CS233, CME251: Geometric and Topological Data Analysis

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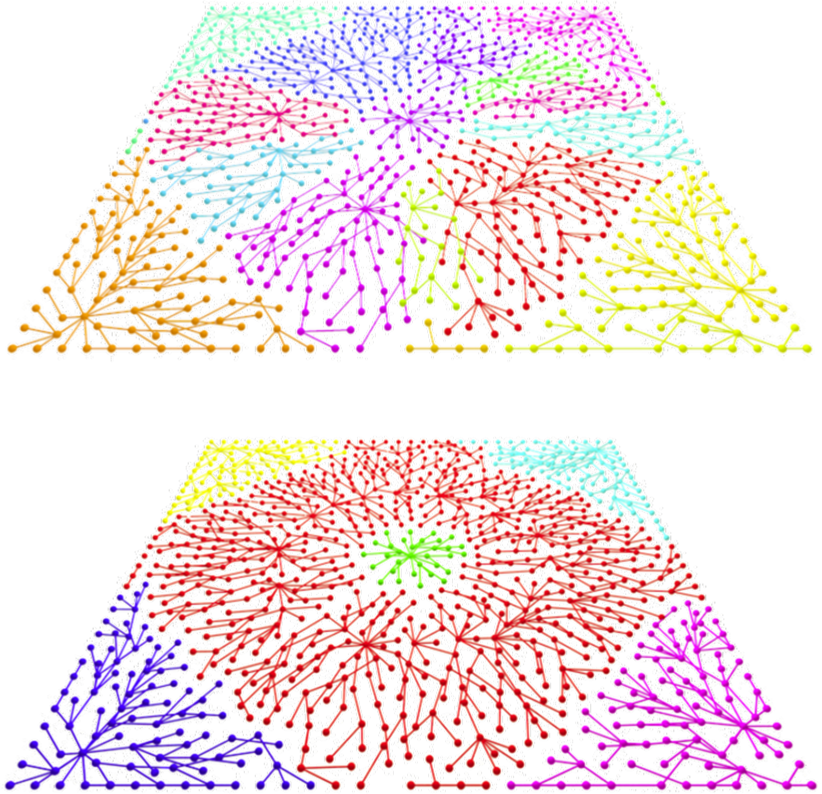


Lecture 7
19 April 2021

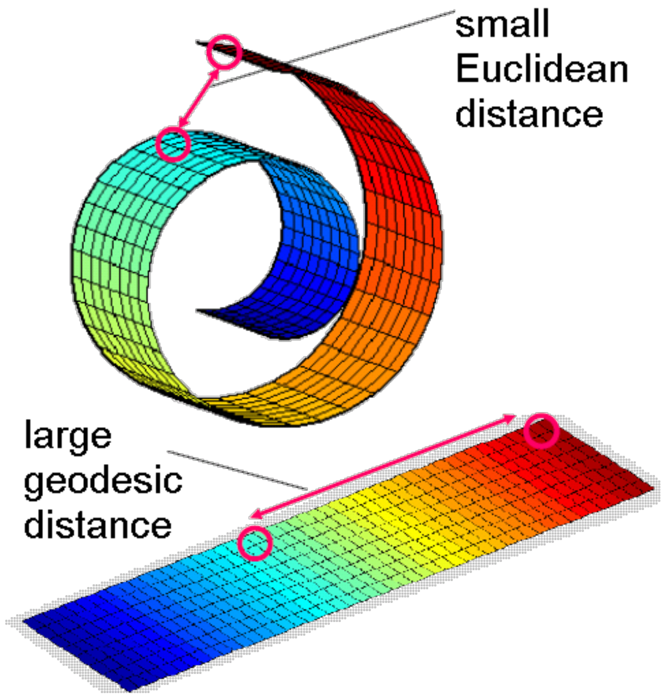
Leonidas Guibas Laboratory
Geometric Computing

