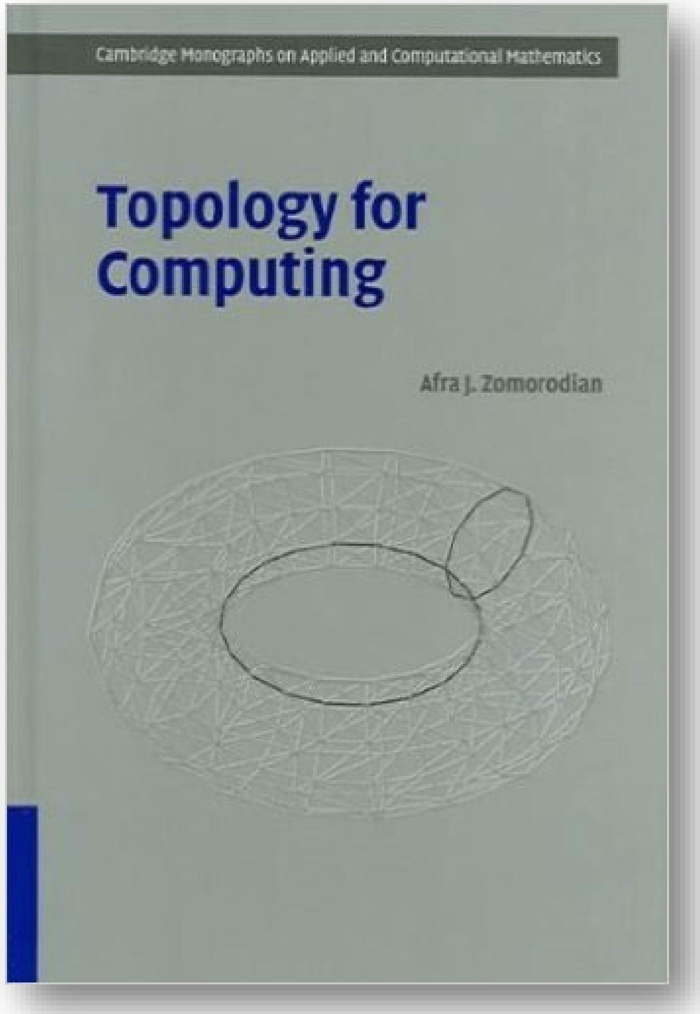
Robert Ghrist

Some Companies



AYASDI

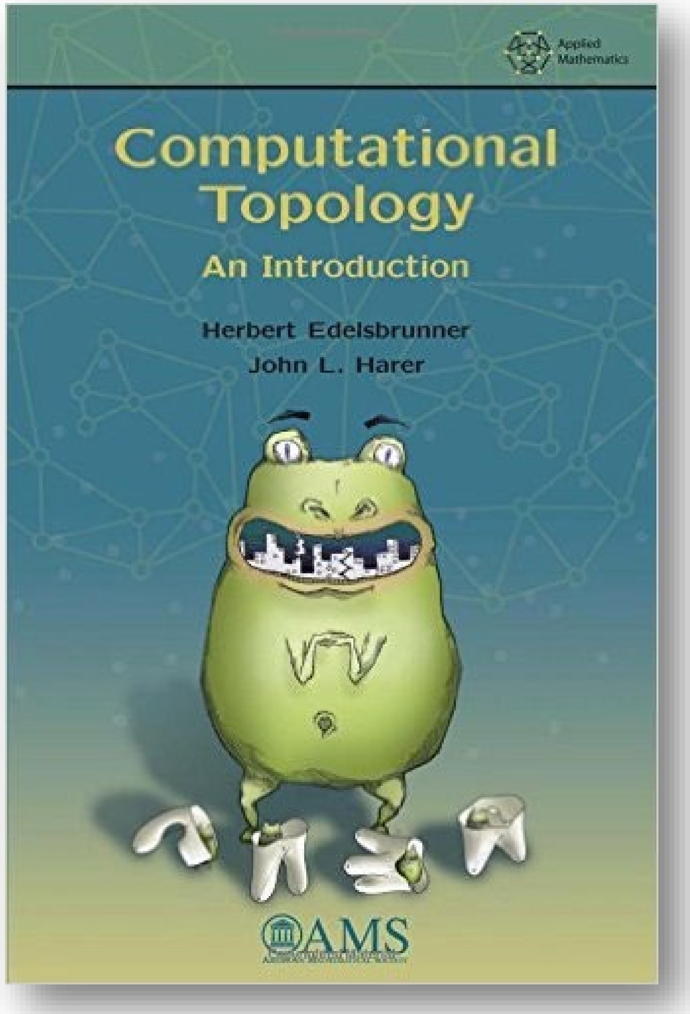
Some Textbooks



Cambridge Monographs on Applied and Computational Mathematics

Topology for Computing

Afra J. Zomorodian



Applied Mathematics

Computational Topology

An Introduction

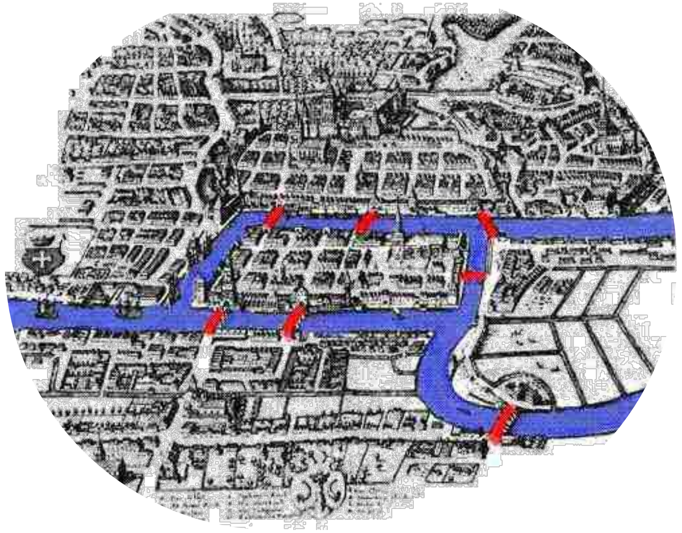
Herbert Edelsbrunner

John L. Harer

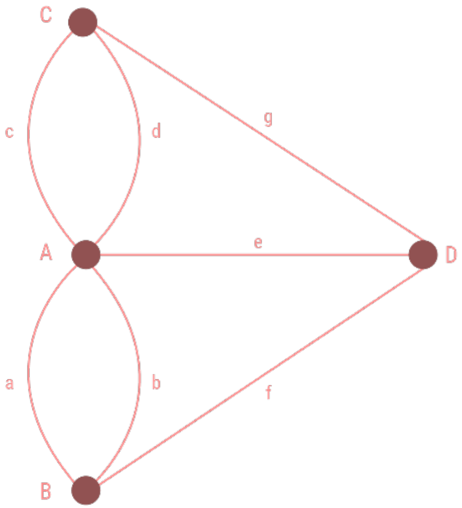
AMS

Topology Studies Connectivity

… and obstructions to connectivity

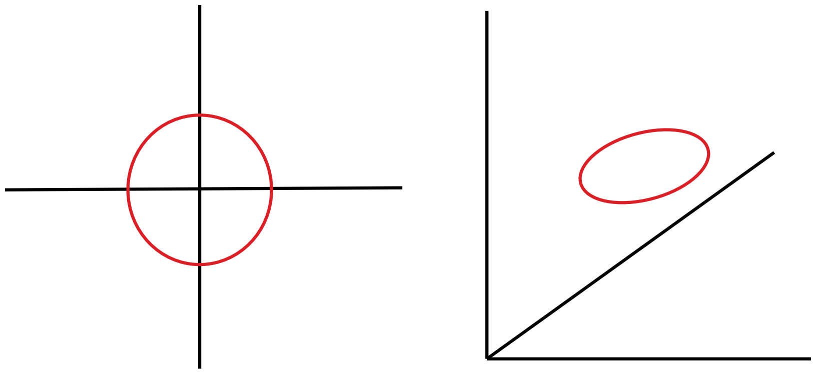


The bridges of
Köningsberg (Kalingrad)



Leonhard Euler
1736

Topological Invariances



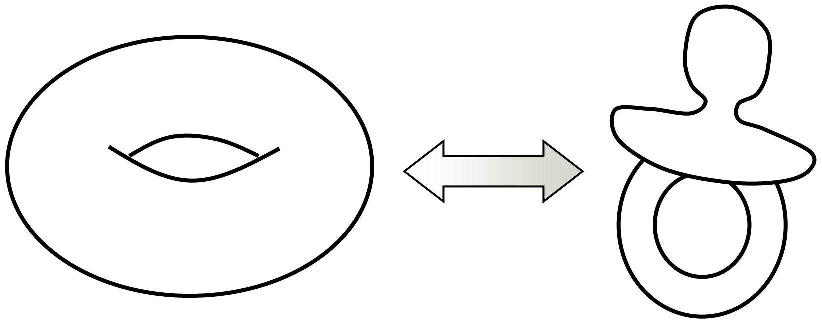
Deformation-invariant

Coordinate-free

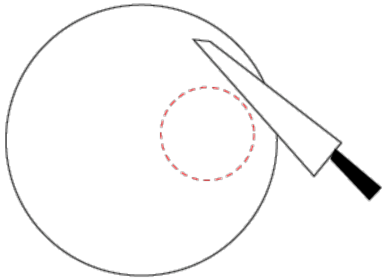
Topology does not take distances too seriously

2-Manifolds: Ordinary Surfaces

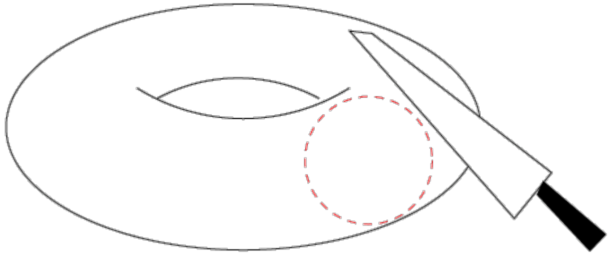
- We are allowed to stretch and shrink
  - Homeomorphism: 1-1, onto, bi-continuous



- But we care about cutting, puncturing, stitching, gluing …



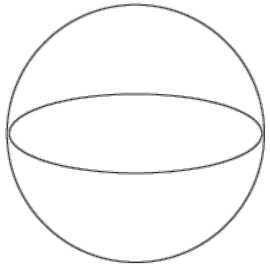
(a) No matter where we cut the sphere, we get two pieces



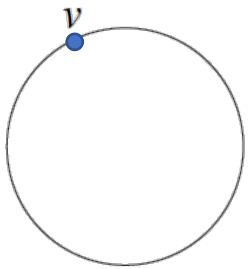
(b) If we're careful, we can cut the torus and still leave it in one piece.

- Note: connectivity information is indexed by dimension

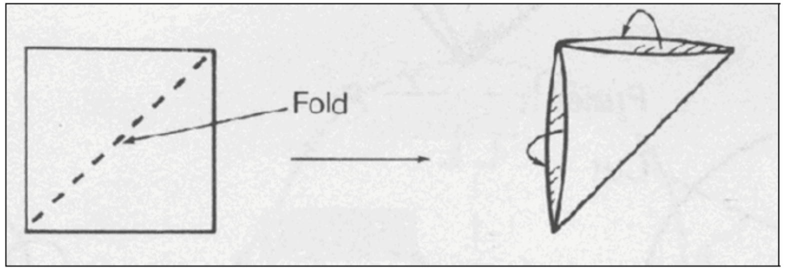
2-Manifold Zoo



(a) $\{x \in \mathbb{R}^3 \mid |x| = 1\}$

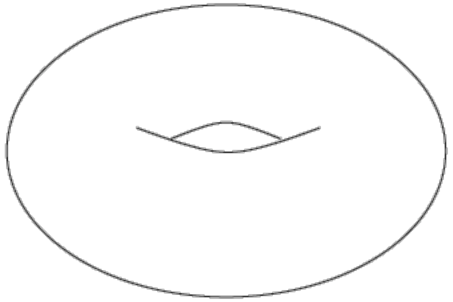


(b) Identify boundary to v

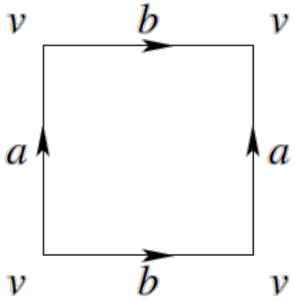


(c) Instructions for a flat sphere

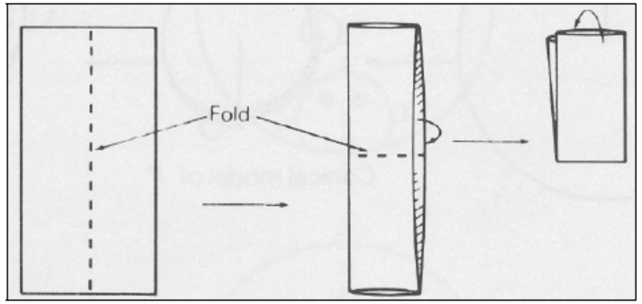
Sphere



(a) Donut surface



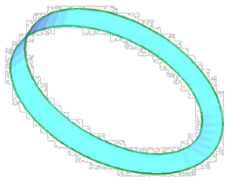
(b) Diagram



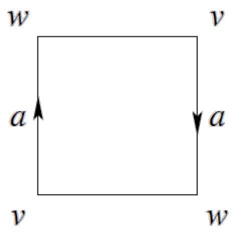
(c) Instructions for a flat torus

Torus

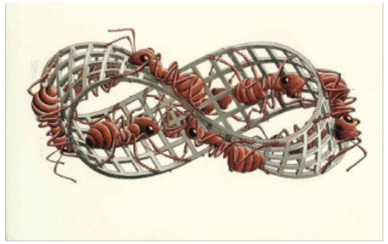
More Exotic Animals



(a) Embedded

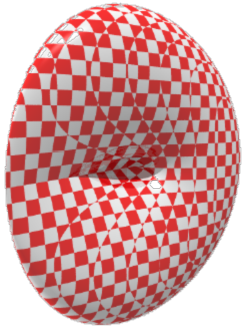


(b) Diagram

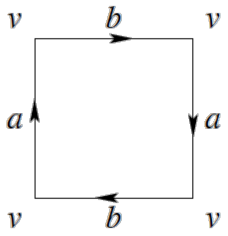


(c) Escher's *Möbius Strip II* (on its side)

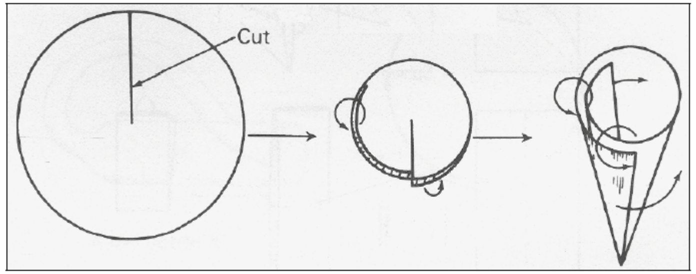
Möbius strip



Cross cap + disk

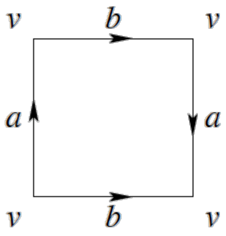


(a) Diagram

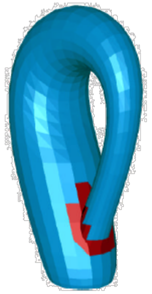


(b) Instructions for a flat $\mathbb{RP}^2$

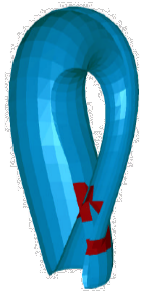
Projective plane



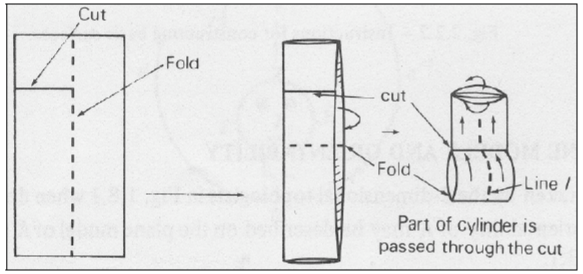
(a) Diagram



(b) An immersion



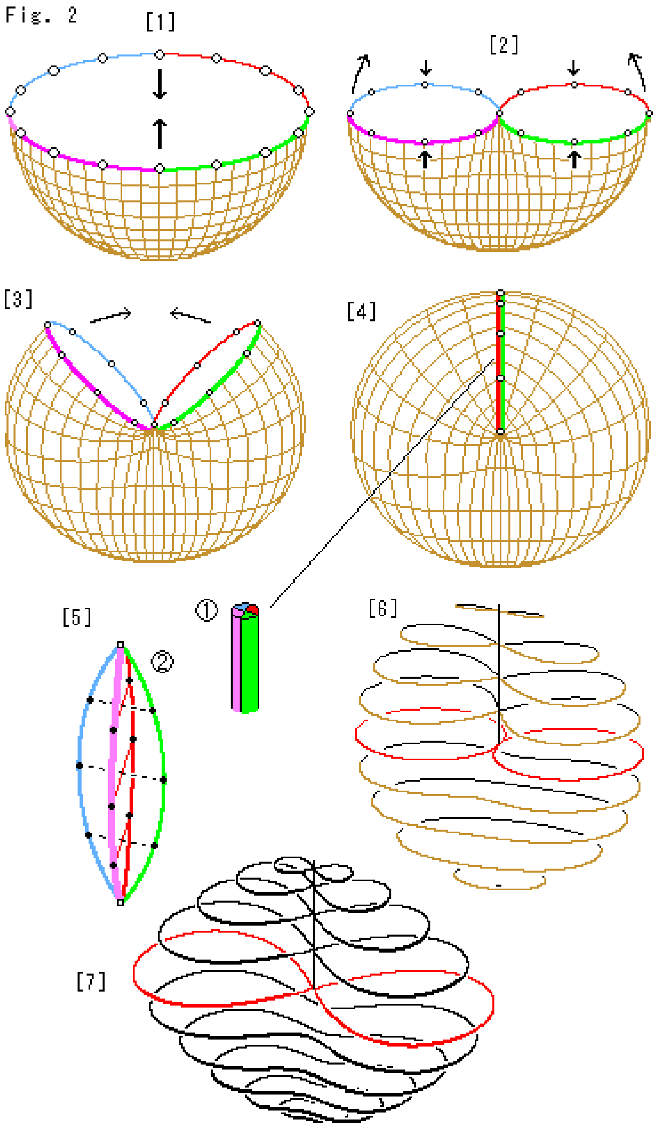
(c) Cut in half (a Möbius strip)



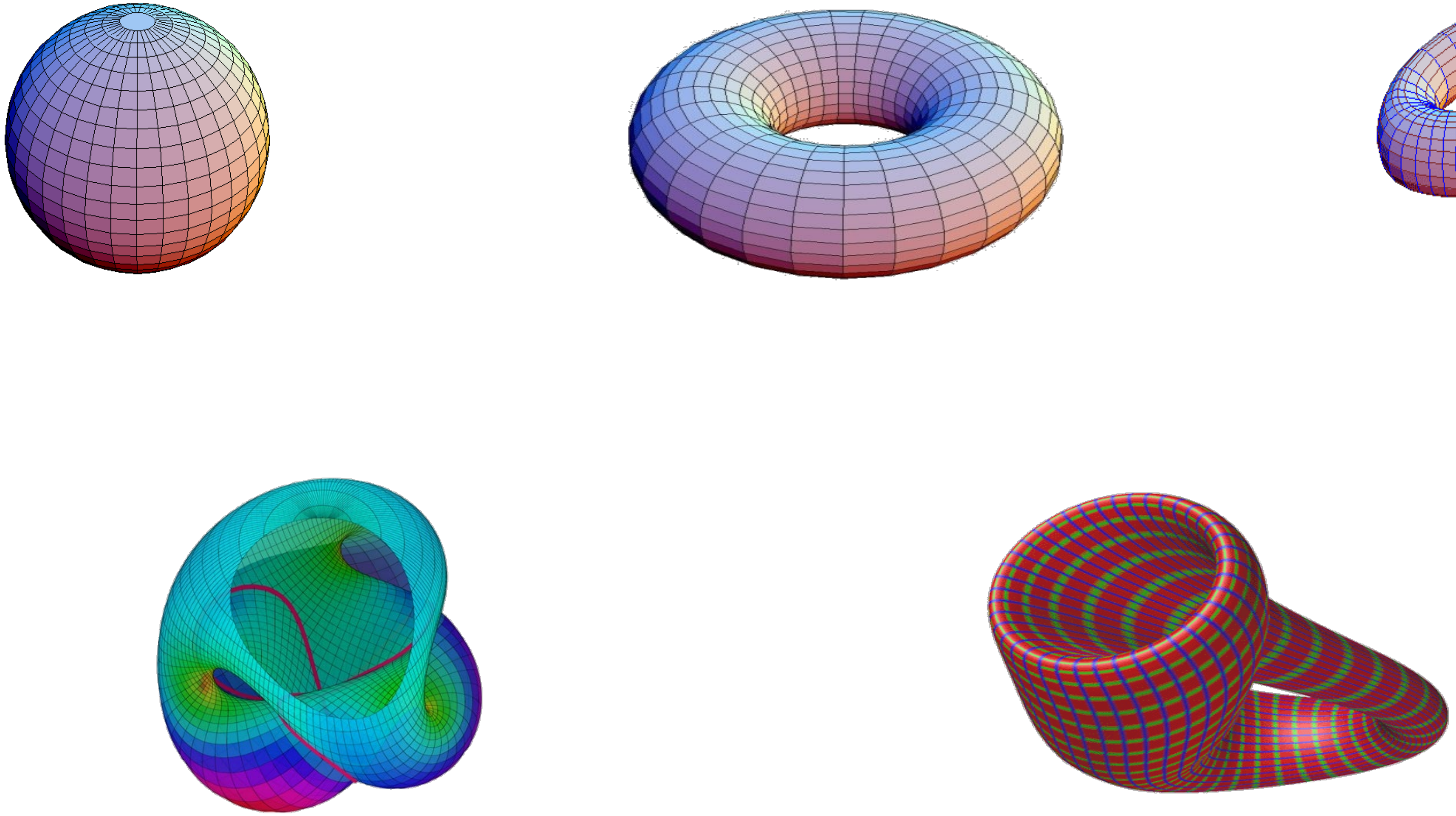
(d) Instructions for a flat $\mathbb{K}^2$

Klein bottle

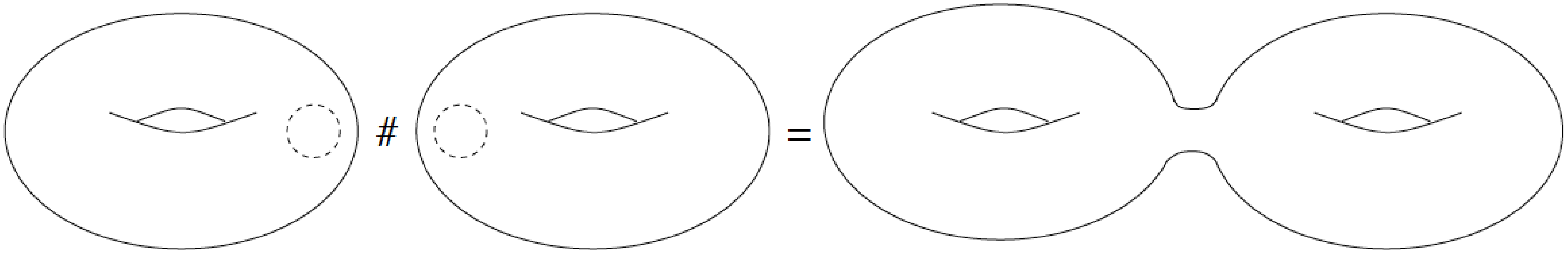
Projective Plane Detail

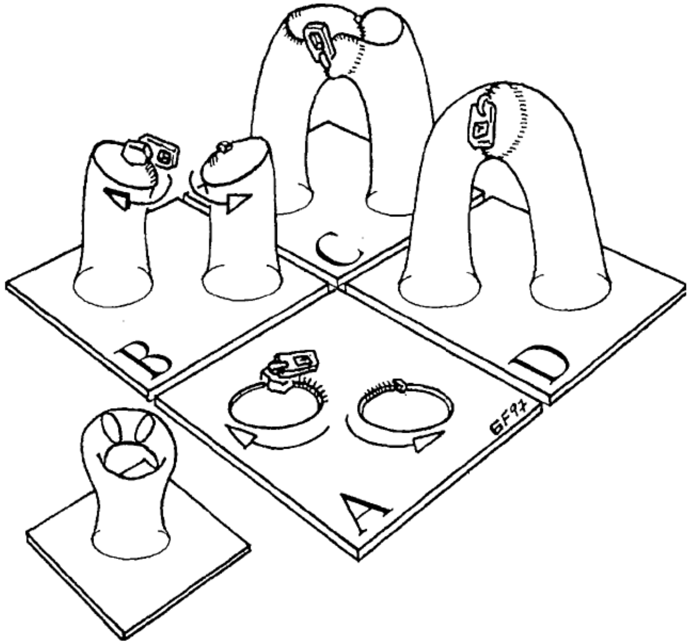


Topology Aims to Classify

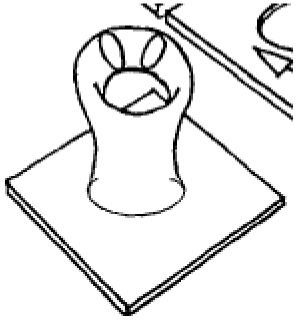


Connected Sums



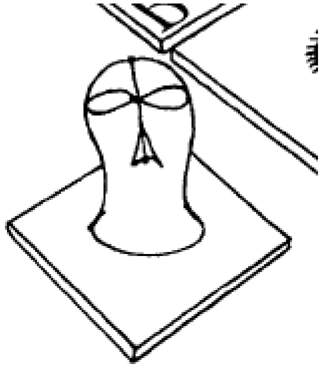


- **Classification Theorem of 2-Manifolds (1860):** Every closed connected compact surface is a connected sum of a sphere with a number of tori and projective planes (sphere + handles + cross cups)



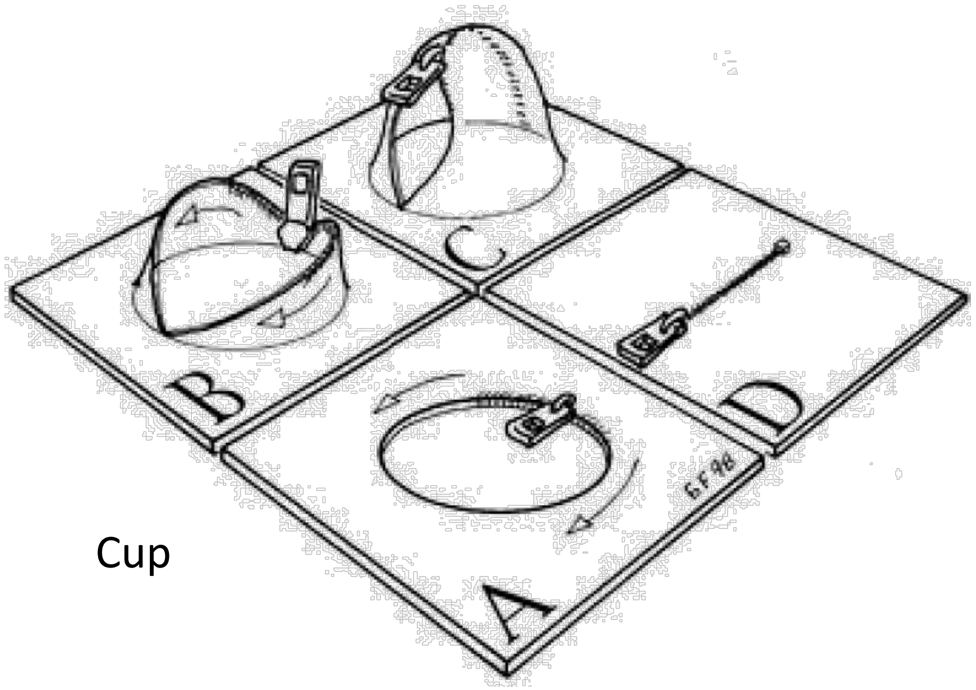
Handle

Klein bottle = sphere + 2 cross cups



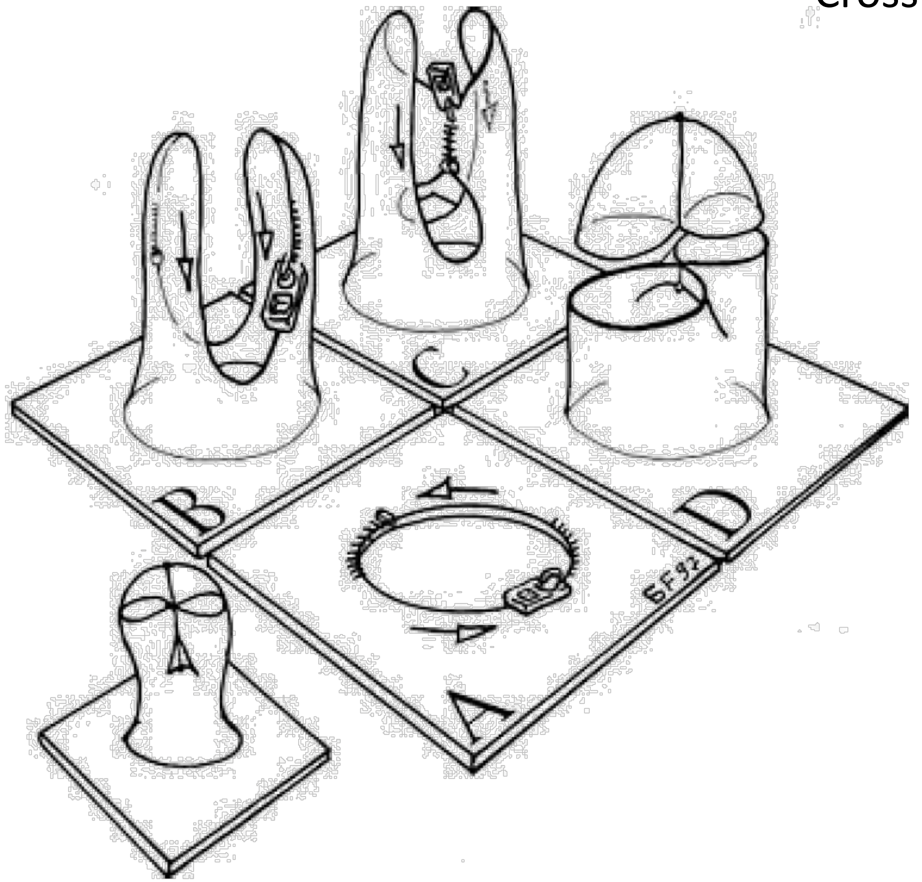
Cross cup

J. Conway’s ZIP Proof (Zero Irrelevancy Proof)



Cup

Cross cup



[Francis and Weeks, AMM, 1999
https://www.maths.ed.ac.uk/~v1ranick/papers/francisweeks.pdf]

Algebraic Topology

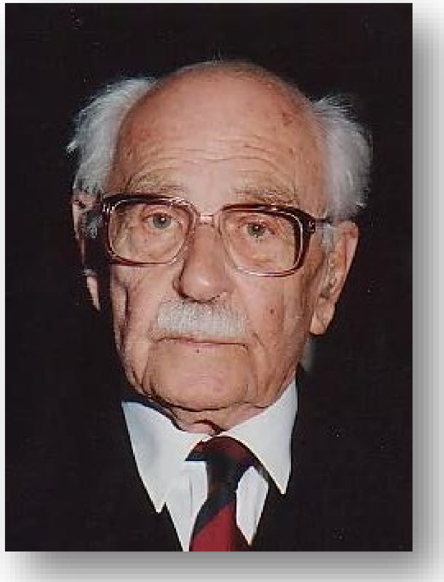
Many other branches: Point set topology, differential topology, combinatorial topology


Henri Poincaré

Luitzen Egbertus Jan Brouwer

Emmy Noether


Heinz Hopf

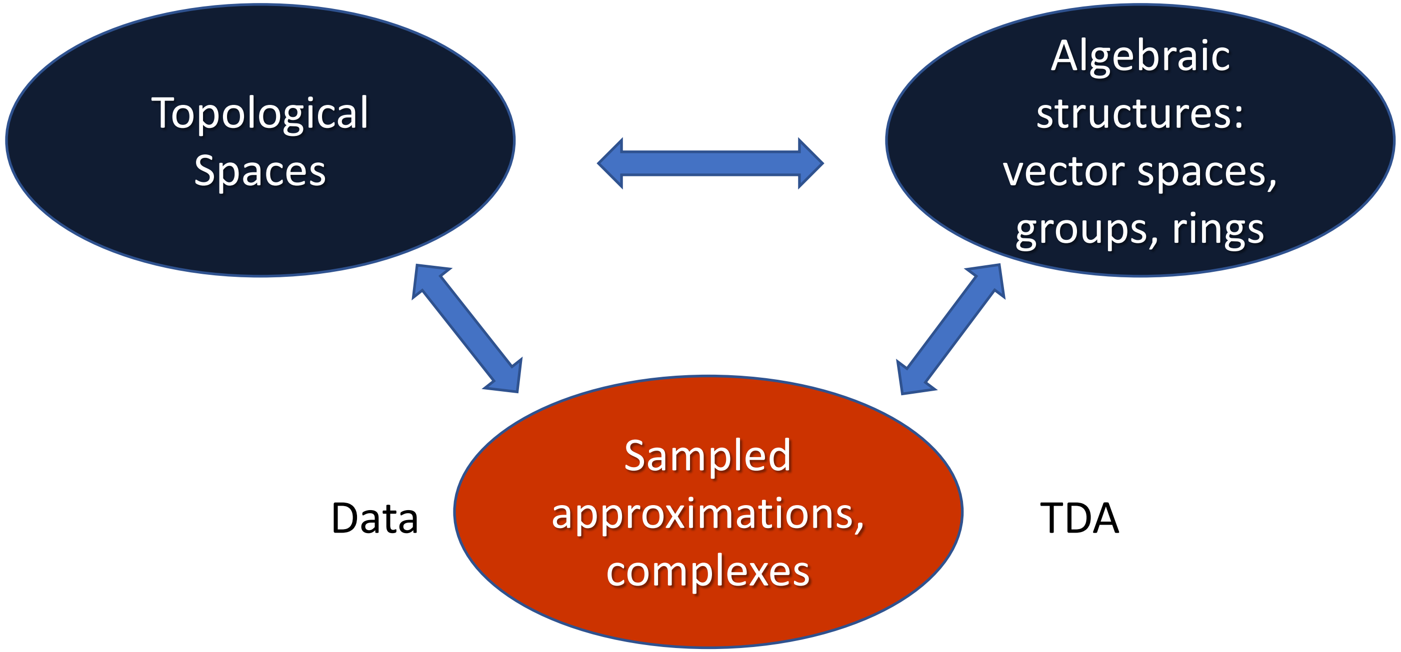

Leopold Vietoris

Topology and Topology Inference

- Unlike geometry, topology studies the global structure of spaces

- Getting to this structure via algebra

algebraic topology



Topological Spaces

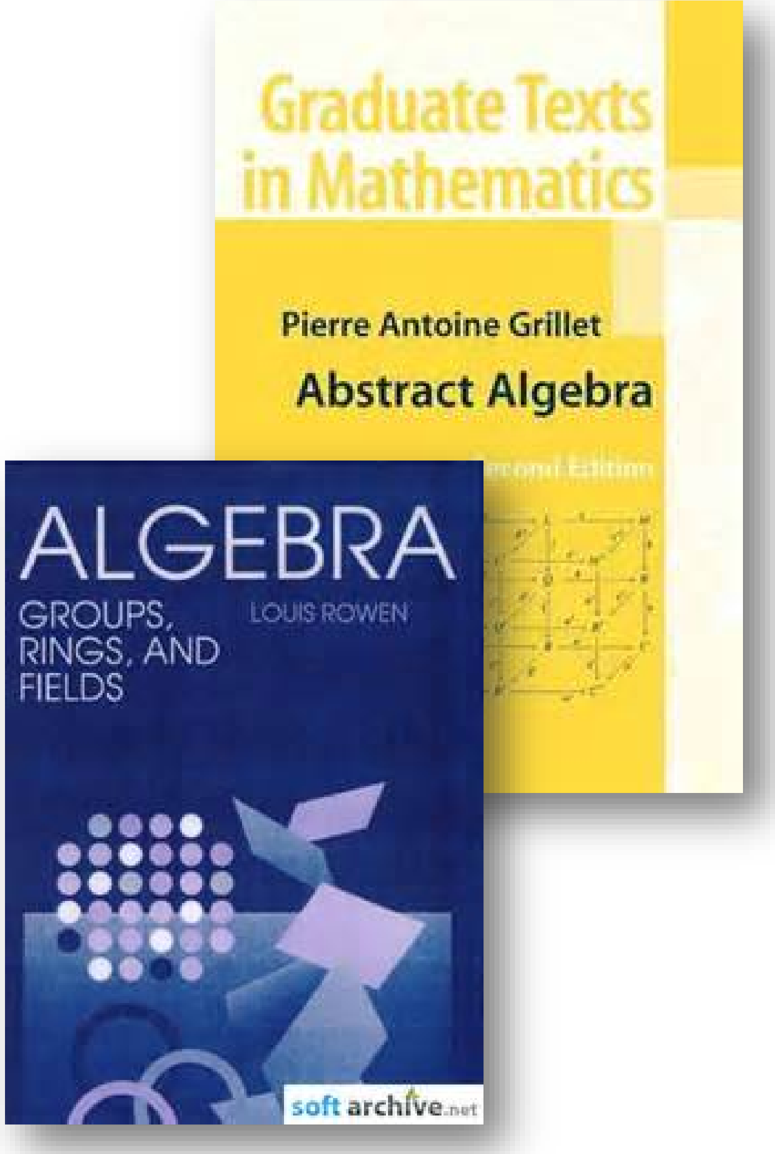
Algebraic structures: vector spaces, groups, rings

Sampled approximations, complexes

Data

TDA

From Data to Algebraic Objects


Graduate Texts in Mathematics

Pierre Antoine Grillet

Abstract Algebra

ALGEBRA

GROUPS, RINGS, AND FIELDS

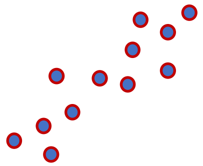
LOUIS ROWEN

Algebraic objects as data descriptors

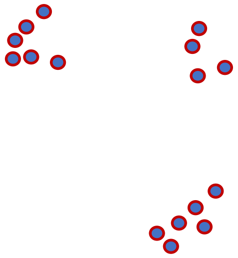
Sampled Spaces

The “Shape of Data”

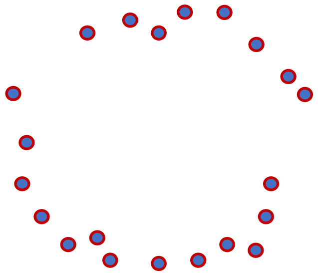
- TDA Studies Sampled Spaces



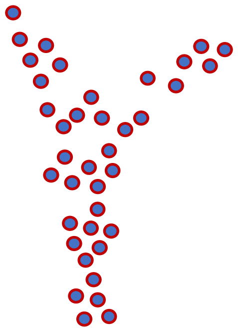
Regression



Cluster

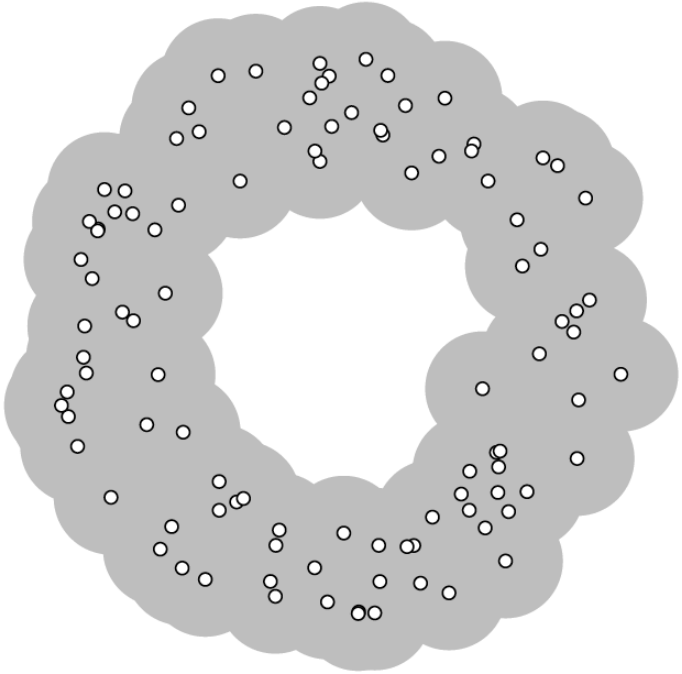


Loop



Flared

Recovering “Shape” from Sampled Data



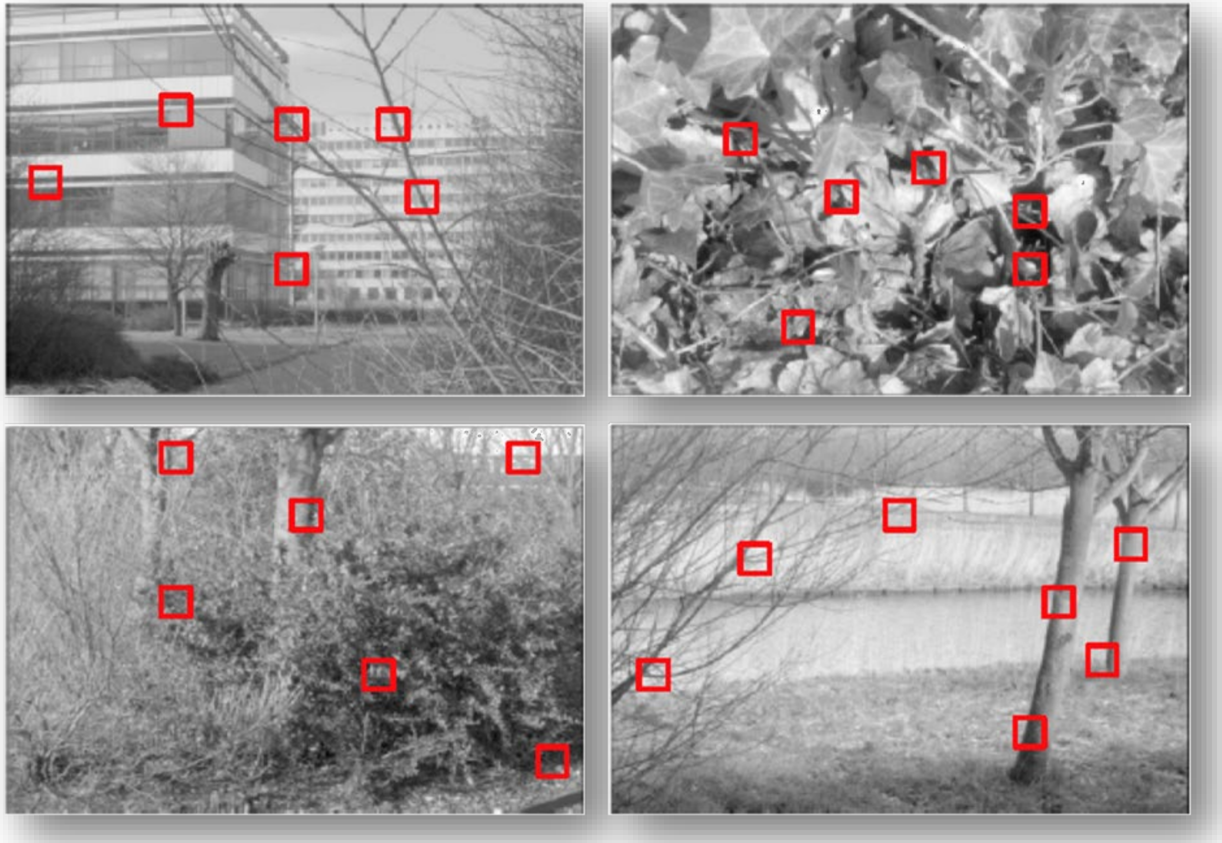
1. Set of points in $\mathbb{R}^2$
2. Looks like an annulus.

What is this?

What does it look like ?

Aim: recover the topology of the underlying space from which the data was sampled

Example: The Space of Natural Images



Example: The Space of Natural Images

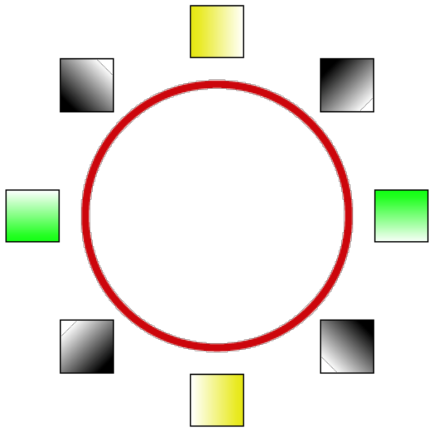
(Carlsson, Ishkanov, de Silva, Zomorodian IJCVC 2008)

- Lee-Mumford-Pedersen (2003) investigated whether a statistically significant difference exists between natural and random images
- Natural images form a “subspace” of all images. Dimension of ambient space e.g. 640 x 480 = 307 200
- This space of natural images should have:
  - high dimension: there are many different images
  - even higher co-dimension: random images look nothing like natural ones
- Data is a collection of black-and-white images used in cognitive science research

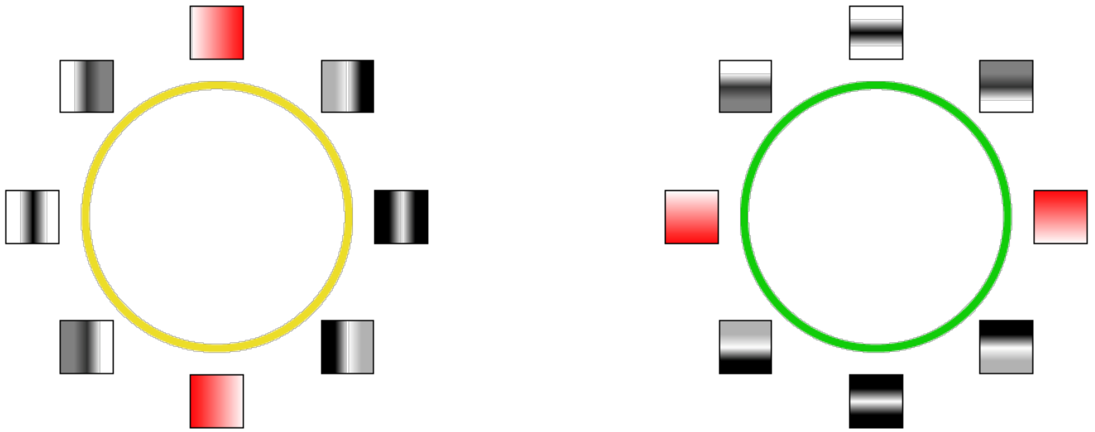
Natural 3 x 3 Patches

- Instead of studying entire images, we consider the distribution of 3 x 3 pixel patches
- Most of these are roughly constant in natural images -- they drown out structure
- L.M.P. chose 8,500,000 patches with high contrast
- Each 3 x 3-patch is considered a vector in R^9
- Normalize brightness: $R^9 \to R^8$
- Normalize contrast: $R^8 \to S^7$