Last Time: Non-Linear Dimensionality Reduction

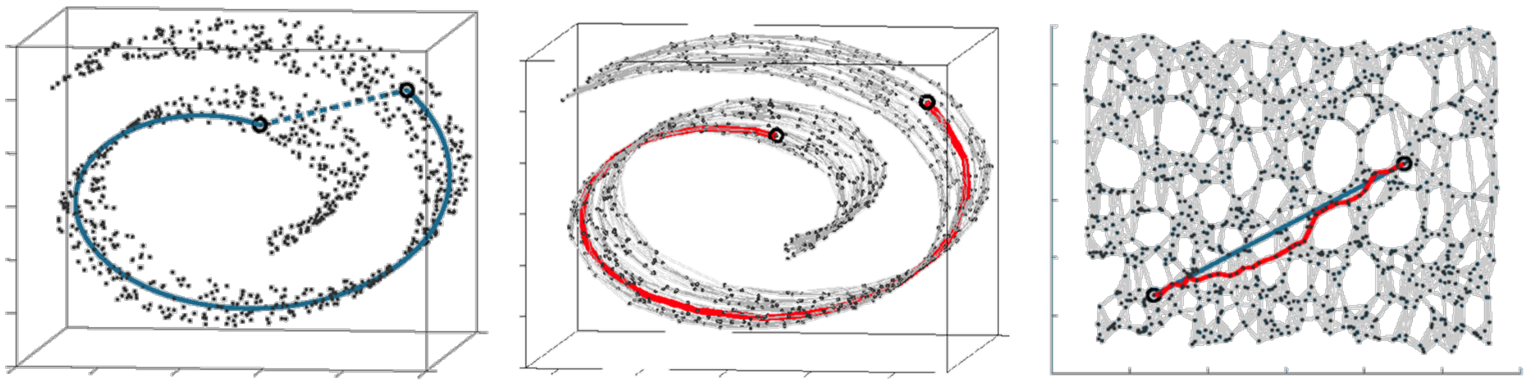
Isomap, Laplacian Eigenmaps, Locally Linear Embeddings, t-SNE, Autoencoders, Self-organizing Maps

ISOMAP (J. B. Tenenbaum, V. de Silva, and J. C. Langford)

- Example of non-linear structure (Swiss roll)
  - Only the geodesic distances reflect the true low-dimensional geometry of the manifold