High-Density Areas

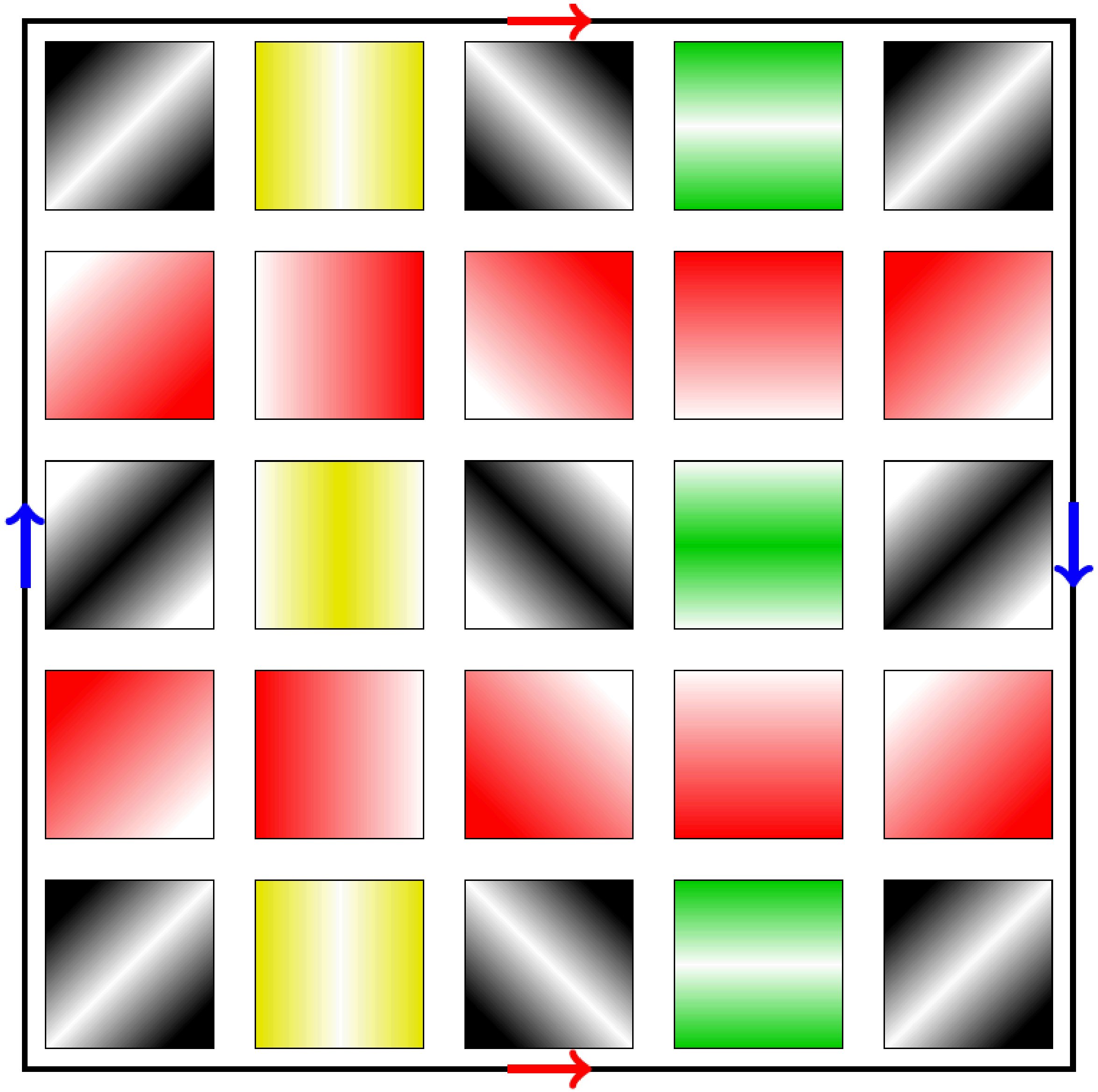


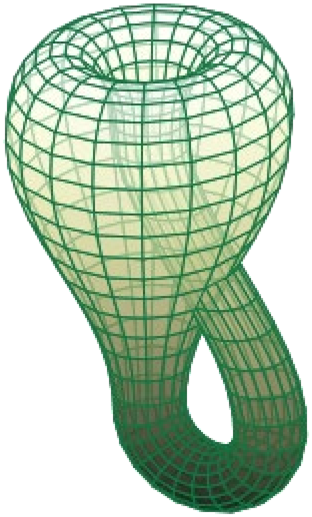
highest density



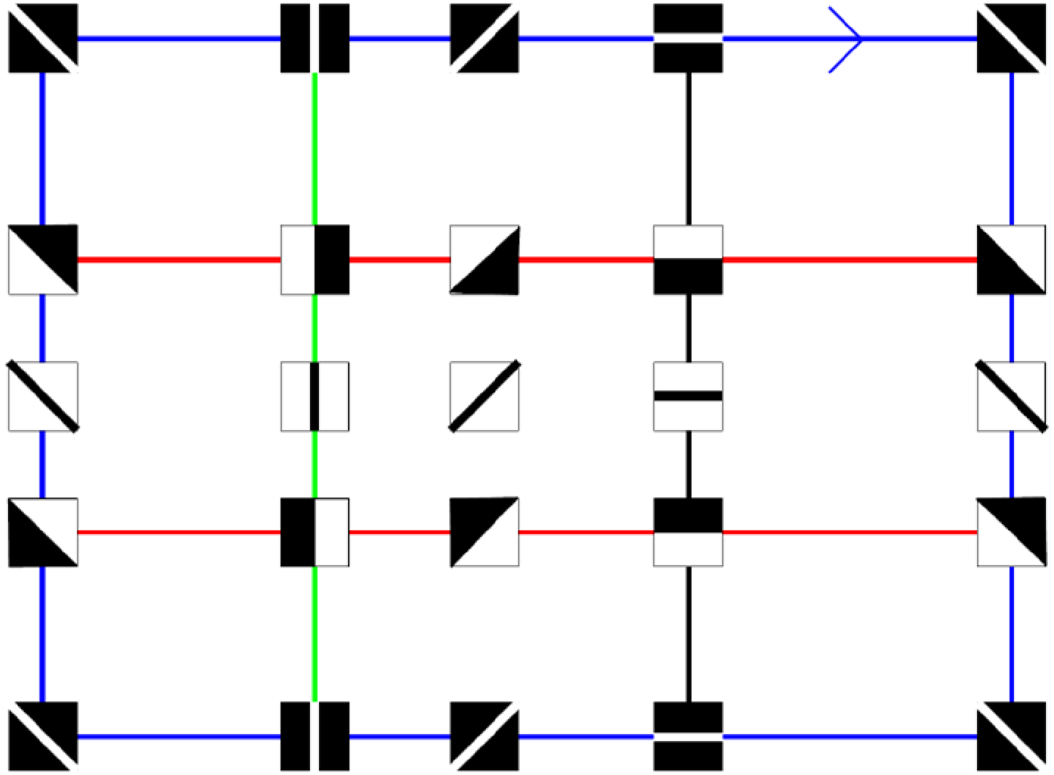
next highest density

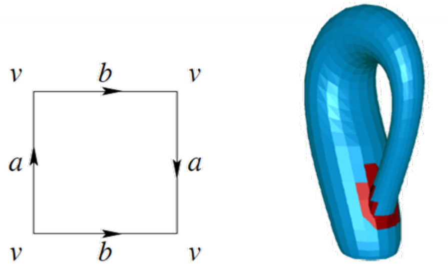
Klein Bottle of Pixel Patches





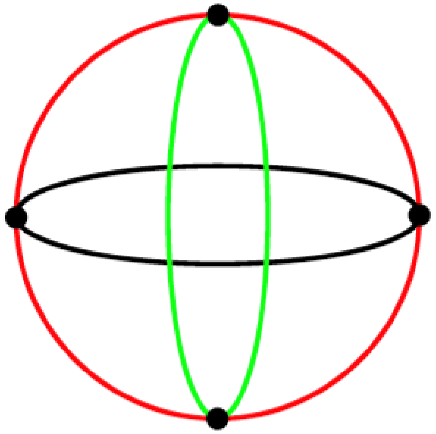
Klein Bottle Structure





(a) Diagram

(b) An immersion



$(\beta_1 = 5)$

(source: [Carlsson, Ishkhanov, de Silva, Zomorodian 2008])

Applications of the Analysis

- An efficient way to parametrize image patches.

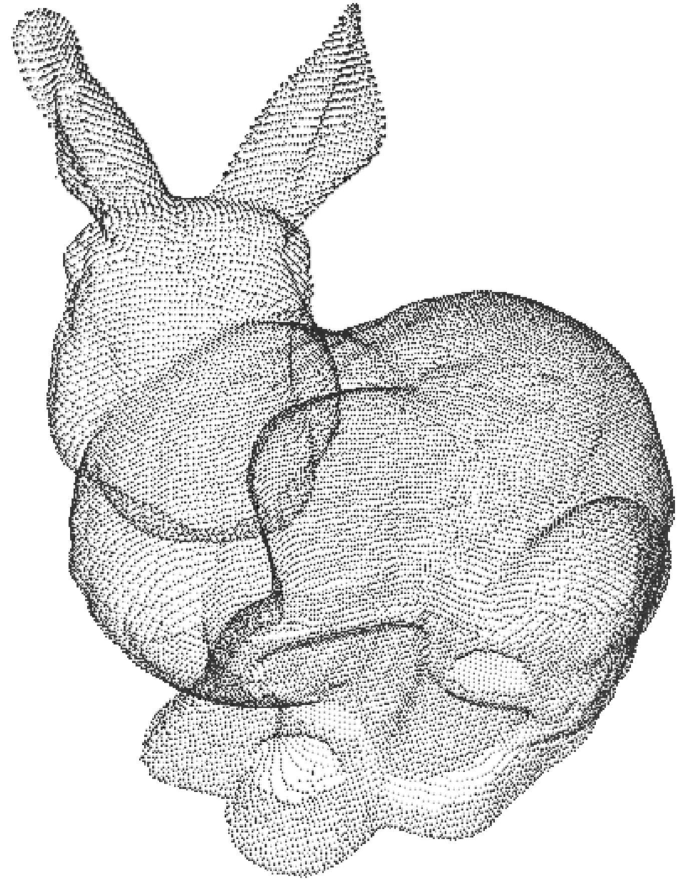
- Image compression: a 3 x 3-cluster may be described using 4 values:
  - 2x : Position of its projection onto the Klein bottle
  - 1x : Original brightness
  - 1x : Original contrast

- Texture analysis: textures yield distributions of occurring patches on the Klein bottle. Rotating the texture corresponds to translating the distribution.

- Deep nets: regularization of initial layers

Simplicial Complexes: Combinatorial Topology

Point Clouds Have No Higher-D Topology

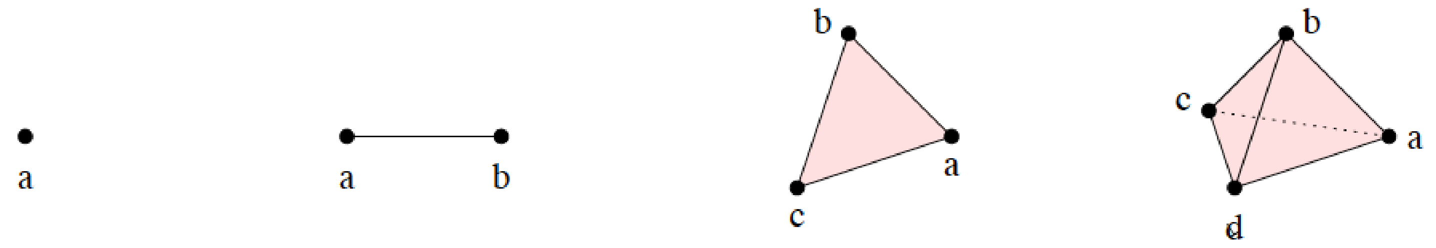


Connect the points into simplicial complexes

Simplicial homology

A Simplex

- A k-simplex is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the vertices of the simplex.
- A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$.



vertex
a

edge
[a, b]

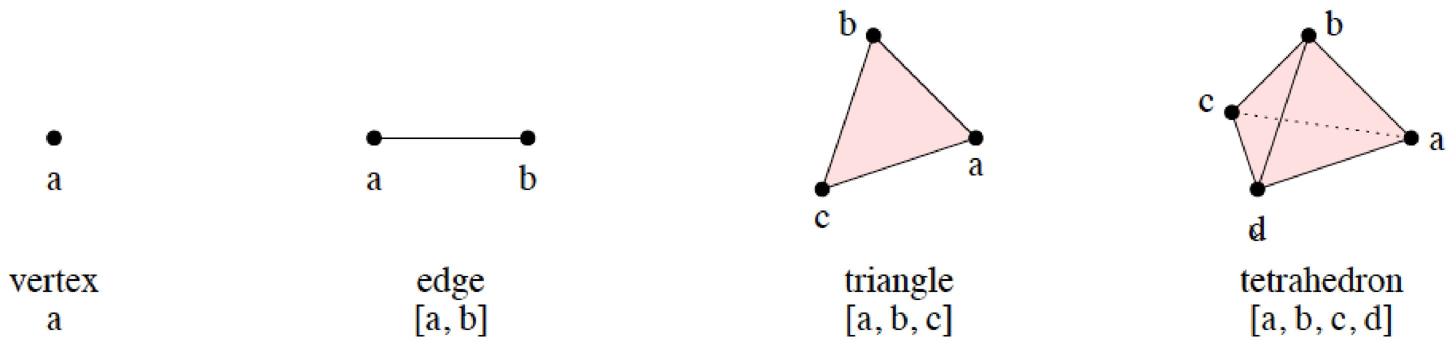
triangle
[a, b, c]

tetrahedron
[a, b, c, d]

d=0 *d*=1 *d*=2 *d*=3

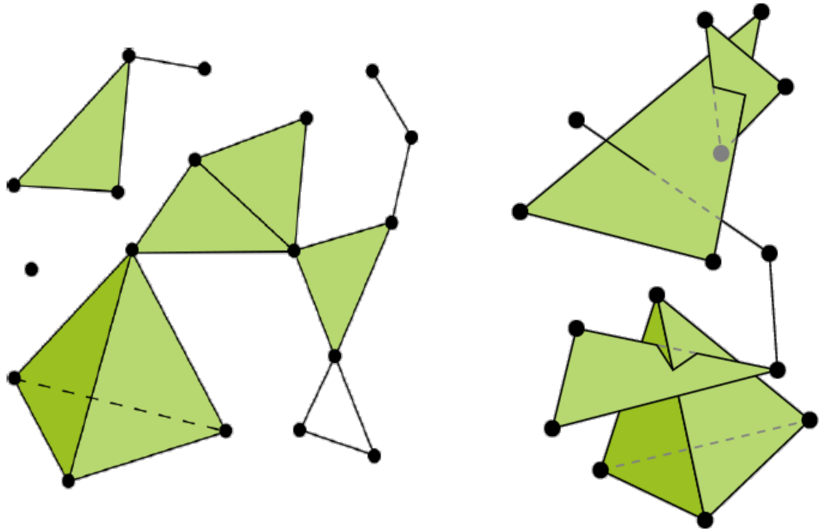
Faces / Subsimplices

- σ: a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$.
- τ defined by $T \subseteq S$ is a face of σ
- σ is its coface.
- $\sigma \geq \tau$ and $\tau \leq \sigma$.
- $\sigma \leq \sigma$ and $\sigma \geq \sigma$.



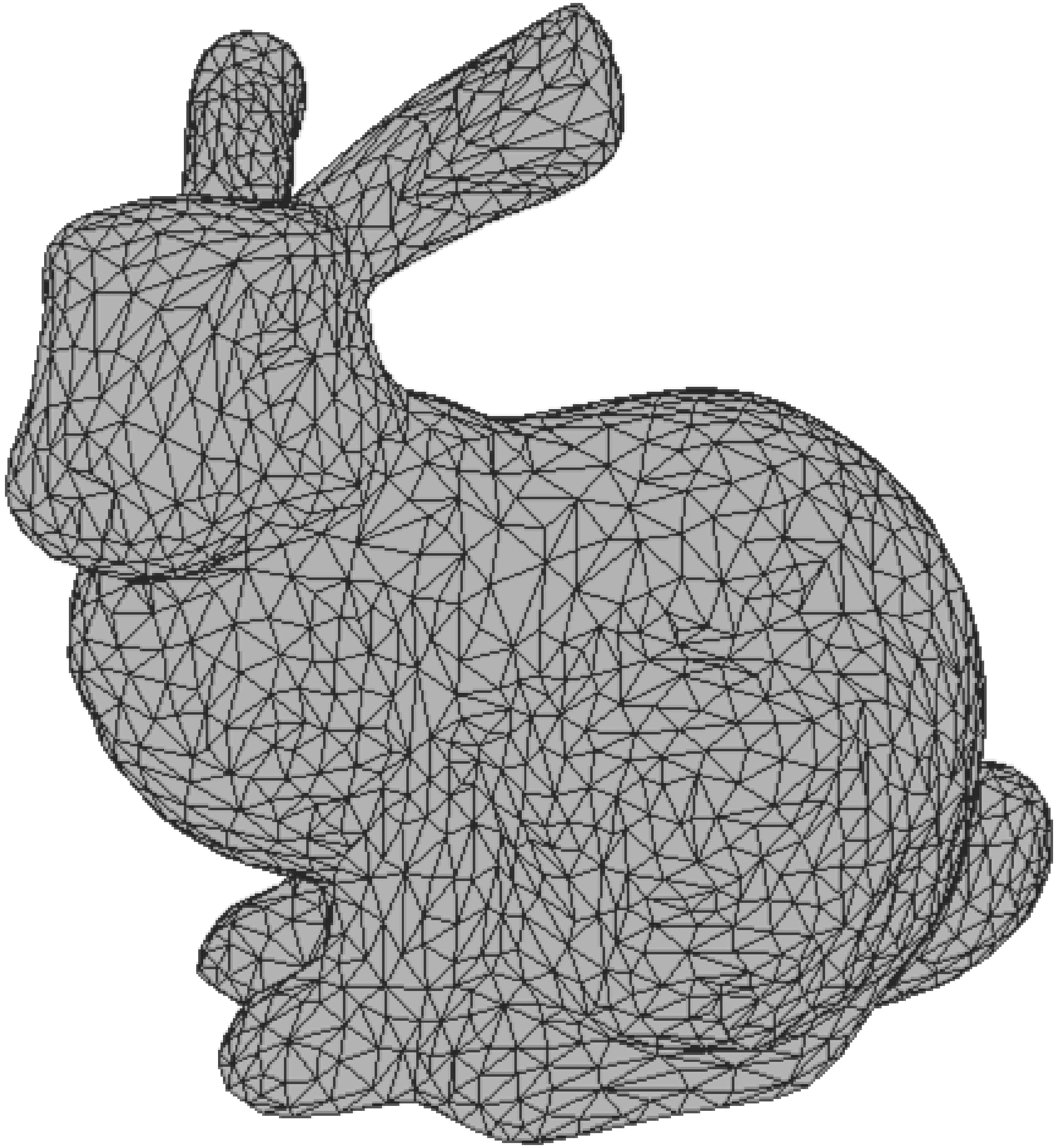
Simplicial Complexes

- A simplicial complex K is a finite set of simplices such that
  1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$,
  2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ or $\sigma \cap \sigma' = \emptyset$.
- The dimension of K is $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$.
- The vertices of K are the zero-simplices in K.
- A simplex is principal if it has no proper coface in K.



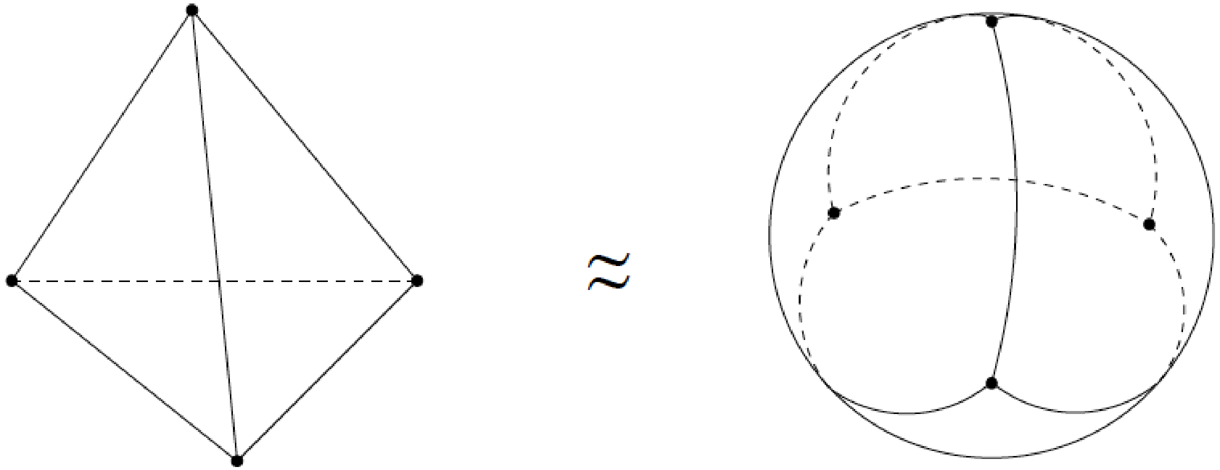
(left) an example (right) a non example

Meshes are 2-D Complexes

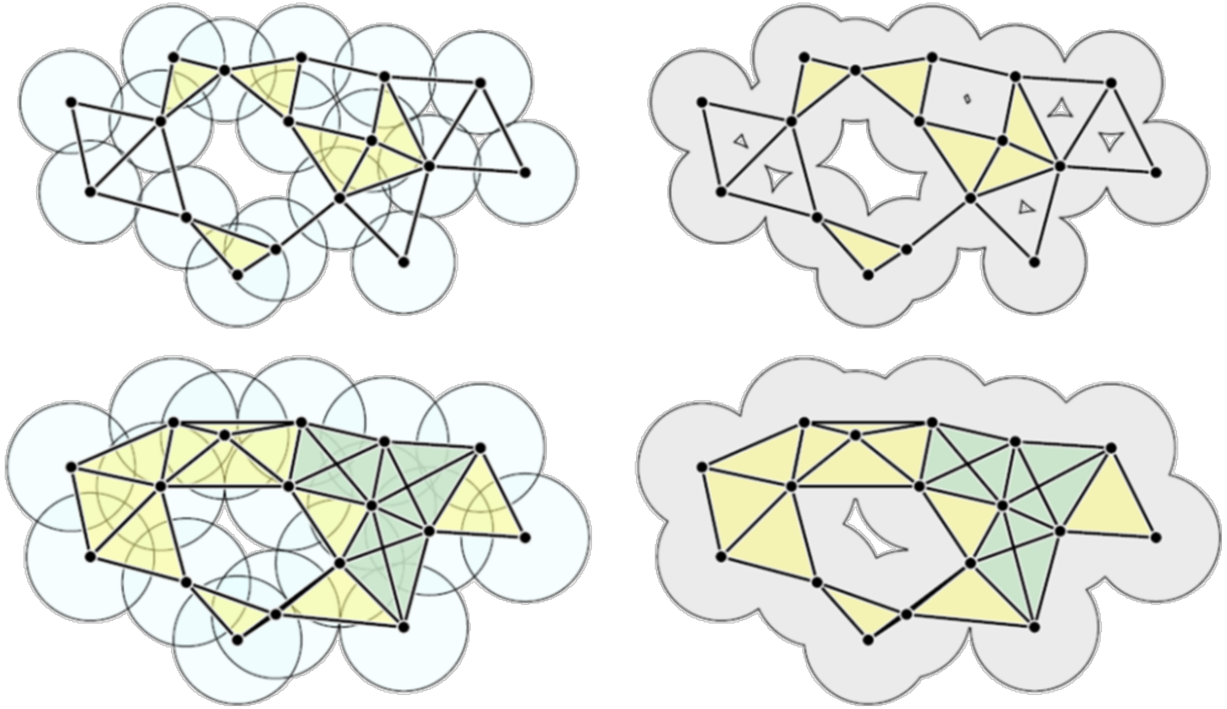


Continuous to Discrete Link: Triangulations

- The underlying space $|K|$ of a simplicial complex K is $|K| = \cup_{\sigma \in K} \sigma$.
- $|K|$ is a topological space.
- A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.



The Nerve of a Finite Cover

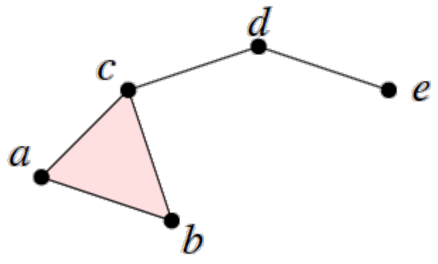


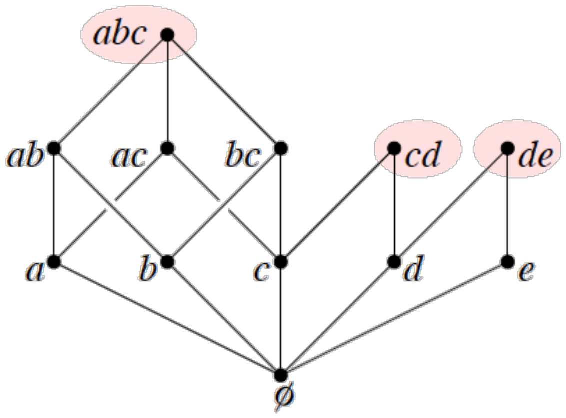
The nerve of $\mathcal{U}$ is the simplicial complex $K(U)$ defined by

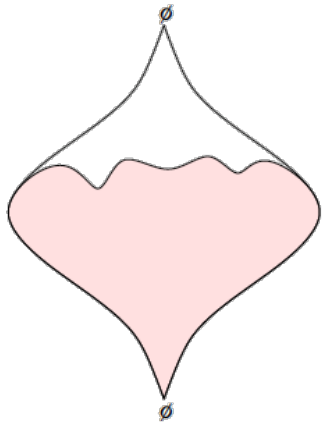
$$\sigma = [U_{i_0}, \ldots, U_{i_k}] \in K(U) \quad \Leftrightarrow \quad \cap_{i=1}^{k} U_{i_j} \neq \emptyset$$

Abstract Simplicial Complexes

- An abstract simplicial complex is a set K, together with a collection $\mathcal{S}$ of subsets of K called (abstract) simplices such that:
  1. For all $v \in K$, $\{v\} \in \mathcal{S}$. We call the sets $\{v\}$ the vertices of K.
  2. If $\tau \subseteq \sigma \in \mathcal{S}$, then $\tau \in \mathcal{S}$.
- We call $\mathcal{S}$ the complex.







Natural partial order structure

Orientability

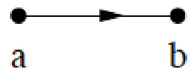
- An orientation of a k-simplex $\sigma \in K$, $\sigma = \{v_0, v_1, \ldots, v_k\}, v_i \in K$ is an equivalence class of orderings of the vertices of σ, where

$$(v_0, v_1, \ldots, v_k) \sim (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(k)})$$

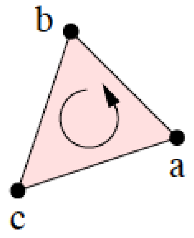
  are equivalent orderings if the parity of the permutation τ is even.

- We denote an oriented simplex, a simplex with an equivalence class of orderings, by $[\sigma]$.

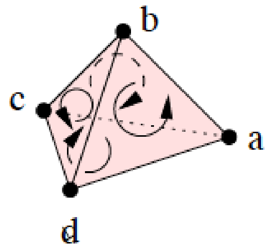
vertex
a



edge
[a, b]



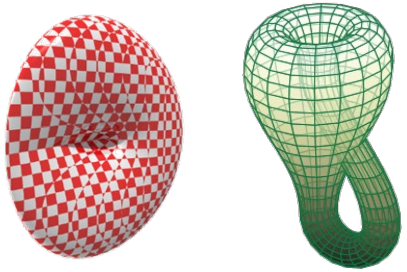
triangle
[a, b, c]

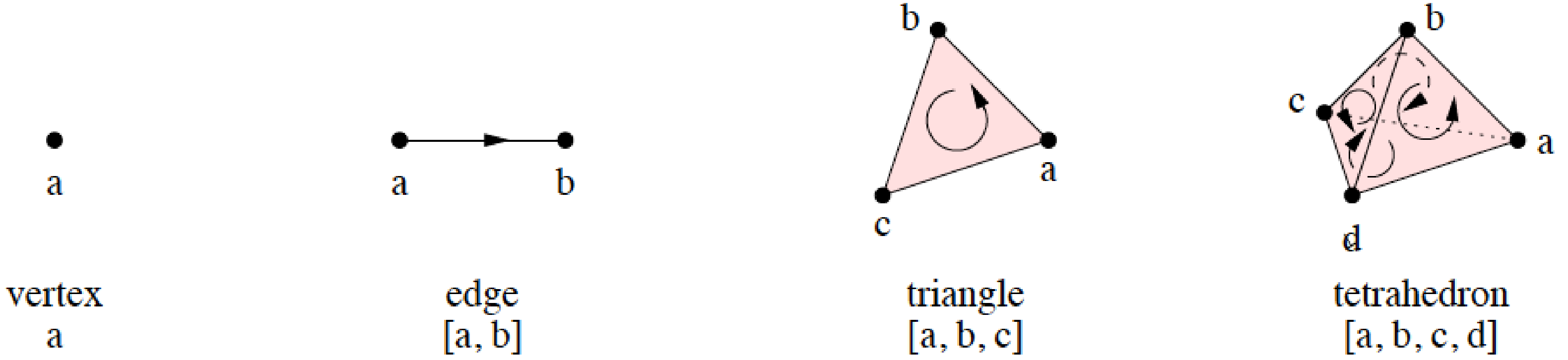


tetrahedron
[a, b, c, d]

Orientability

- Two k-simplices sharing a $(k-1)$-face σ are consistently oriented if they induce different orientations on σ.
- A triangulable d-manifold is orientable if all d-simplices can be oriented consistently.
- Otherwise, the d-manifold is non-orientable





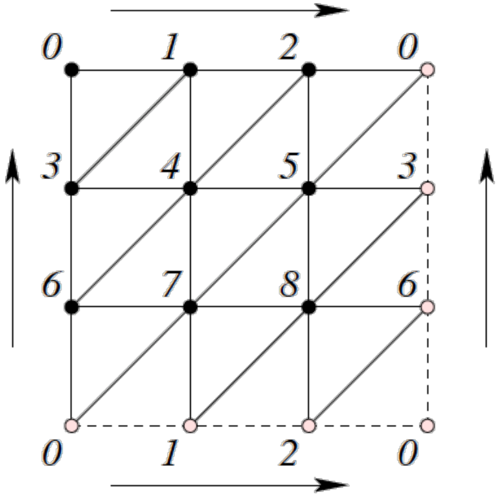
Euler Characteristic: A Topological Invariant

- K a simplicial complex with s_k k-simplices.
- The Euler characteristic $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}.$$

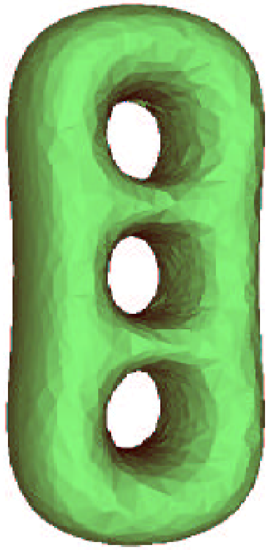
- $v - e + f = 1$ (Graph Theory)
- Invariant for $|K|$
- Any triangulation gives the same answer!
- Intrinsic property

More on Euler



2-Manifold	χ
Sphere $\mathbb{S}^2$	2
Torus $\mathbb{T}^2$	0
Klein bottle $\mathbb{K}^2$	0
Projective plane $\mathbb{RP}^2$	1

- (Theorem) For compact surfaces $\mathbb{M}_1, \mathbb{M}_2$, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.
- $\chi(g\mathbb{T}^2) = 2 - 2g$
- $\chi(g\mathbb{RP}^2) = 2 - g$
- The connected sum of g tori is called a surface with genus g.



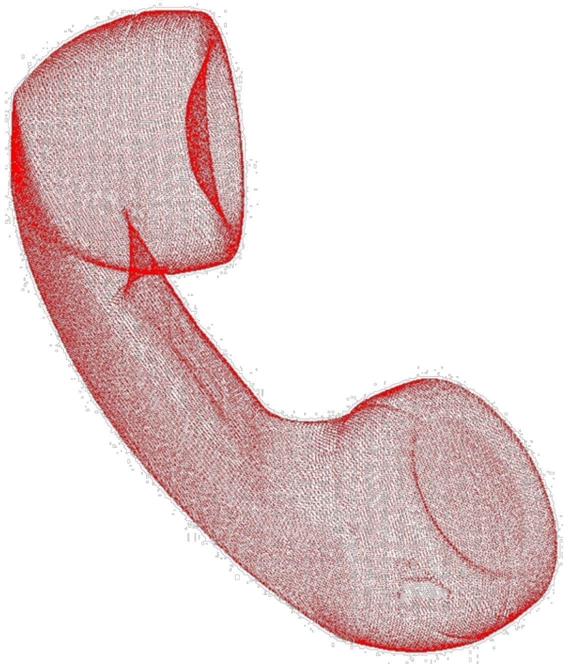
# denotes connected sum

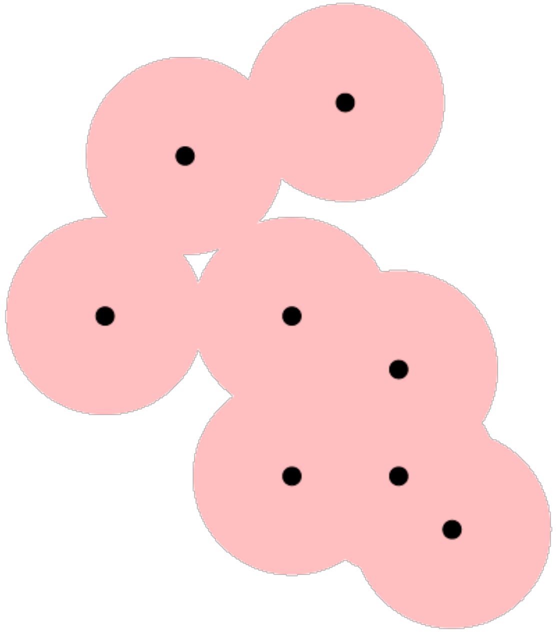
Topological Classification via Invariants

- (Theorem) Closed compact surfaces $\mathbb{M}_1$ and $\mathbb{M}_2$ are homeomorphic, $\mathbb{M}_1 \approx \mathbb{M}_2$ iff
  1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ and
  2. either both surfaces are orientable or both are non-orientable.
- "iff" so full answer. We're done!
- Higher dimensions?

Useful Complexes on Point Clouds

ϵ-Balls





- ϵ-ball: $B_\epsilon(x) = \{y \mid d(x,y) < \epsilon\}$.
- Open sets and topology
- Manifold is $\tilde{\mathbb{M}} = \bigcup_{m_i \in M} B_\epsilon(m_i)$

A Model Space

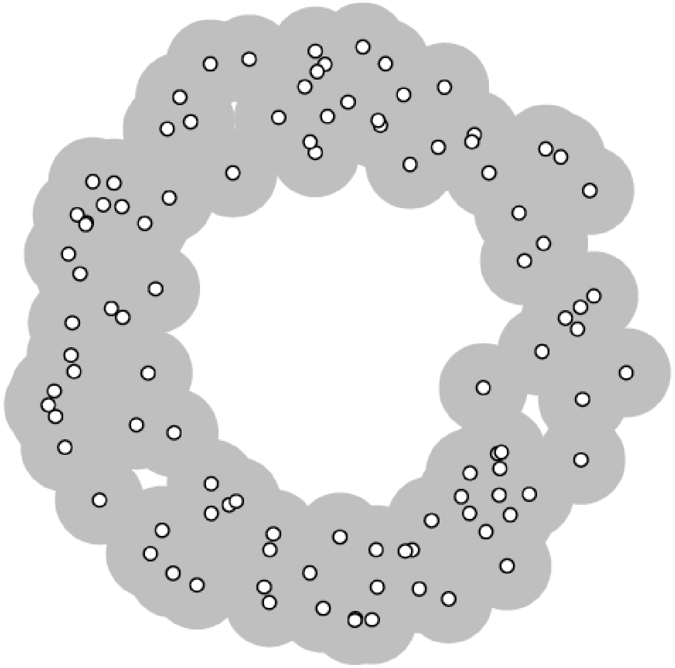
For a dataset X we study the topology of the *union of balls*

$$M_\epsilon = \bigcup_{x \in X} B_\epsilon(x)$$

Two Issues:

Scale: No natural choice of ϵ!

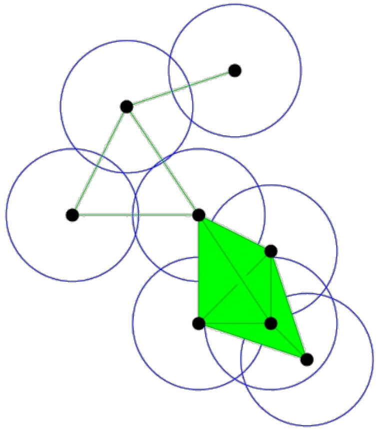
Conception: How to encode M_ϵ on computer?



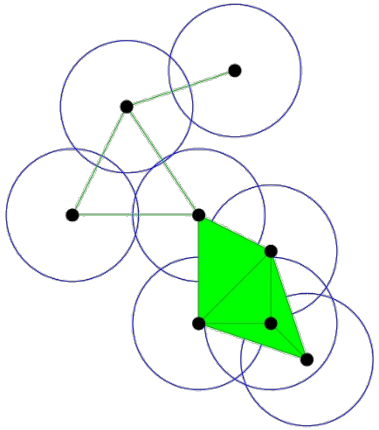
Complex Zoo

Must choose which simplices to introduce

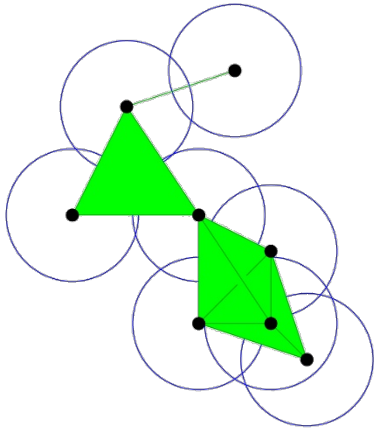
Čech



Alpha

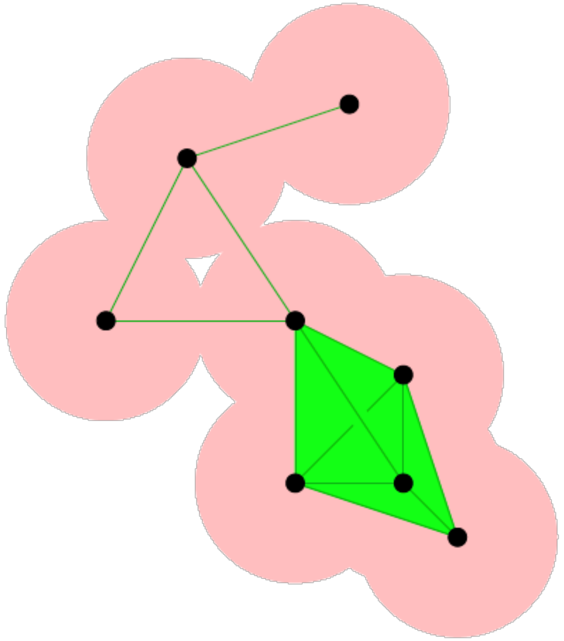


Rips



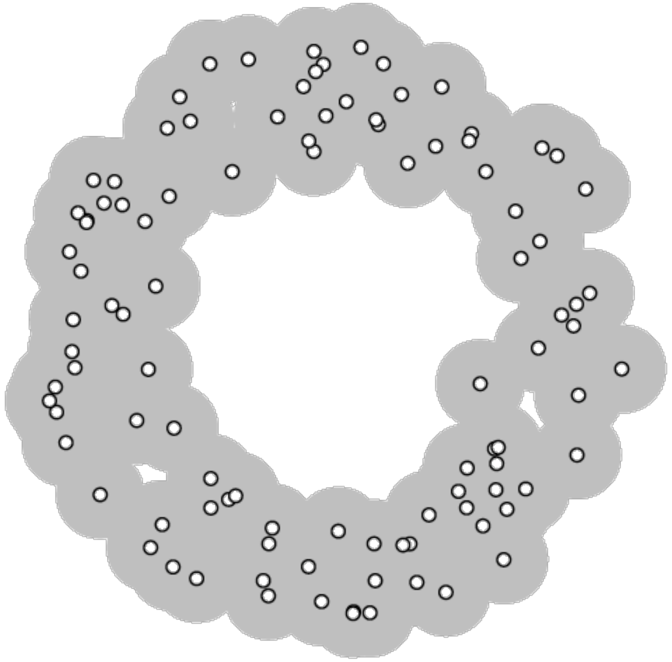
Combinatorial complexes provide discrete representations of the underlying space

Čech Complex



- $C_\epsilon(M) = \left\{\operatorname{conv} T \mid T \subseteq M, \bigcap_{m_i \in T} B_\epsilon(m_i) \neq \emptyset\right\}.$
- $\sum_{k=0}^{m} \binom{m}{k} = 2^{m+1} - 1$
- $C_\epsilon(M) \simeq \tilde{\mathbb{M}}$

Čech Complex

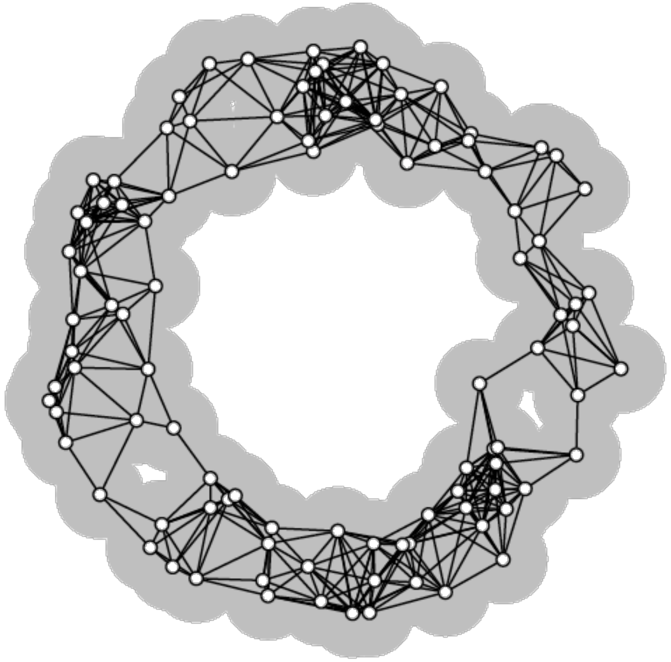


$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

Čech Complex



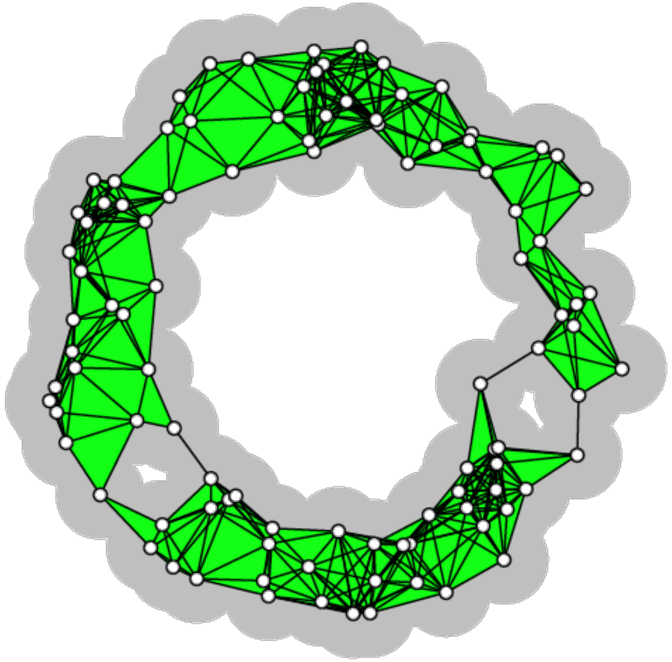
$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

Pairwise intersections as *edges* (1-cells)

Čech Complex



$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

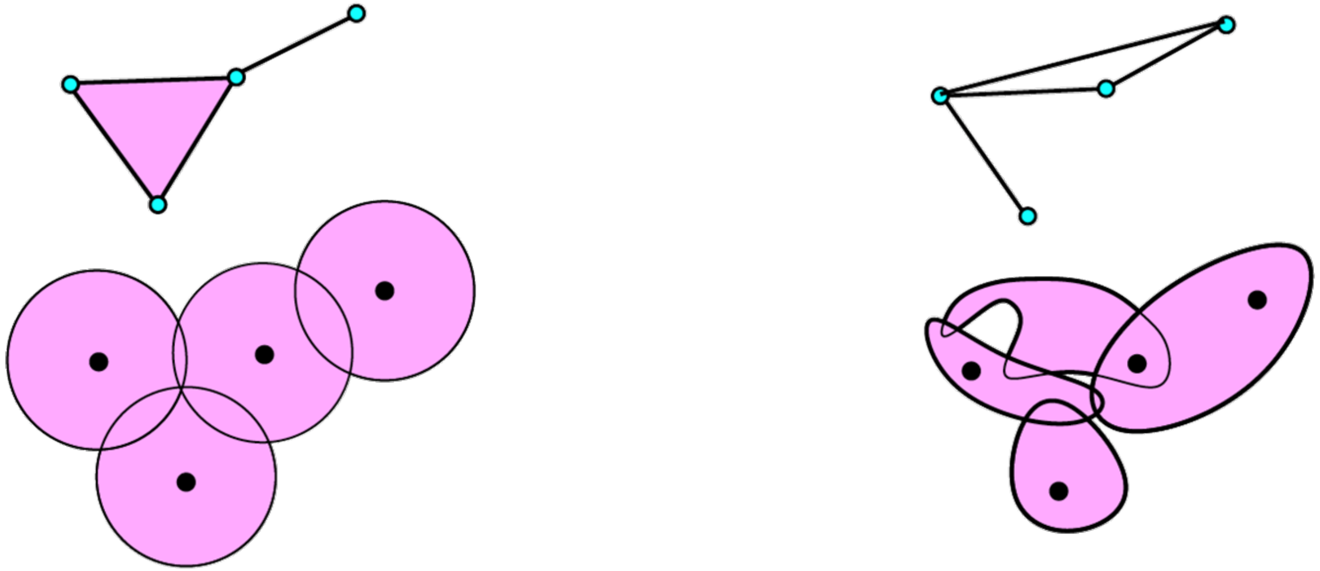
Pairwise intersections as *edges* (1-cells)

Threeway intersections as *triangles* (2-cells)

k-way intersections as *(k+1)-cells*

Can be hard to compute …

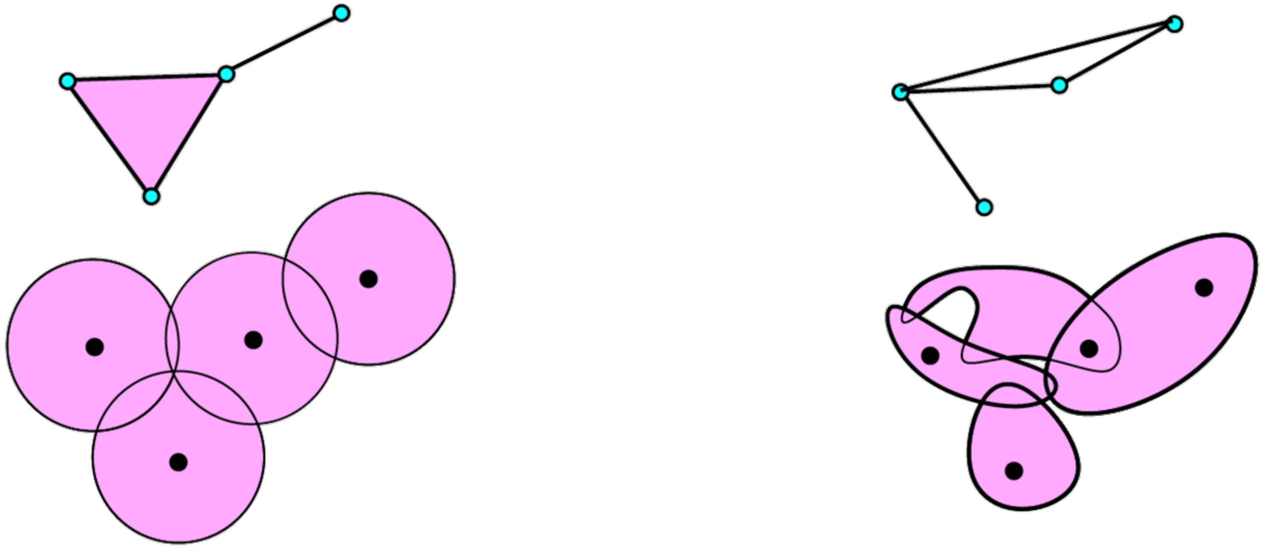
General Čech Complex = Nerve



- Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of a topological space X by open sets: $X = \cup_{i \in I} U_i$.
- The Cěch complex $C(\mathcal{U})$ associated to the covering $\mathcal{U}$ is the simplicial complex defined by:
  - the vertex set of $C(\mathcal{U})$ is the set of the open sets U_i
  - $[U_{i_0}, \cdots, U_{i_k}]$ is a k-simplex in $C(\mathcal{U})$ iff $\cap_{j=0}^{k} U_{i_j} \neq \emptyset$.

General Čech Complex

Lemma (Nerve Lemma, Leray '45)
$\check{C}_\epsilon$ is topologically equivalent to M_ϵ.

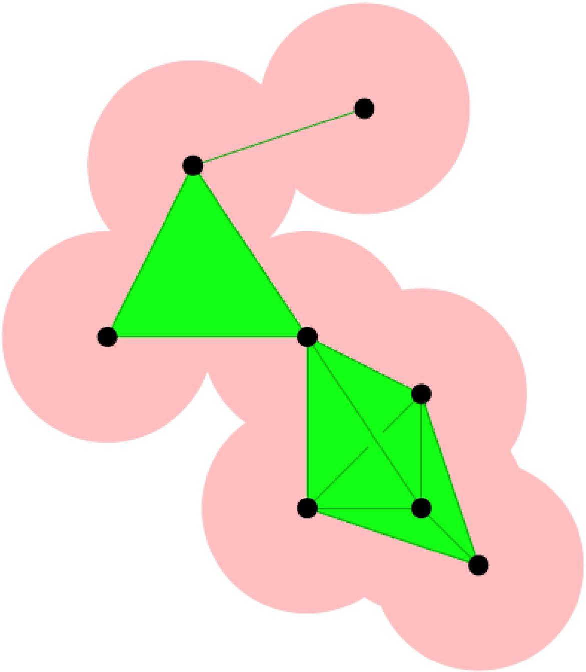


Nerve theorem (Leray): If all the intersections between opens in $\mathcal{U}$ are either empty or contractible then $C(\mathcal{U})$ and $X = \cup_{i \in I} U_i$ are homotopy equivalent.

$\Rightarrow$ The combinatorics of the covering (a simplicial complex) carries the topology of the space.

Warning: even when the open sets are euclidean balls, the computation of the Cěch complex is a very difficult task!

Rips-Vietoris Complex



The “poor man’s” alternative
to the Čech

This is a common complex
For computations

- $R_\epsilon(M) = \{\operatorname{conv} T \mid T \subseteq M,\, d(m_i, m_j) < \epsilon, m_i, m_j \in T\}\,.$
- Still $O\left(\binom{m}{k}\right)$ for the kth skeleton
- Need $(k+1)$st skeleton for computing H_k

Rips vs. Čech

Rips vs Čech

Let $L = \{p_0, \cdots p_n\}$ be a (finite) point cloud (in a metric space).

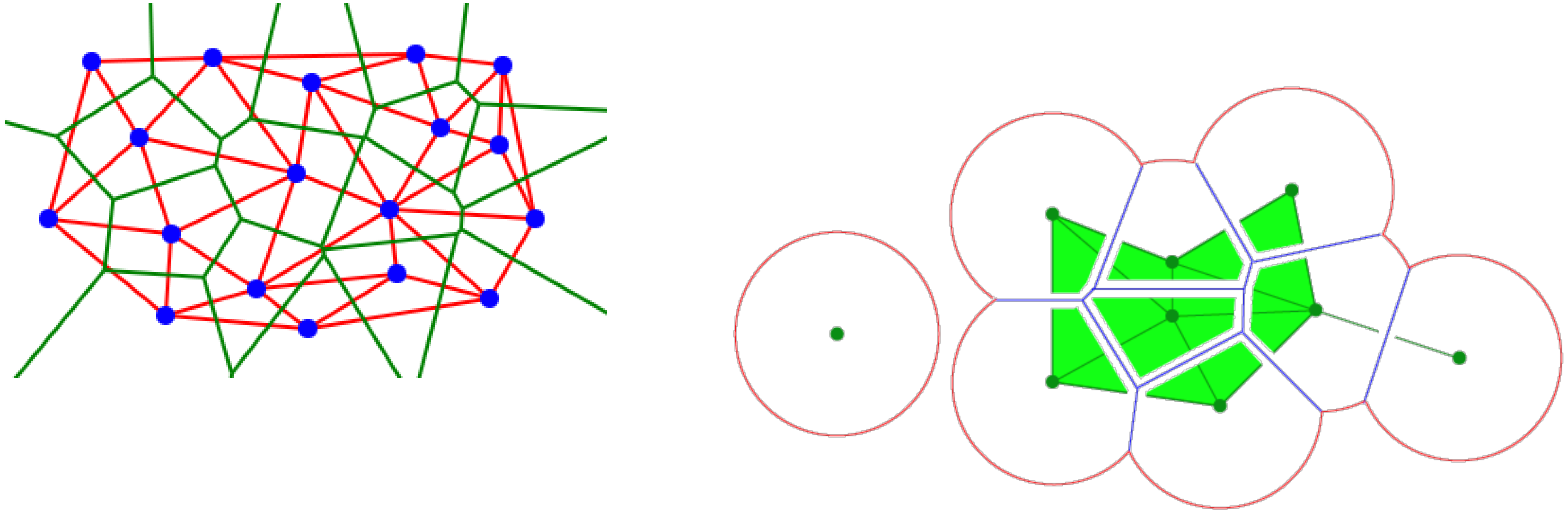
The Rips complex $\mathcal{R}^\alpha(L)$: for $p_0, \cdots p_k \in L$,

$$\sigma = [p_0 p_1 \cdots p_k] \in \mathcal{R}^\alpha(L) \ \text{ iff } \ \forall i, j \in \{0, \cdots k\},\ d(p_i, p_j) \leq \alpha$$

- Easy to compute and fully determined by its 1-skeleton
- Rips-Čech interleaving: for any $\alpha > 0$,

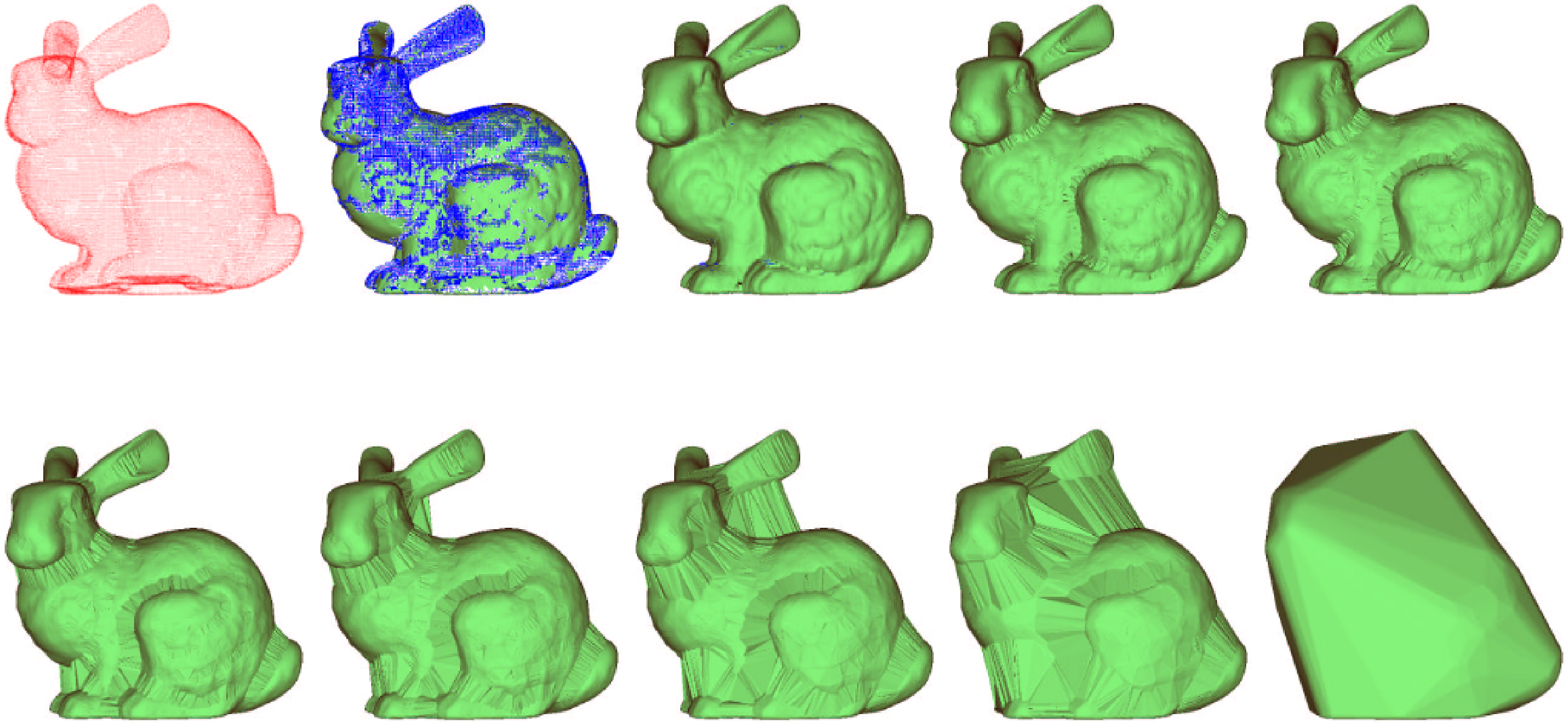
$$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L) \subseteq \mathcal{R}^{2\alpha}(L) \subseteq \cdots$$

Alpha Complex



- $V(m_i) = \{x \in \mathbb{R}^3 \mid d(x, m_i) \leq d(x, m_j)\ \forall m_j \in M\}$
- $\hat{V}(m_i) = B_\epsilon(m_i) \cap V(m_i)$
- $A_\epsilon = \left\{ \operatorname{conv} T \mid T \subseteq M, \bigcap_{m_i \in T} \hat{V}(m_i) \neq \emptyset \right\}$
- $A_\epsilon(M) \simeq \tilde{\mathbb{M}}$, $A_\epsilon \subseteq D$, the Delaunay complex
- $O(n \log n + n^{\lceil d/2 \rceil})$

Alpha Complexes on the Stanford Bunny



- 34,834 points, 1,026,111 complexes

Algebraic Structures: Group Theory

Groups

- A group $\langle G, *\rangle$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:
  - (a) $*$ is associative.
  - (b) G has an identity e element for $*$ such that $e * x = x * e = x$ for all $x \in G$.
  - (c) any element a has an inverse a' with respect to the operation $*$, i.e. $\forall a \in G, \exists a' \in G$ such that $a' * a = a * a' = e$.
- If G is finite, the order of G is $|G|$.
- We often omit the operation and refer to G as the group.
- $\langle \mathbb{Z}, +\rangle$, $\langle \mathbb{R}, \cdot\rangle$, $\langle \mathbb{R}, +\rangle$, are all groups.
- A group G is abelian if its binary operation $*$ is commutative.

Subgroups

- Let $\langle G, * \rangle$ be a group and $S \subseteq G$. If S is closed under $*$, then $*$ is the induced operation on S from G.
- A subset $H \subseteq G$ of group $\langle G, * \rangle$ is a subgroup of G if H is a group and is closed under $*$. The subgroup consisting of the identity element of G, $\{e\}$ is the trivial subgroup of G. All other subgroups are nontrivial.
- (Theorem)$H \subseteq G$ of a group $\langle G, * \rangle$ is a subgroup of G iff:
  1. H is closed under $*$,
  2. the identity e of G is in H,
  3. for all $a \in H$, $a^{-1} \in H$.
- Example: subgroups of $\mathbb{Z}_4$

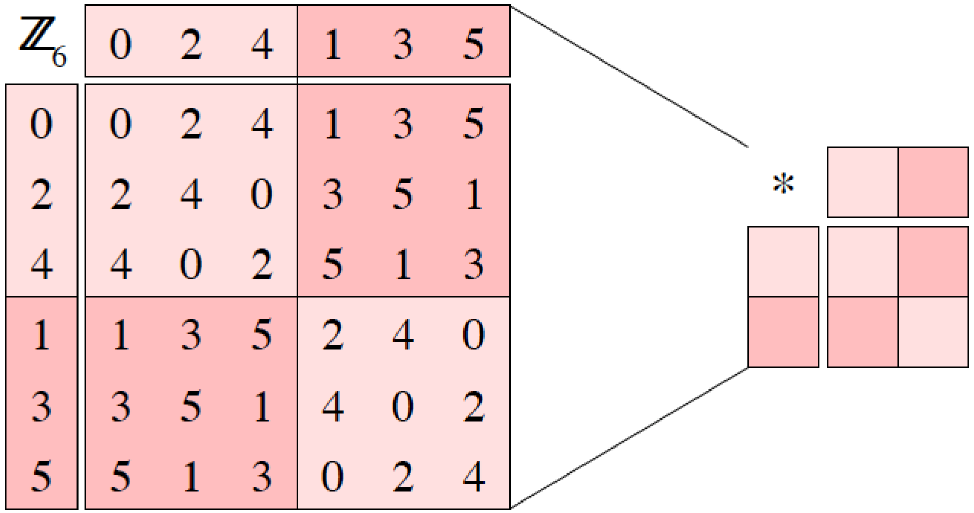
Cosets

- Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by: $a \sim_L b$ iff $a^{-1}b \in H$. Let $\sim_R$ be defined by: $a \sim_R b$ iff $ab^{-1} \in H$. Then $\sim_L$ and $\sim_R$ are both equivalence relations on G.
- Let H be a subgroup of group G. For $a \in G$, the subset $aH = \{ah \mid h \in H\}$ of G is the left coset of H containing a, and $Ha = \{ha \mid h \in H\}$ is the right coset of H containing a.
- If left and right cosets match, the subgroup is normal.
- All subgroups H of an abelian group G are normal, as $ah = ha, \forall a \in G, h \in H$
- $\{0, 2\}$ is a subgroup of $\mathbb{Z}_4$. It is normal. The coset of 1 is $1 + \{0, 2\} = \{1, 3\}$. That's all folks!

Factor / Quotient Groups

- Let H be a normal subgroup of group G.
- Left coset multiplication is well-defined by the equation $(aH)(bH) = (ab)H$
- The cosets of H form a group G/H under left multiplication
- G/H is the factor group (or quotient group) of G modulo H.
- The elements in the same coset of H are congruent modulo H.

Example



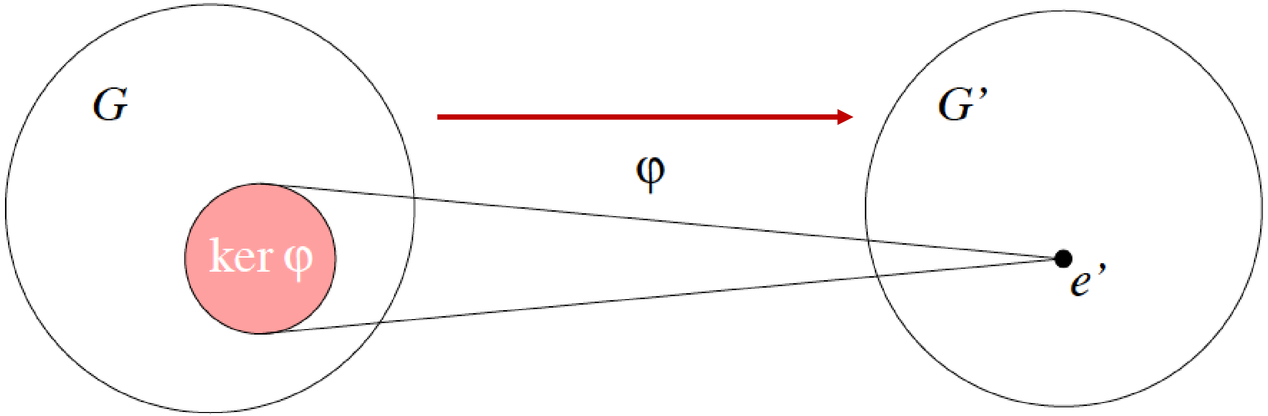
$\mathbb{Z}_6$	0	2	4	1	3	5
0	0	2	4	1	3	5
2	2	4	0	3	5	1
4	4	0	2	5	1	3
1	1	3	5	2	4	0
3	3	5	1	4	0	2
5	5	1	3	0	2	4

*	□	■
□	□	■
■	■	□

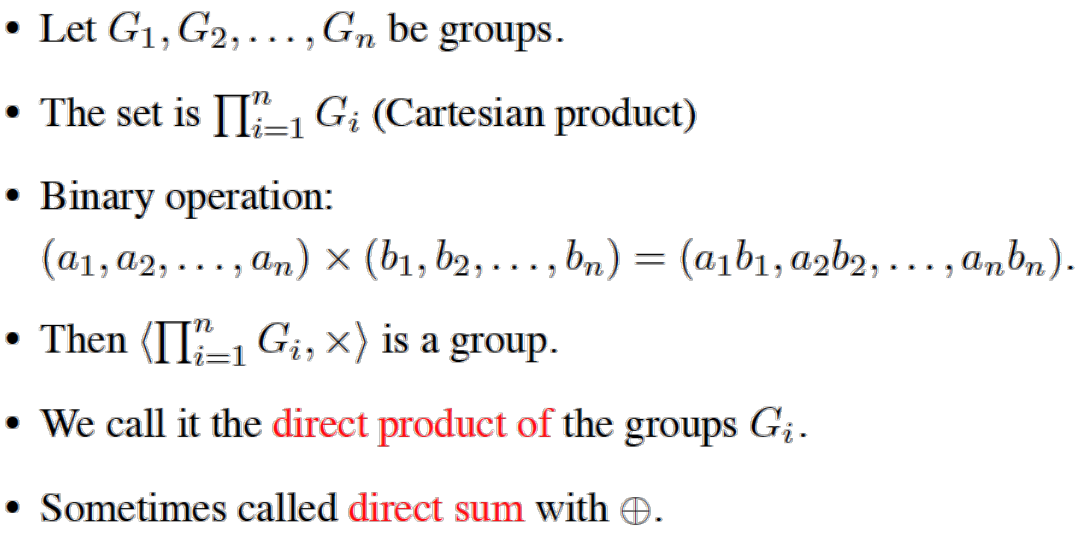
- $\{0, 2, 4\}$ is a normal subgroup
- Cosets $\{0, 2, 4\}, \{1, 3, 5\}$
- $\mathbb{Z}_6 / \{0, 2, 4\} \cong \mathbb{Z}_2$

Homomorphisms

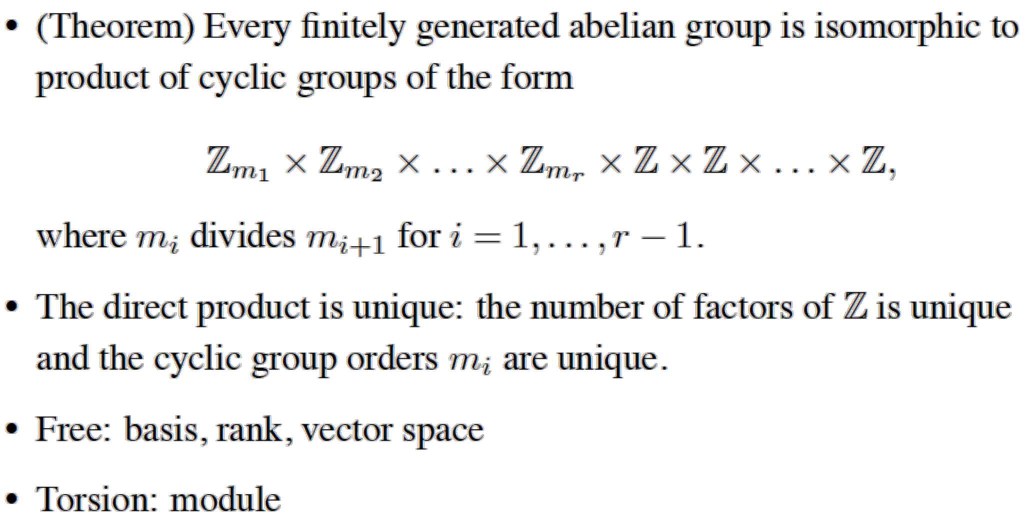
- A map φ of a group G into a group G' is a *homomorphism* if $\boxed{\varphi(ab) = \varphi(a)\varphi(b)}$ for all $a, b \in G$.
- If e is the identity in G, then $\varphi(e)$ is the identity e' in G'.
- If $a \in G$, then $\varphi(a^{-1}) = \varphi(a)^{-1}$.
- If H is a subgroup of G, then $\varphi(H)$ is a subgroup of G'.
- If K' is a subgroup of G', then $\varphi^{-1}(K')$ is a subgroup of G.
- The normal subgroup $\ker \varphi = \varphi^{-1}(\{e'\}) \subseteq G$, is the kernel of φ.



Finitely Generated Abelian Groups



- Let $G_1, G_2, \ldots, G_n$ be groups.
- The set is $\prod_{i=1}^{n} G_i$ (Cartesian product)
- Binary operation:
  $(a_1, a_2, \ldots, a_n) \times (b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$.
- Then $\langle \prod_{i=1}^{n} G_i, \times \rangle$ is a group.
- We call it the direct product of the groups G_i.
- Sometimes called direct sum with $\oplus$.



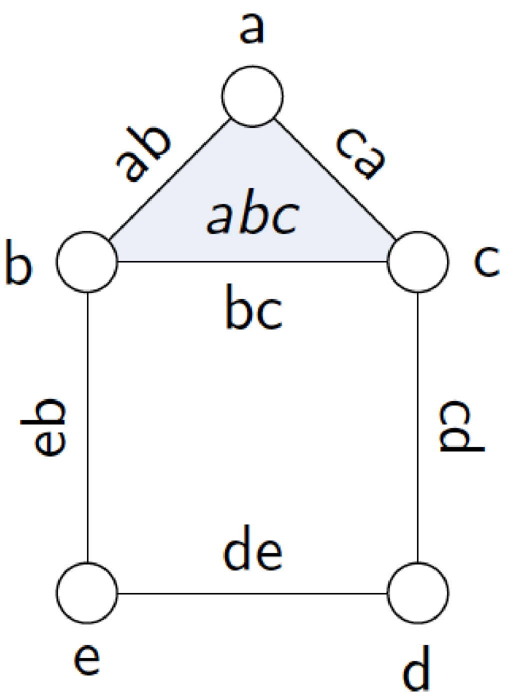
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

  $$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

  where m_i divides m_{i+1} for $i = 1, \ldots, r-1$.
- The direct product is unique: the number of factors of $\mathbb{Z}$ is unique and the cyclic group orders m_i are unique.
- Free: basis, rank, vector space
- Torsion: module

Algebraic Topology: Homology

Topology of Simplicial Complexes



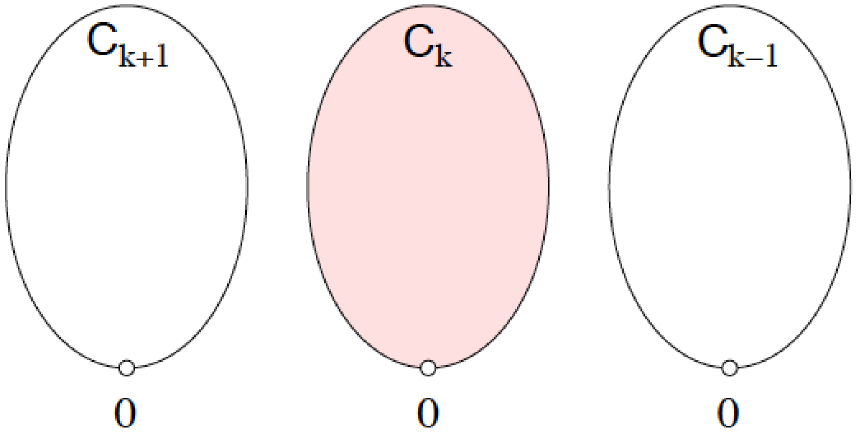
A simplicial complex is a collection of simplices

- Each simplex has a dimension.
- Collection is closed under subset relation.
- Simplices of dimension d have $d + 1$ vertices
- Each simplex represented by an ordered list of vertices

Chain Groups

Other coefficient fields/rings also OK

- Simplicial complex K
- k-chain: $c = \sum_i n_i[\sigma_i], n_i \in \mathbb{Z}, \sigma_i \in K$ (like a path)
- $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ have different orientations.
- The kth chain group C_k of K is the free abelian group on its set of oriented k-simplices
- $\operatorname{rank} \mathsf{C}_k = ?$



We take linear combinations of simplices of a given dimension k

Boundary Operator

- The boundary operator $\partial_k : \mathsf{C}_k \rightarrow \mathsf{C}_{k-1}$ is a homomorphism defined linearly on a chain c by its action on any simplex $\sigma = [v_0, v_1, \ldots, v_k] \in c$,

$$\partial_k \sigma \quad = \quad \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k],$$

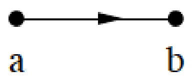
  where $\hat{v}_i$ indicates that v_i is deleted from the sequence.

- $\partial_1 [a, b] = b - a.$
- $\partial_2 [a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b].$
- $\partial_3 [a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c].$

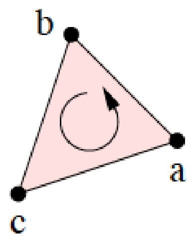
Boundary Examples

- $\partial_1[a,b] = b - a.$
- $\partial_2[a,b,c] = [b,c] - [a,c] + [a,b] = [b,c] + [c,a] + [a,b].$
- $\partial_3[a,b,c,d] = [b,c,d] - [a,c,d] + [a,b,d] - [a,b,c].$
- $\partial_1\partial_2[a,b,c] = [c] - [b] - [c] + [a] + [b] - [a] = 0.$

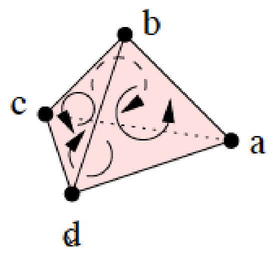
vertex
a



edge
[a, b]

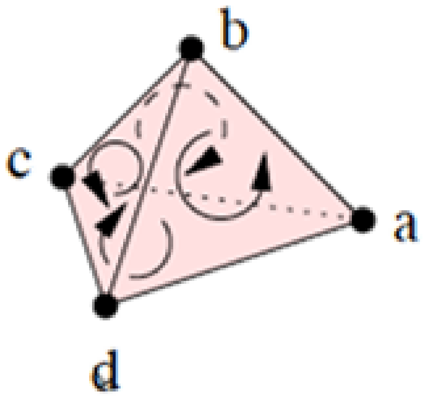


triangle
[a, b, c]



tetrahedron
[a, b, c, d]

Boundary Theorem



- (Theorem) $\partial_{k-1}\partial_k = 0$, for all k.
- Proof:

$\partial_{k-1}\partial_k[v_0, v_1, \ldots, v_k] =$

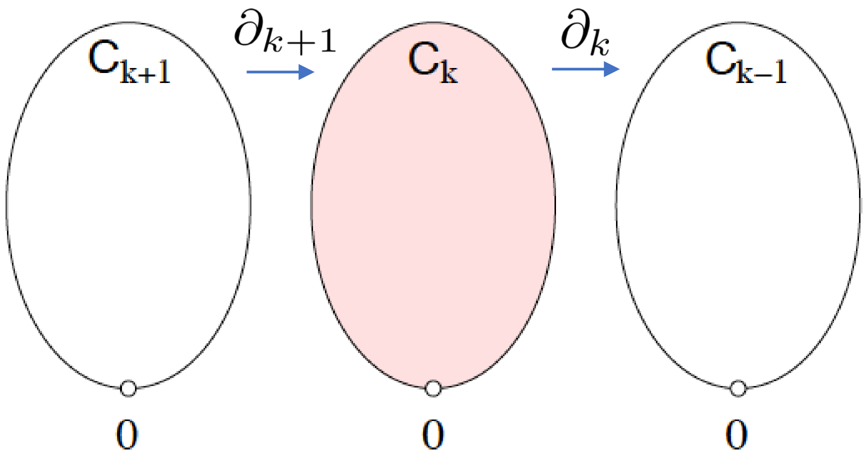
$$\begin{aligned} &= \partial_{k-1}\sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k] \\ &= \sum_{j<i} (-1)^i(-1)^j [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k] \\ &\quad + \sum_{j>i} (-1)^i(-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k] \\ &= 0, \end{aligned}$$

as switching i and j in the second sum negates the first sum.

Chain Complex

- The boundary operator connects the chain groups into a chain complex C_*:

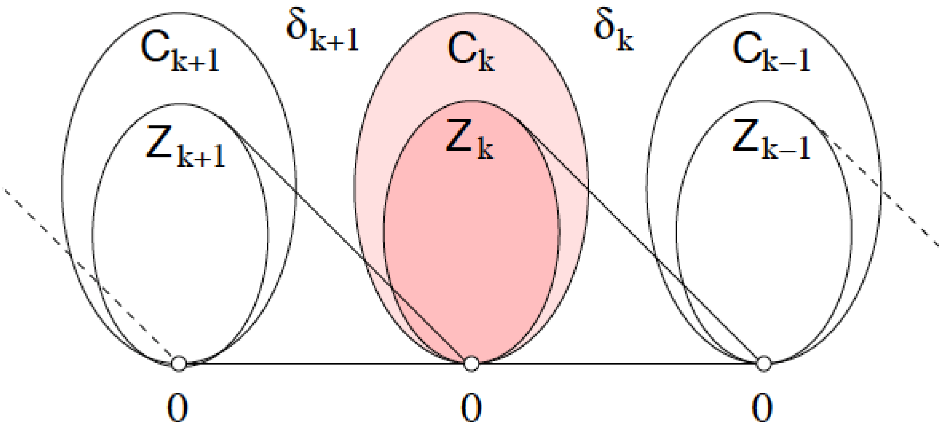
$$\dots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \dots .$$

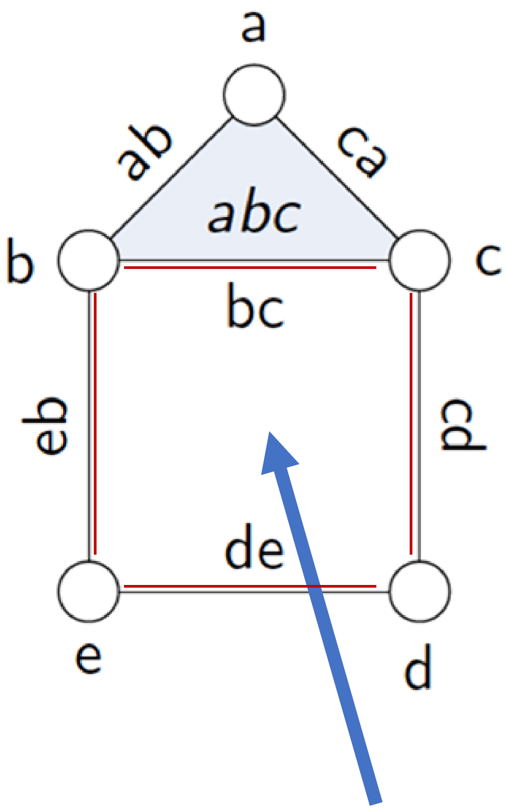


Cycle Group

- Let c be a k-chain
- If it has no boundary, it is a k-cycle (zycle?)
- $\partial_k c = \emptyset$, so $c \in \ker \partial_k$
- The kth cycle group is

$$\mathsf{Z}_k = \ker \partial_k = \{c \in \mathsf{C}_k \mid \partial_k c = \emptyset\}.$$



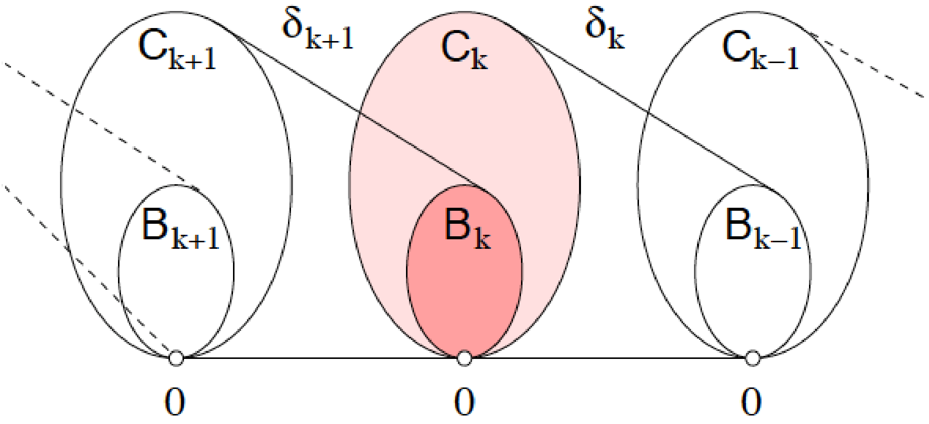


$\partial(bc + cd + de + eb)$
$= c - b + d - c + e - d + b - e = 0.$

Boundary Group

- Let b be a k-chain
- If b is a boundary of something, it is a k-boundary.
- The kth boundary group is

$$\mathsf{B}_k = \operatorname{im} \partial_{k+1} = \{c \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1} : c = \partial_{k+1} d\}.$$



Boundaries are Cycles!

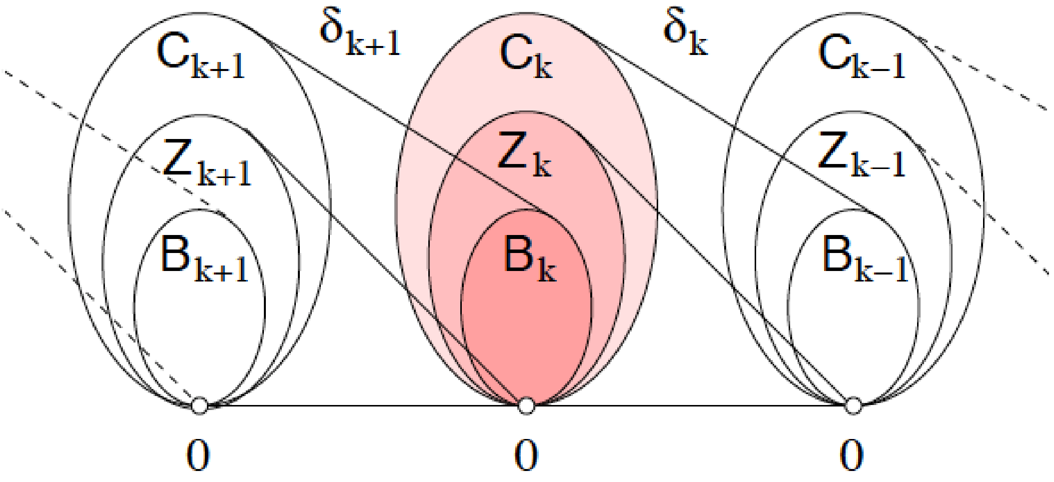
- Let b be a k-boundary.
- Then, $\exists c \in \mathbf{C}_{k+1}$, such that $b = \partial_{k+1} c$.
- What is the boundary of b?

$$\partial_k b = \partial_k \partial_{k+1} c = \emptyset,$$

by the boundary theorem.

- That is, every boundary is a cycle!

- $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$



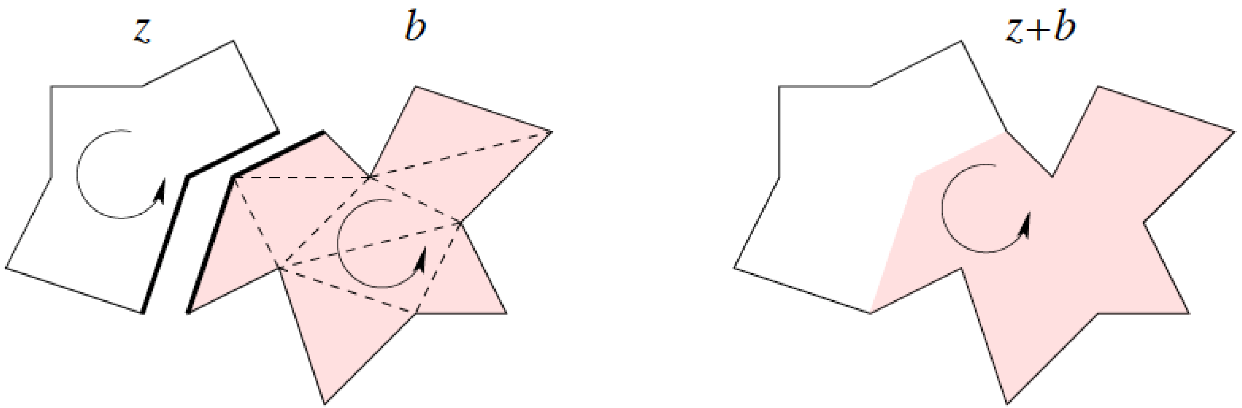
Nesting behavior

Equivalent Cycles

- z is a k-cycle
- b is a k-boundary
- We would like to have $z + b$ be equivalent to z

Cosets!

- That is, if $z_1 - z_2 = b$ where b is a boundary, then $z_1 \sim z_2$
- Any boundary would do!

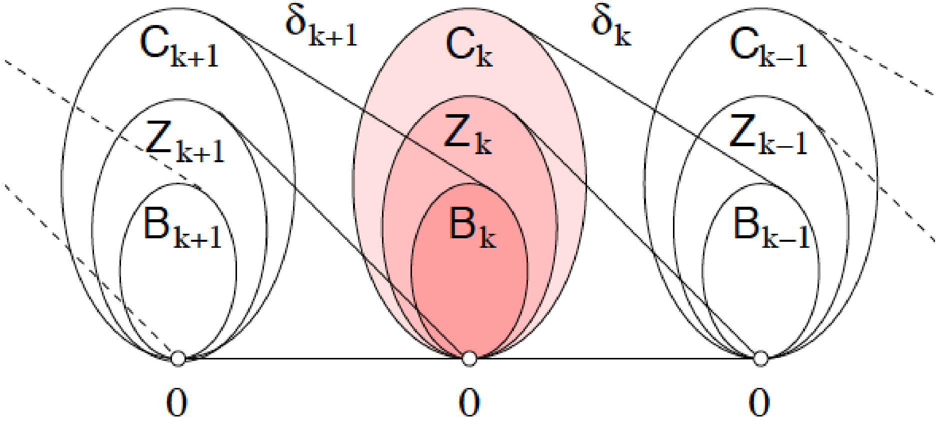


Simplicial Homology

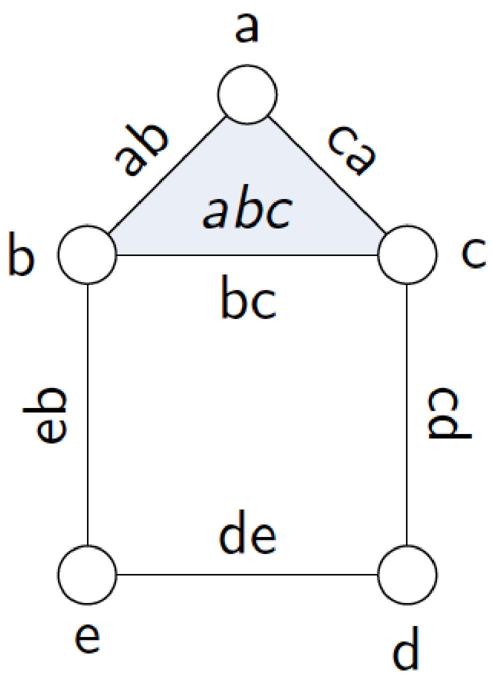
- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

- If $z_1 = z_2 + \mathsf{B}_k$, $z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are homologous
- $z_1 \sim z_2$.



To Repeat



In other words..

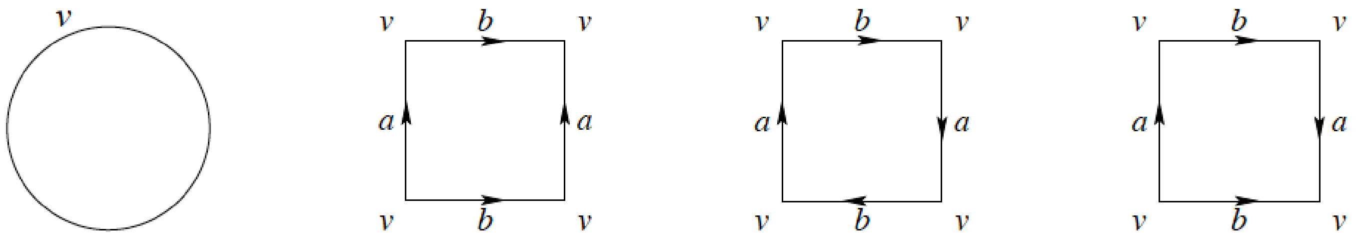
- The kernel (null space) of ∂_k is the vector space of cycles in dimension k.
- The image of ∂_k is the subspace of boundary cycles in dimension $k-1$.

Homology of a space X is the quotient:

$$H_k(X) = \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$$

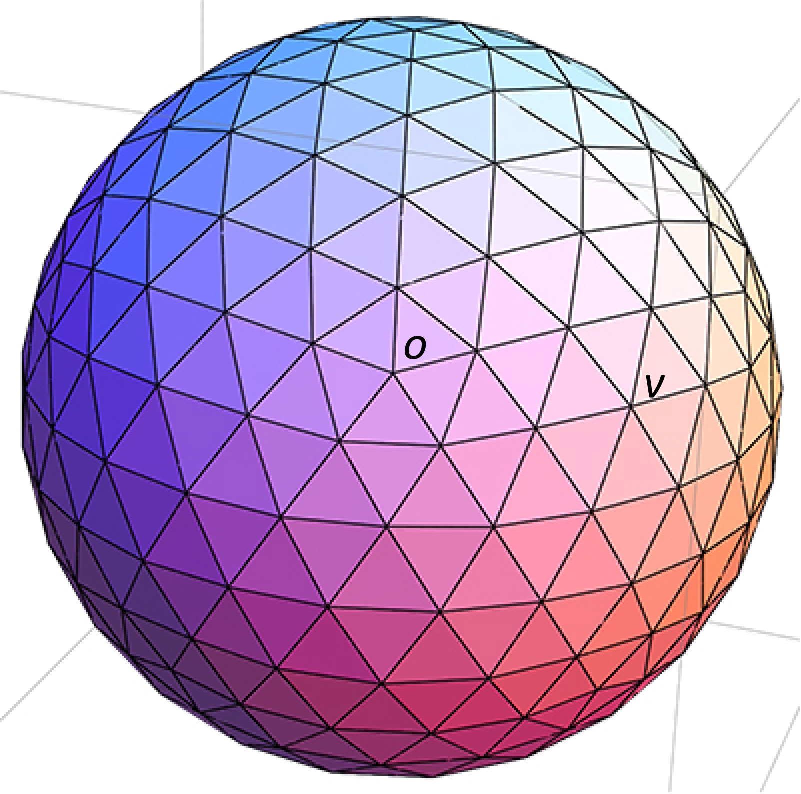
Homology of 2-Manifolds

2-manifold	H_0	H_1	H_2
sphere	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$
torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$



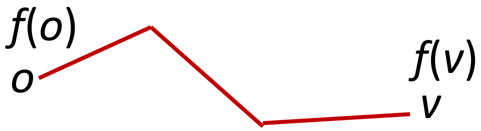
(a) Sphere (b) Torus (c) Projective plane (d) Klein bottle

Why is $H_0(S^2) = Z$?



A 0-chain is an assignment of integers f(v) to each mesh vertex v.

The mesh is connected, so all pairs of vertices are homologous.



So we can pick a specific vertex o and use paths from o to each other v to “cancel” f at v.

Homology Groups

- Homology groups are finitely generated abelian.
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

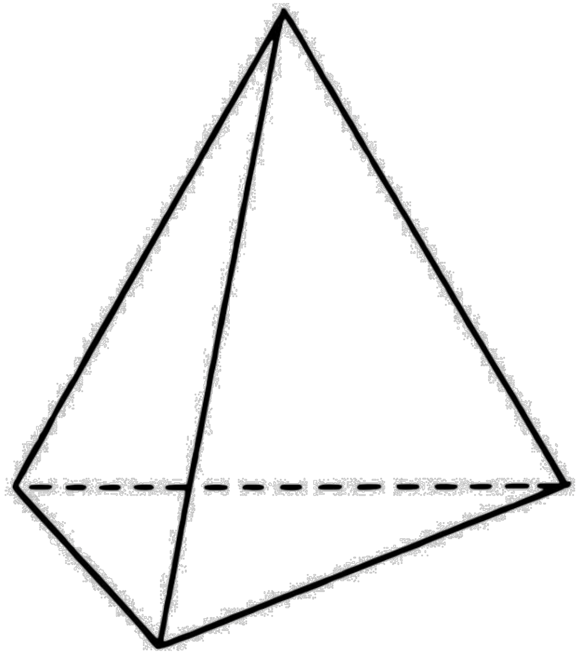
$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

- The kth Betti number β_k of a simplicial complex K is $\beta_k = \beta(\mathsf{H}_k)$, the rank of the free part of H_k.
- Torsion coefficients

- Alexander Duality:
  - β_0 measures the number of components of the complex.
  - β_1 is the rank of a basis for the tunnels.
  - β_2 counts the number of voids in the complex.

Betti Numbers β_i

- Ranks of the free part of homology groups H_i
- β_0 counts the number of connected components
- β_1 counts the number of independent loops
- β_2 counts the number of independent voids
- …

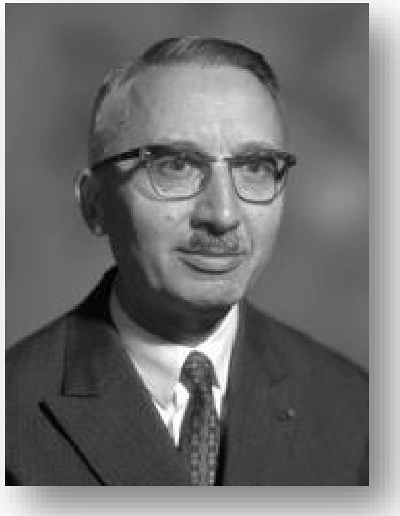


Topology is fundamentally a tool for classification

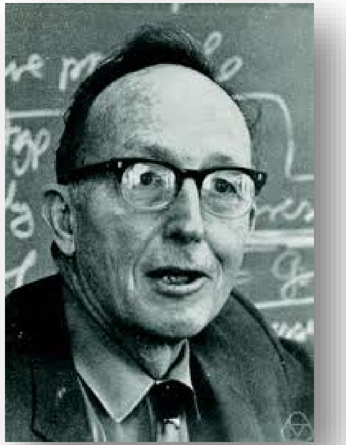
Invariance of Homology Groups

- (Hauptvermutung) Any two triangulations of a topological space have a common refinement (Poincaré 1904)
  - True for polyhedra of dimension ≤ 2 (Papakyriakopoulos 1943)
  - True for 3-manifolds (Moïse 1953)
  - False in dimensions ≥ 6 (Milnor 1961)
  - False for manifolds of dimension ≥ 5 (Kirby and Siebenmann 1969)
- Singular homology

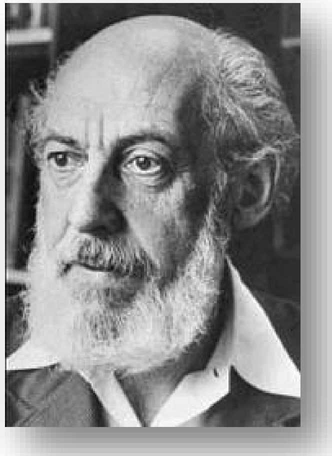
Homological Algebra: Functors and Categories



Herni Cartan



Saunders MacLane

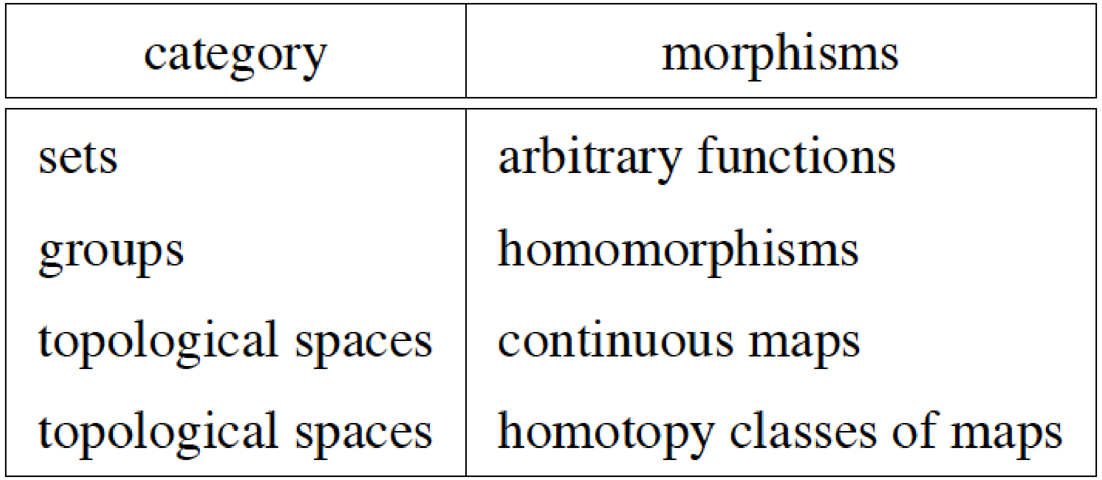


Samuel Eilenberg

Categories

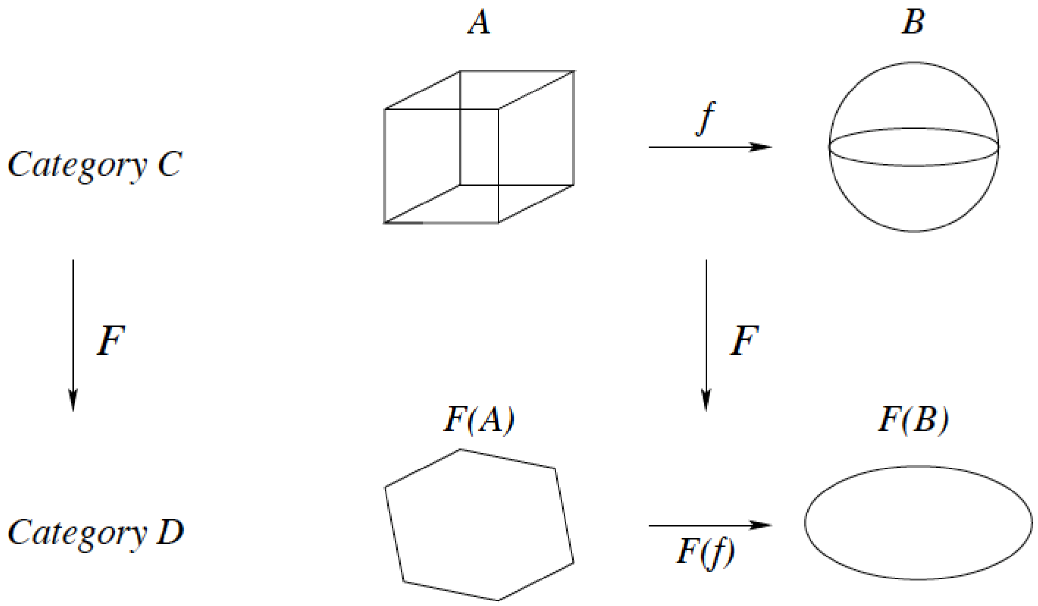
- A collection $\mathrm{Ob}(\mathcal{C})$ of objects
- Sets $\mathrm{Mor}(X, Y)$ of morphisms for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$
- An identity morphism $1 = 1_X \in \mathrm{Mor}(X, X)$ for each X.
- a composition of morphisms function $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, satisfying $f \circ 1 = 1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.
- A category $\mathcal{C}$

Example Categories



category	morphisms
sets	arbitrary functions
groups	homomorphisms
topological spaces	continuous maps
topological spaces	homotopy classes of maps

Functors



- $X \in \mathcal{C}, F(X) \in \mathcal{D},$
- $f \in \mathrm{Mor}(X, Y), F(f) \in \mathrm{Mor}(F(X), F(Y))$
- $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$
- F is a (covariant) functor

Functoriality

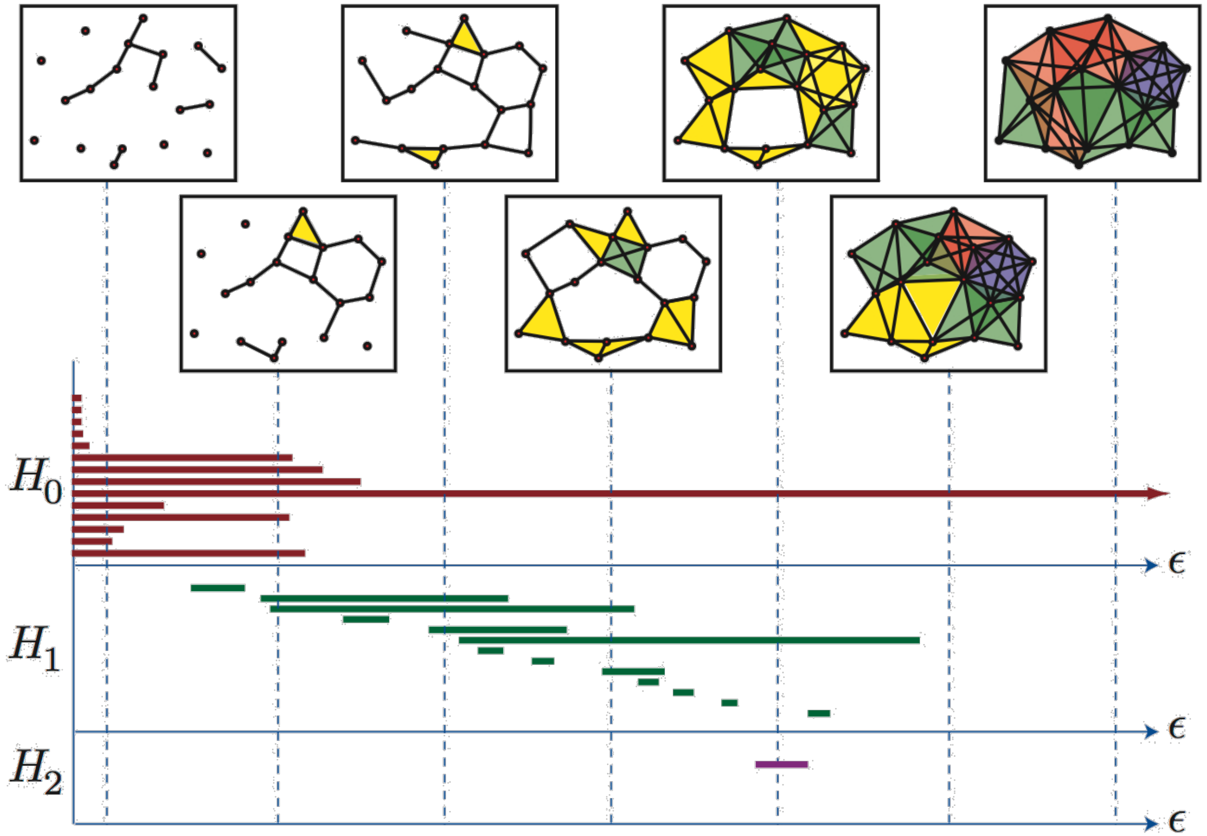
transformation of input ⇒ transformation of output

Specifically, this is a commutative diagram:

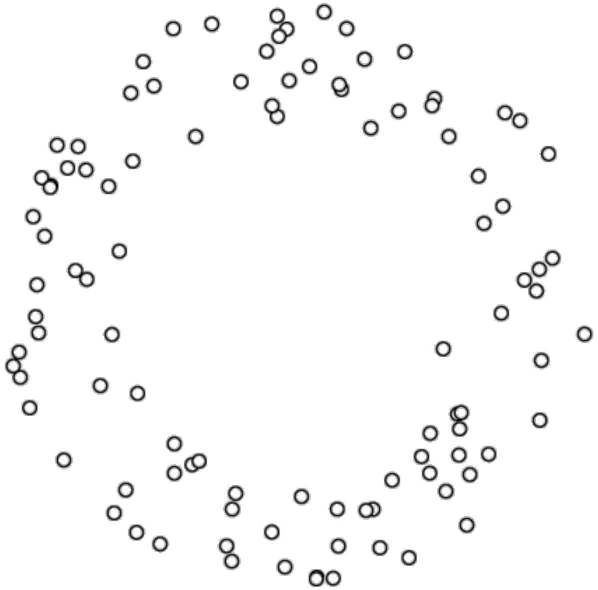
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "H_*"'] & Y \arrow[d, "H_*"] \\
H_*(X) \arrow[r, "H_*(f)"] & H_*(Y)
\end{tikzcd}$$

Moral: Invariants are not artifacts of arbitrary choices!

Persistent Homology



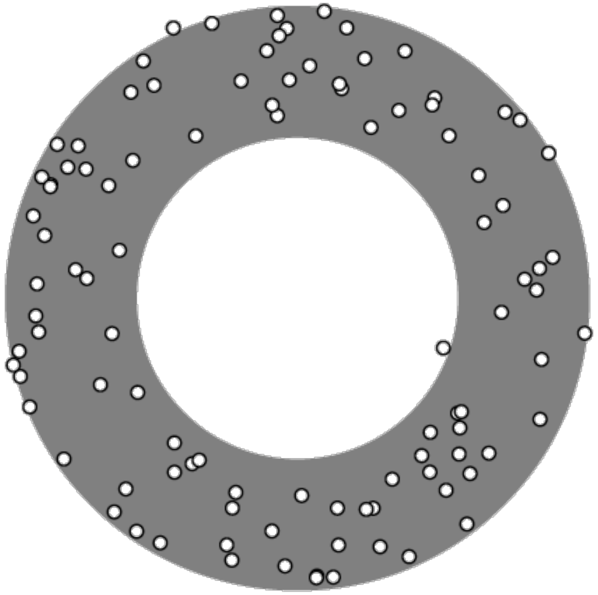
Sampled Data Has “Shape”



2-dimensional

Approximates annulus

Sampled Data Has “Shape”



2-dimensional

Approximates annulus

Topological features of annulus:

1 component ($\beta_0 = 1$)

1 loop ($\beta_1 = 1$)

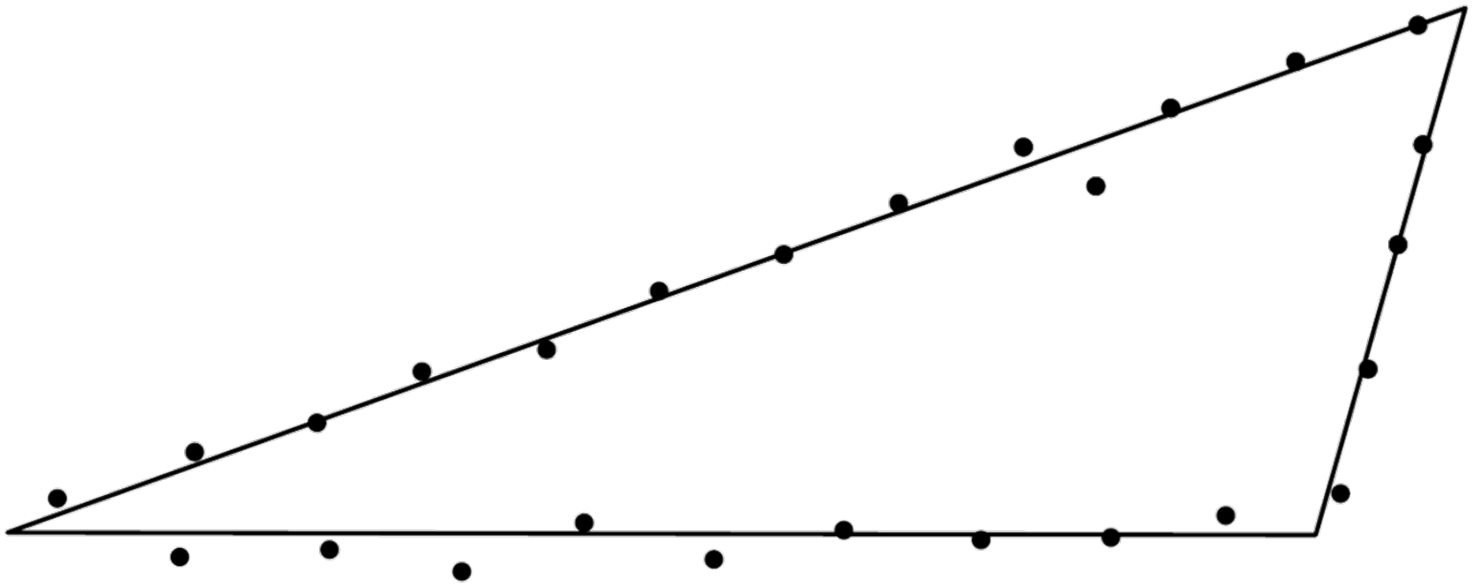
Goal: Recover *topology* of annulus from point cloud

We do so by building various complexes on the point cloud

Filtrations

How to Choose ϵ?

- How to determine the topology of the underlying space from a point cloud approximation?



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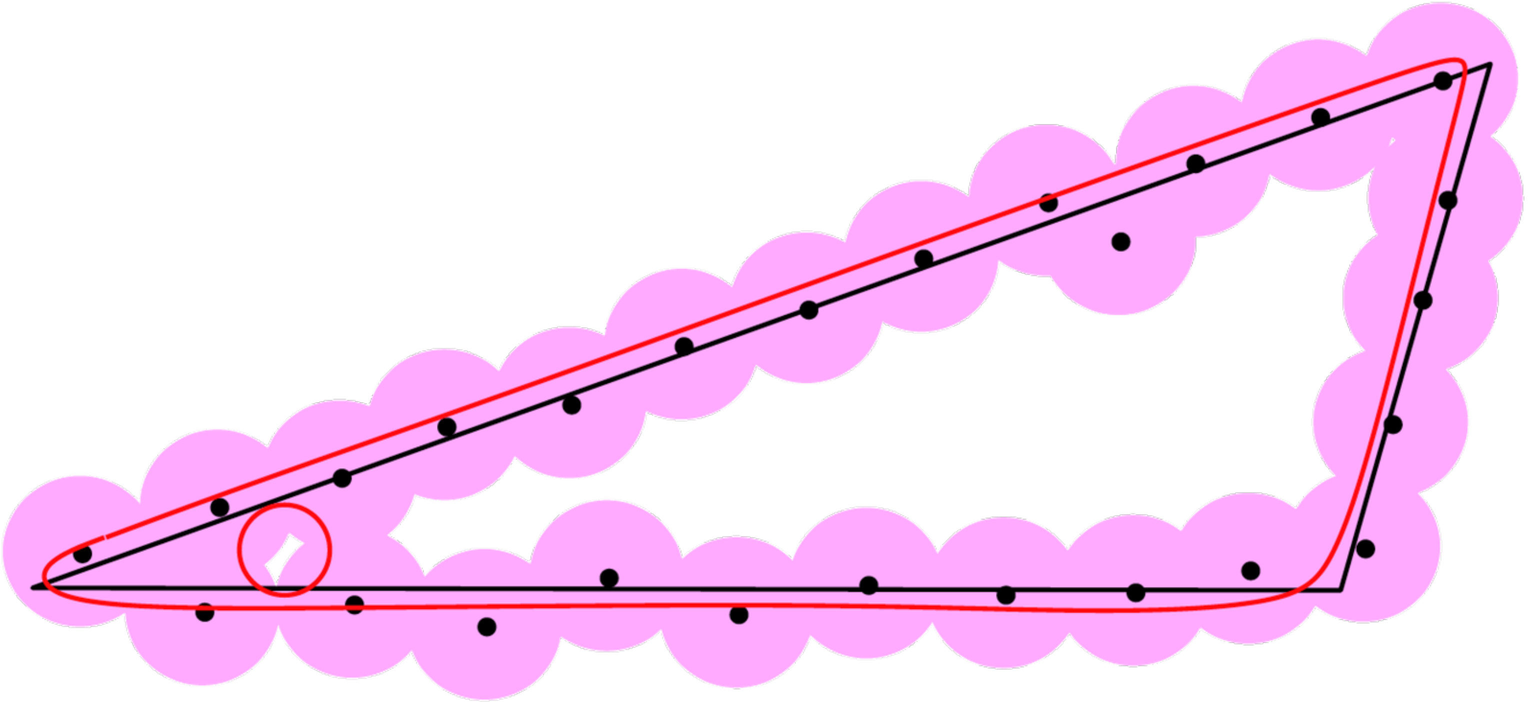

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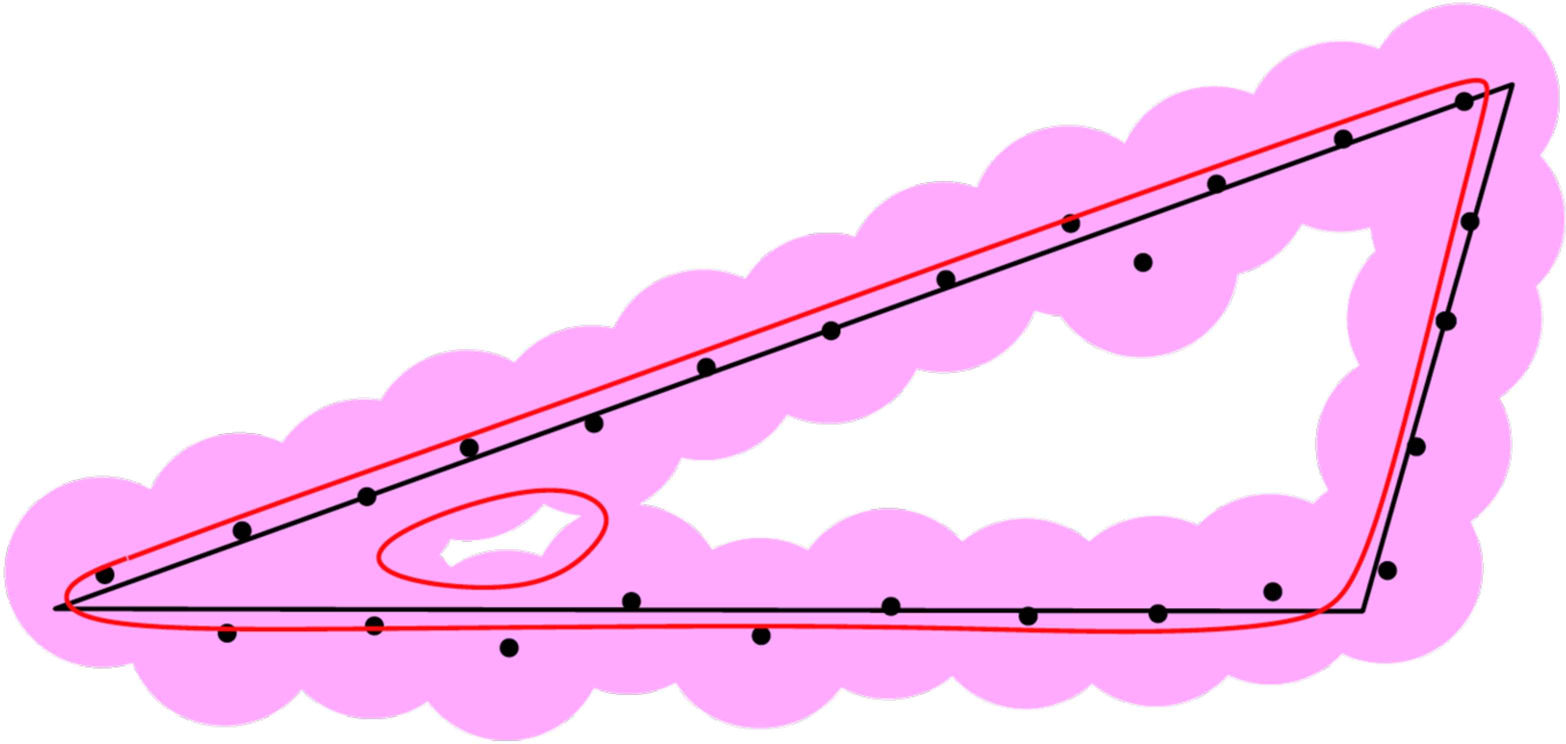

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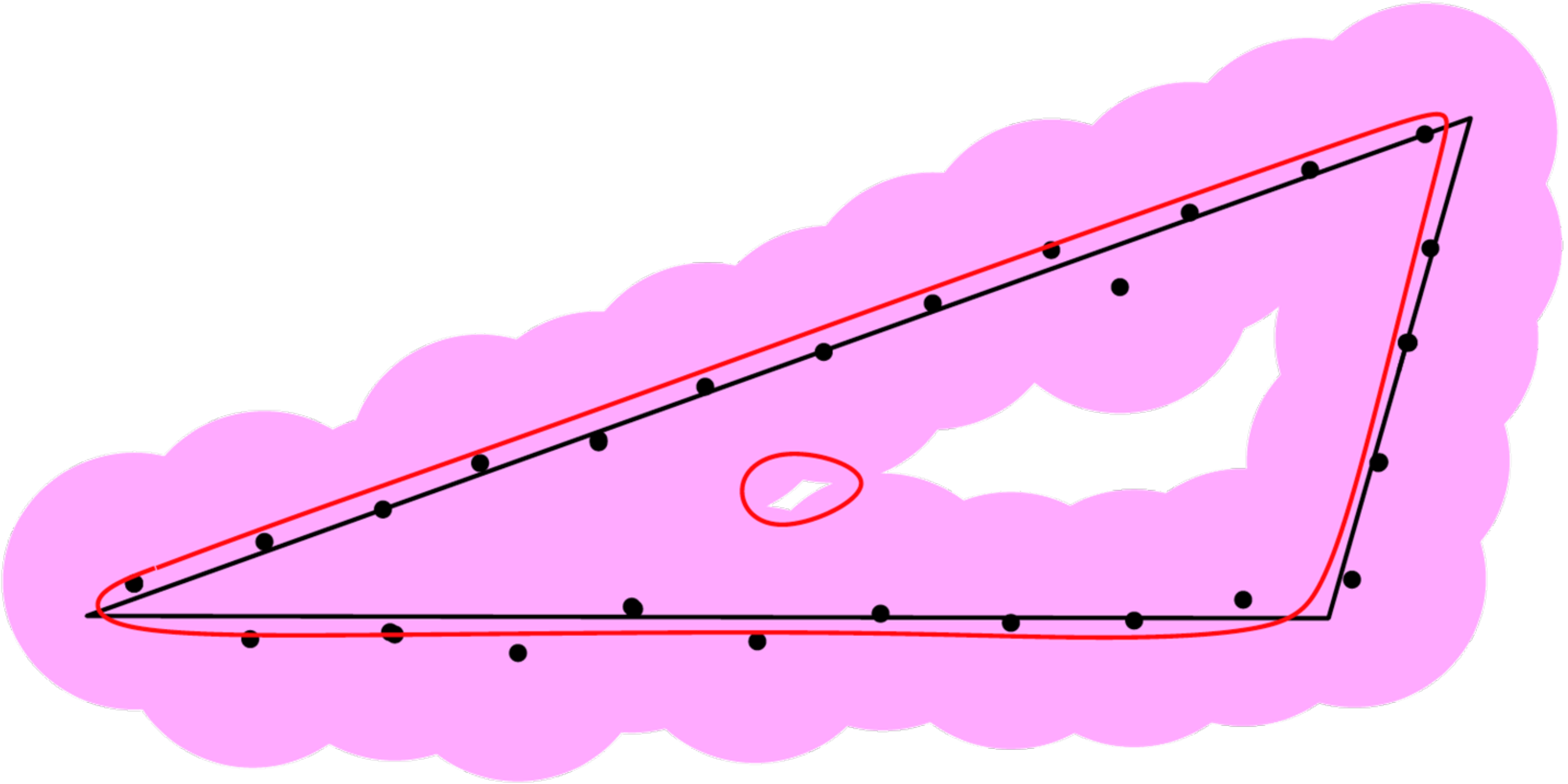
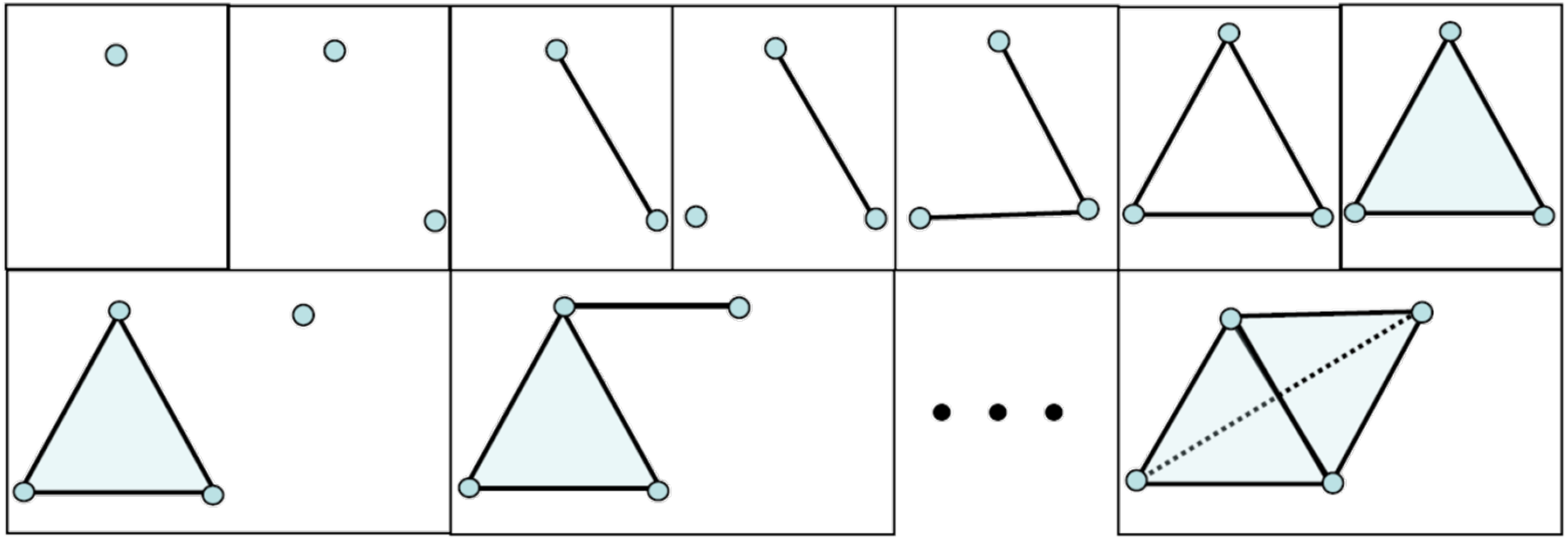


Filtrations



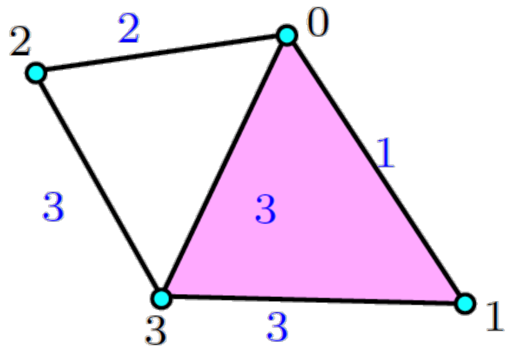
A filtration of a (finite) simplicial complex K is a sequence of subcomplexes such that

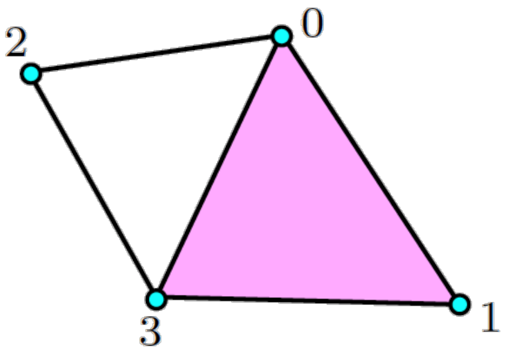
i) $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$,

ii) $K^{i+1} = K^i \cup \sigma^{i+1}$ where σ^{i+1} is a simplex of K.

Sub-simplices of a simplex must be added before the simplex!

The Sub-Level Set Filtration

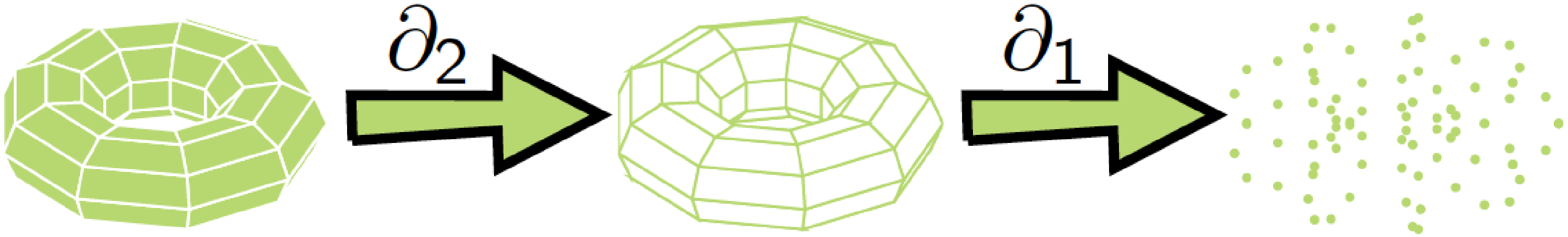




- f a real valued function defined on the vertices of K
- For $\sigma = [v_0, \cdots, v_k] \in K$, $f(\sigma) = \max_{i=0,\cdots,k} f(v_i)$
- The simplices of K are ordered according increasing f values (and dimension in case of equal values on different simplices).

Persistent Homology: Do not choose an ϵ!

Standard Homology



C_2 vector space of faces

C_1 vector space of edges

C_0 vector space of vertices

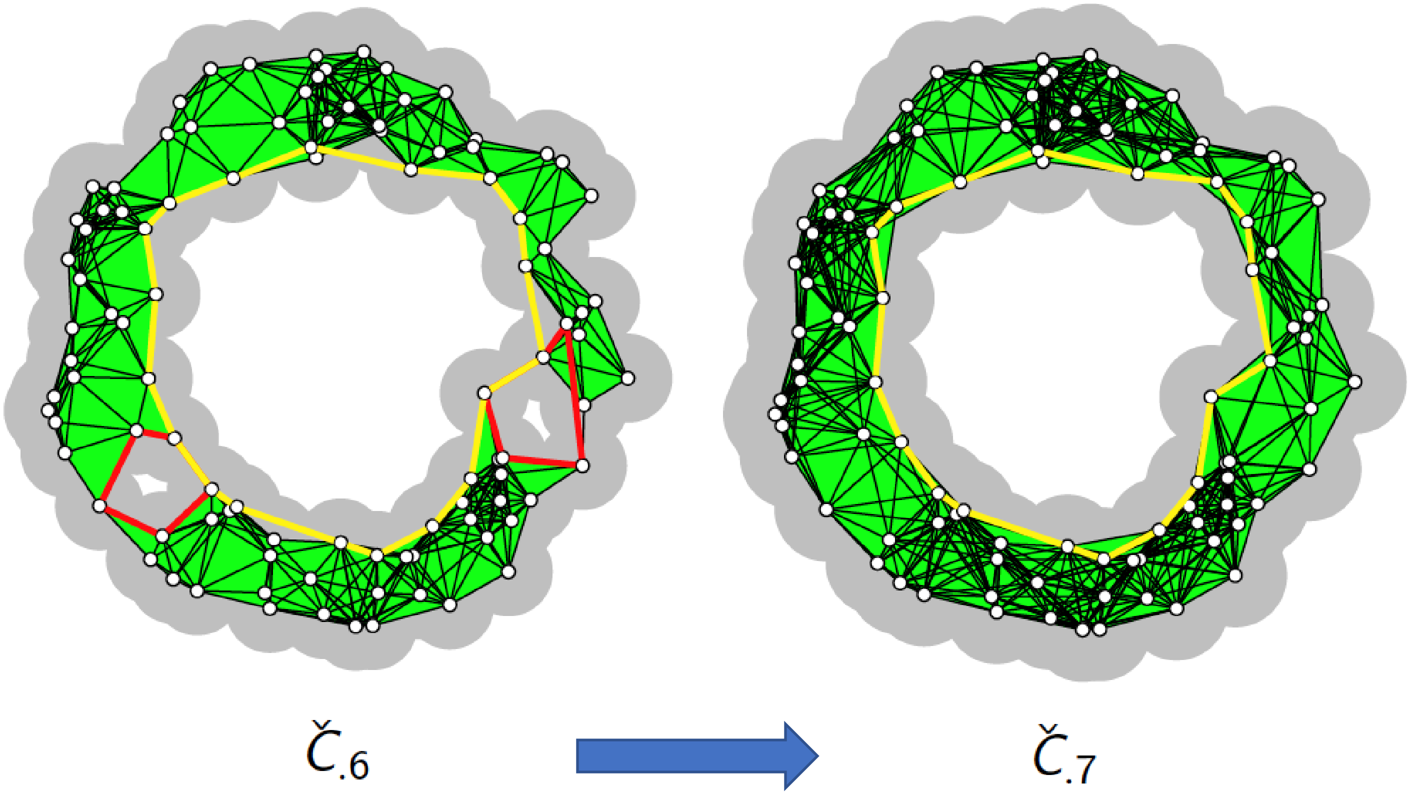
Take the linear extension of the boundary operator:

$$\partial_d([v_0, \dots v_d]) = \sum_{i=0}^{d} (-1)^i [v_0 \dots, \hat{v}_i, \dots v_d]$$

Fact: $\partial_{d-1} \circ \partial_d \equiv 0 \Rightarrow \operatorname{Im} \partial_d \subseteq \ker \partial_{d-1}$

Definition: $H_d(K) = \ker \partial_d / \operatorname{Im} \partial_{d+1}$

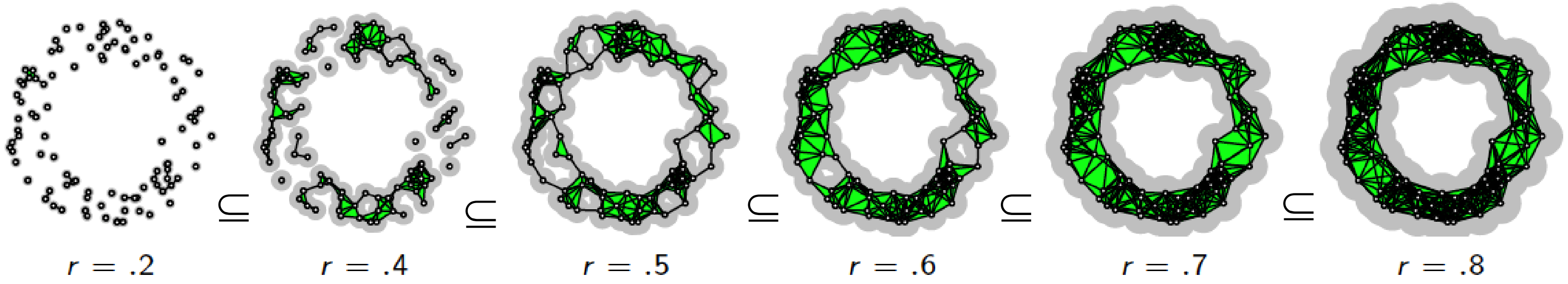
We Can Track Topological Features in a Filtration



Functorality allows us to systematically track holes over time!

The inclusion map among the complexes translates to a homomorphism between the homology groups

Persistent Homology is Functorial Homology

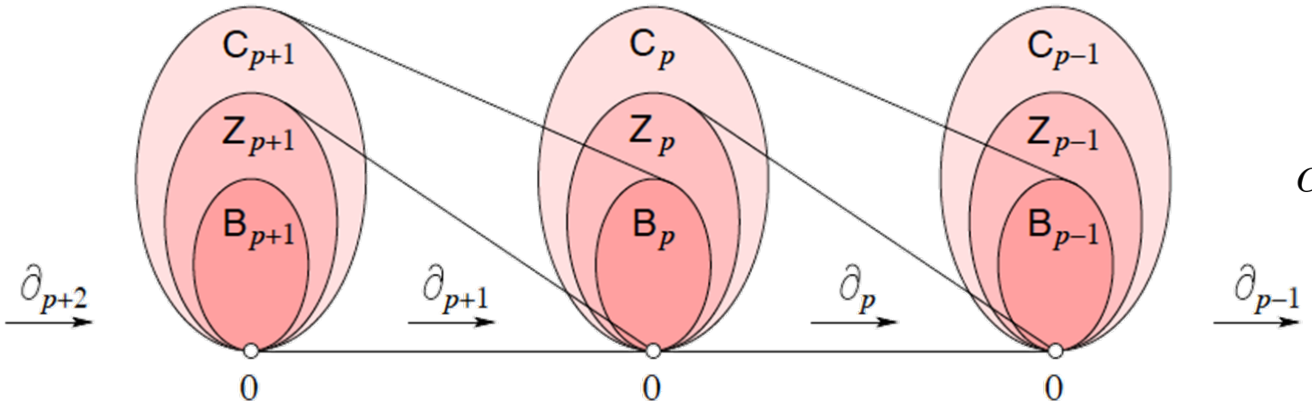


$$H_d(\check{C}_*) = \bigoplus_{\epsilon} H_d(\check{C}_\epsilon)$$

Homology of the entire filtration

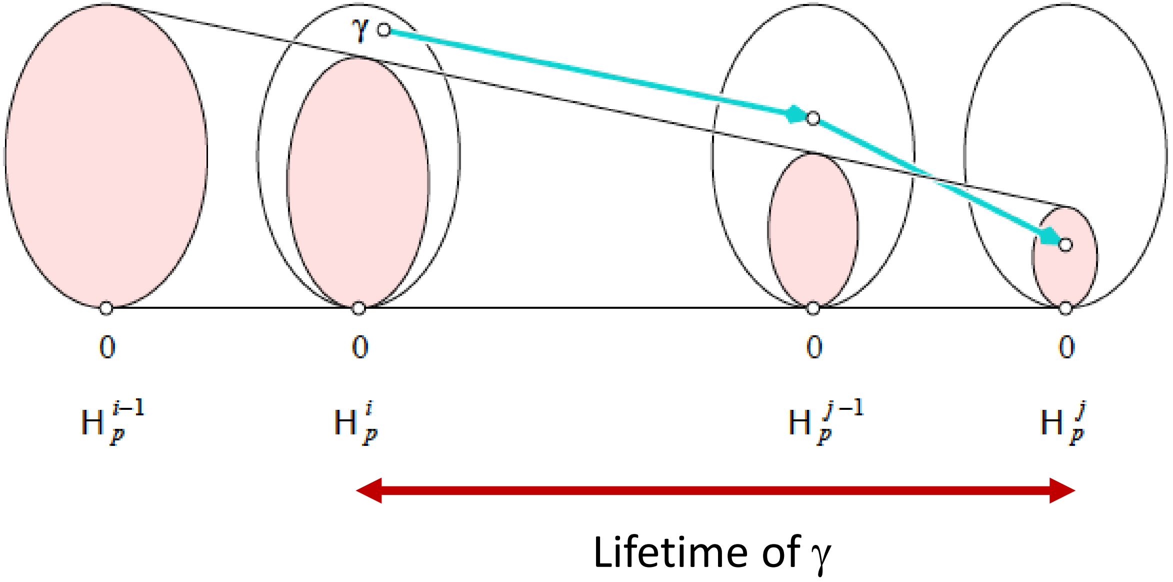
Homomorphisms at the homology level allow us
to track homology classes – i.e., topological features

Persistence of Homology Classes



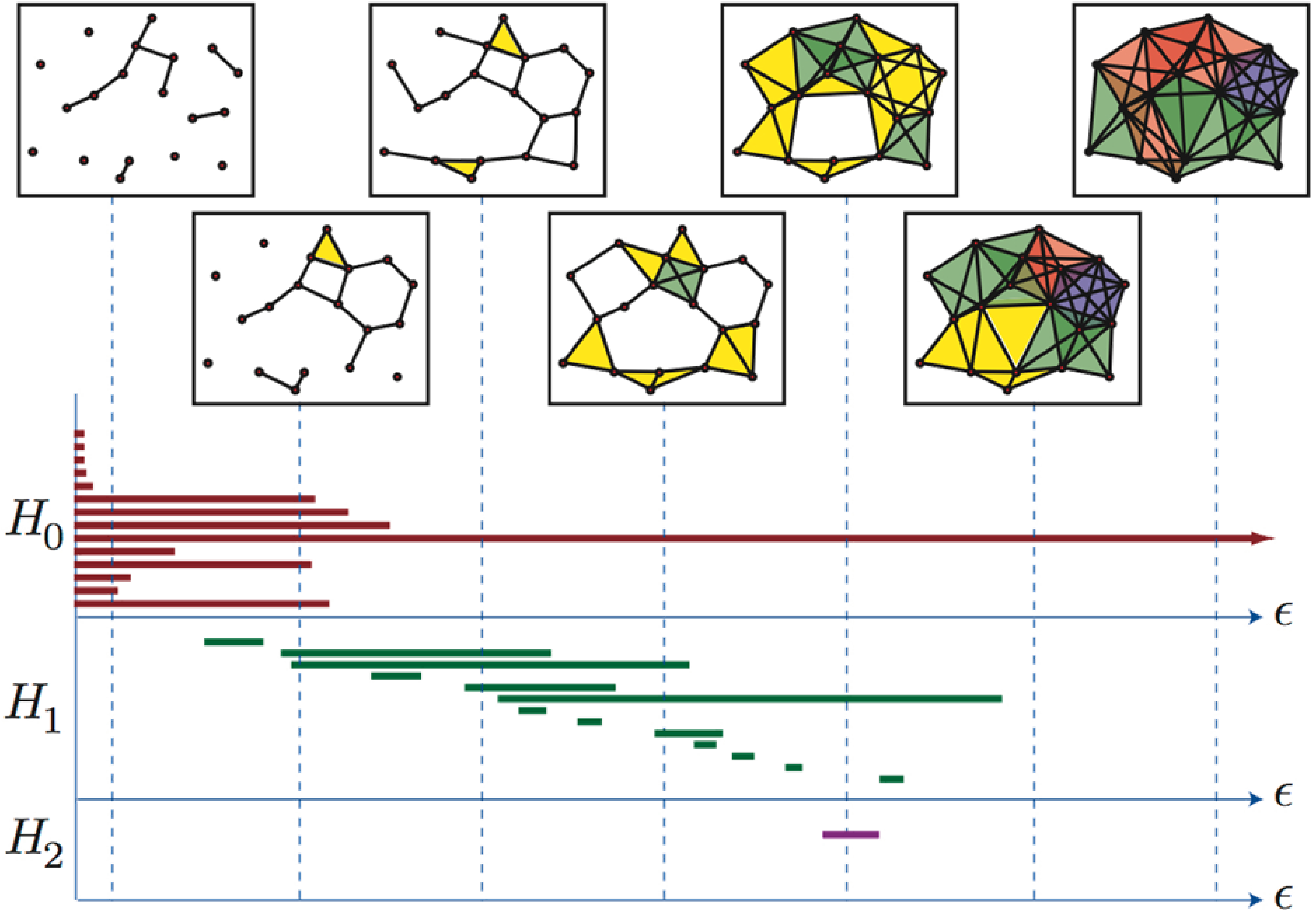
$$c_p = z_p + b_{p-1}$$

$$G/\ker(f) \equiv \operatorname{im}(f)$$



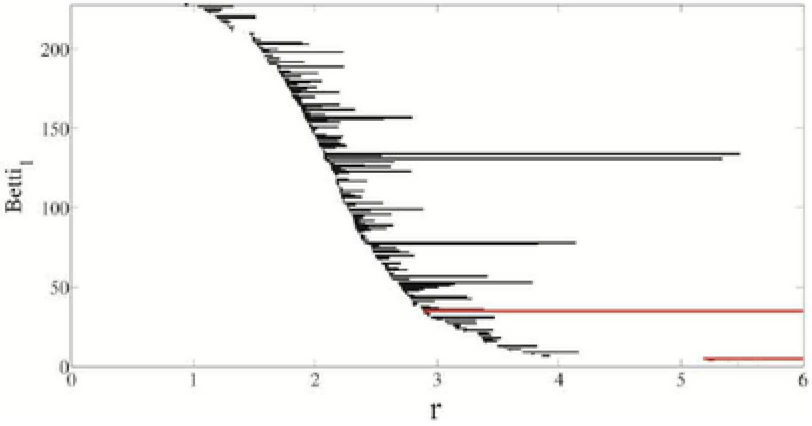
Lifetime of γ

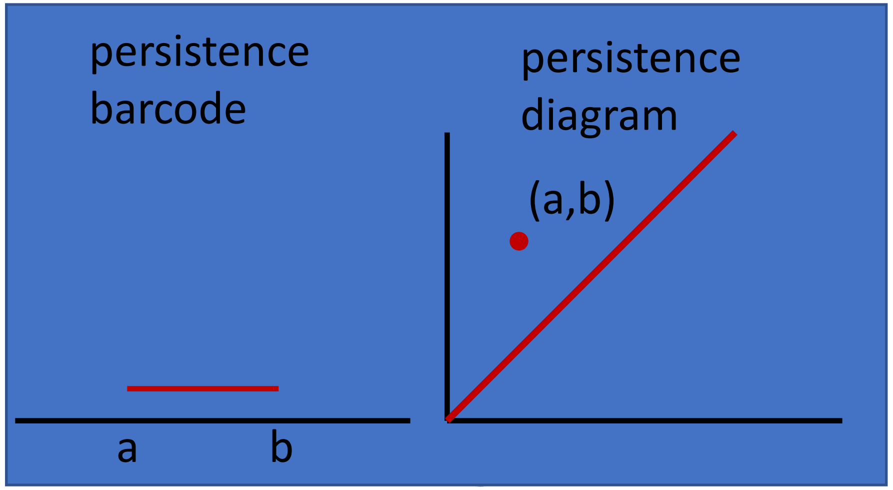
Barcodes are the Lifetimes of Topological Features



Barcodes are the output of persistent homology

Another View: Persistence Diagrams




persistence barcode

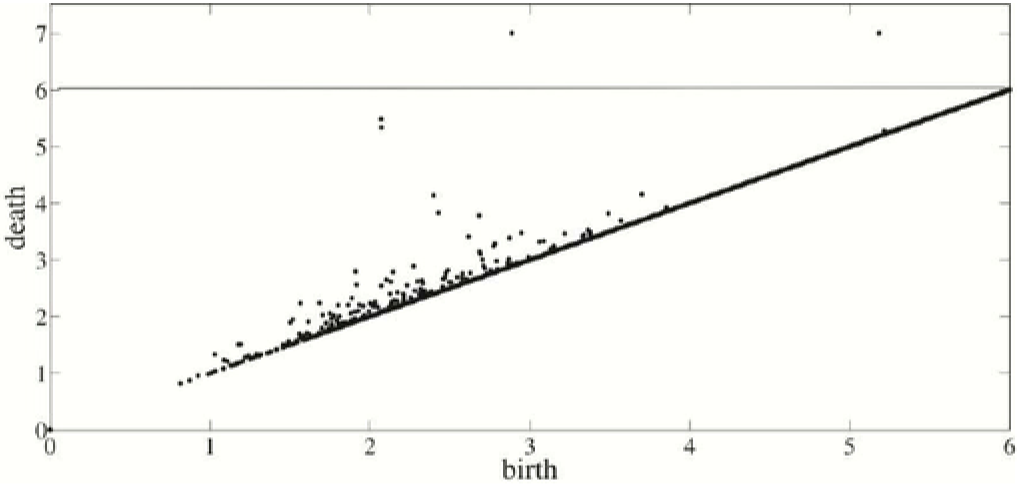
persistence diagram

(a,b)

a

b

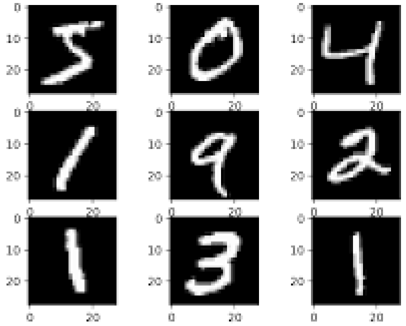
long barcodes = points away from the diagonal = robust features

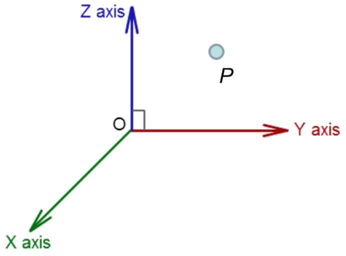


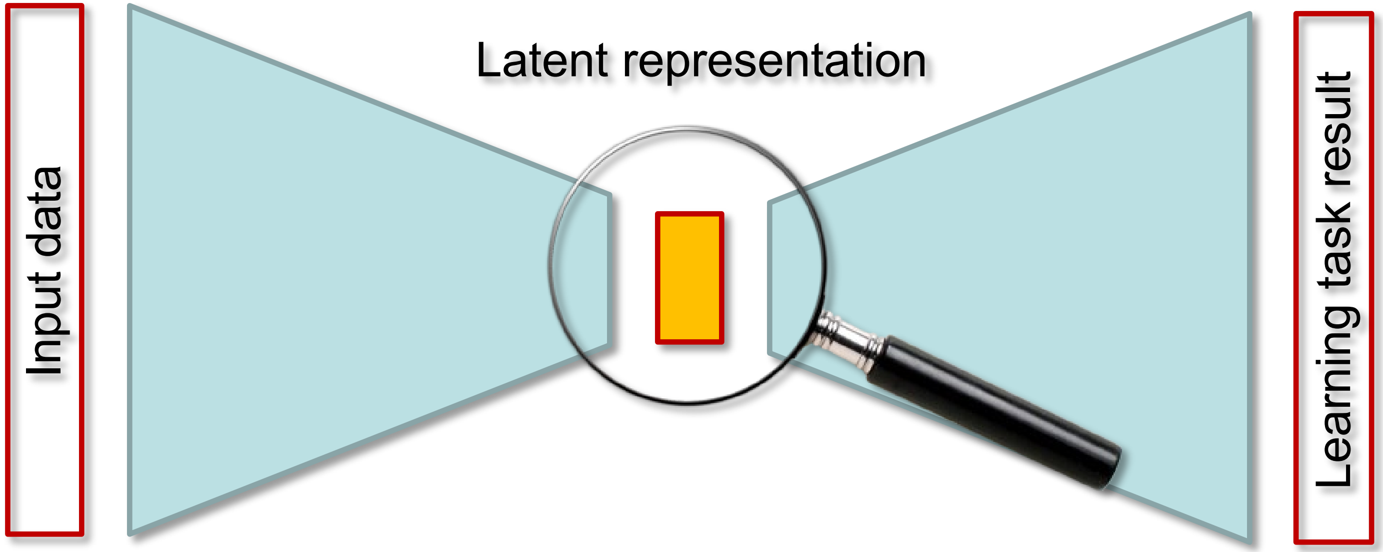
Short barcodes = points near the diagonal = noise

Map 1-D intervals to points in 2-D

Algebraic Descriptors for Data







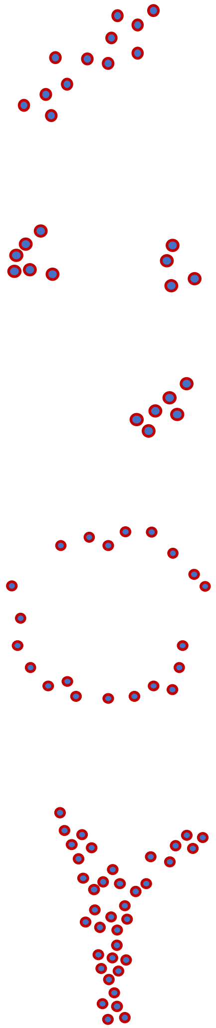
Input data

Latent representation

Learning task result

Feature extraction, encoding

Output generation, decoding



The End