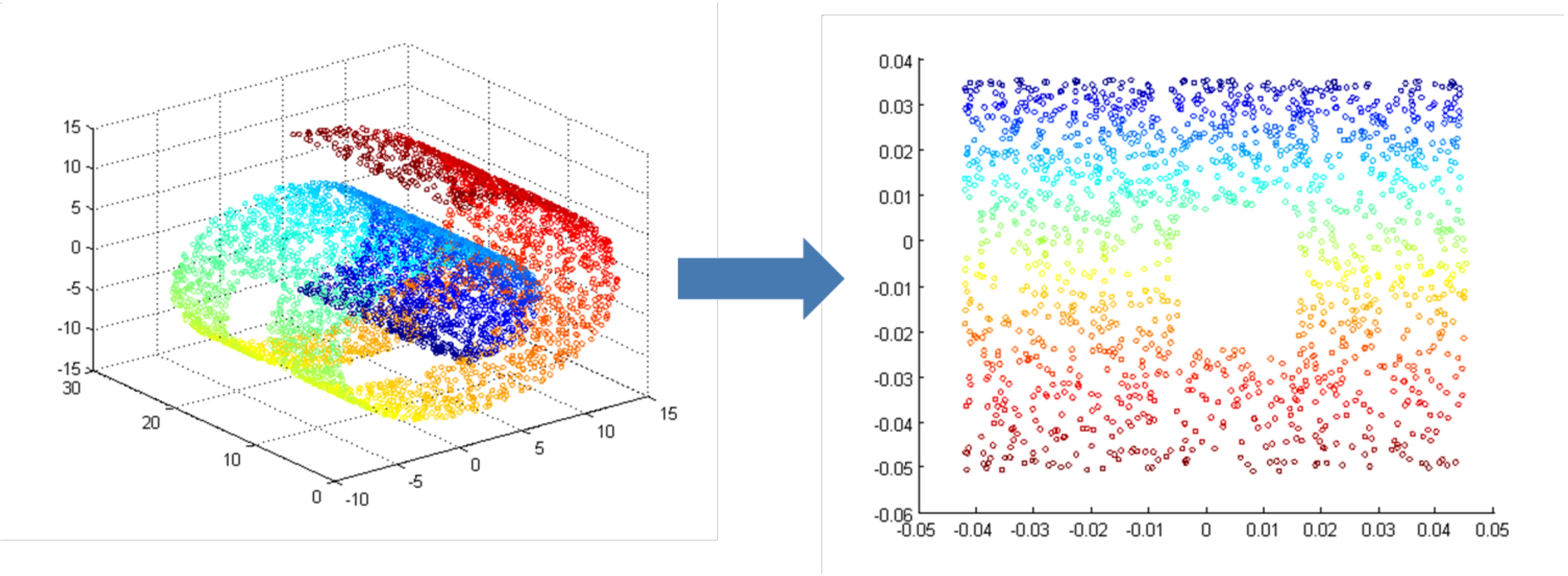


- ISOMAP (Isometric Feature Mapping)
  - Uses the geodesic manifold distances between all pairs
  - Preserves the intrinsic geometry of the data -- Preserves the geodesic distances
  - Estimate geodesic distances between point pairs
  - Use MDS for an embedding



Laplacian Eigenmaps (M. Belkin and P. Niyogi)

- Start same as Isomap, but use a spectral embedding in lieu of MDS



Hole distorts long geodesic distances, but affects less diffusion distances

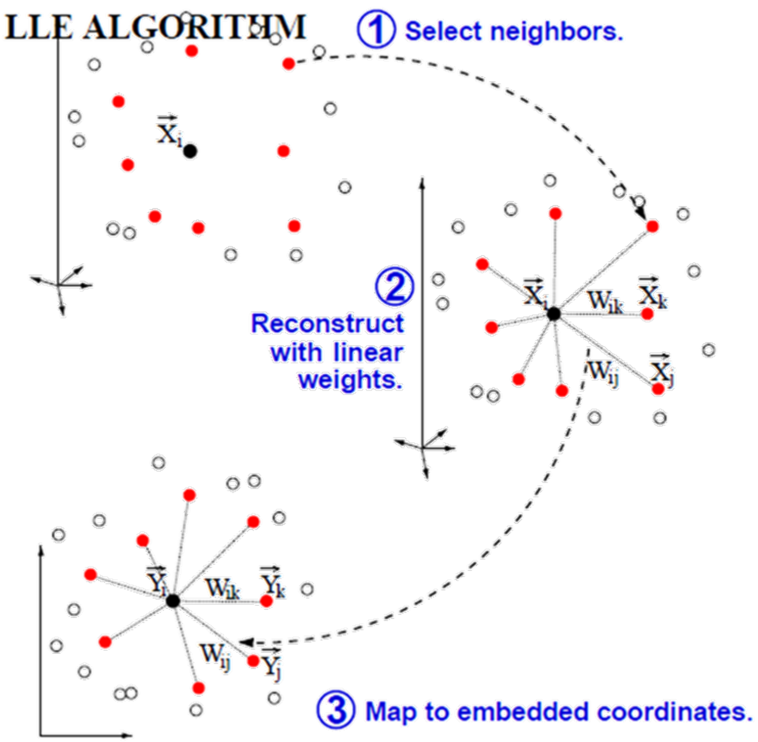
Locally Linear Embeddings (LLE) (S. T. Roweis and L. K. Saul)

- Define neighborhood relations between points (build NN graph)
  - *k* nearest neighbors
  - ε-balls
- Find weights that reconstruct each data point from its neighbors:

$$\min_{\sum_j w_{ij}=1} \left\| \mathbf{x_i} - \sum_{j \in N(i)} w_{ij} \mathbf{x_j} \right\|^2$$

- Find low-dimensional coordinates so that the same weights hold: $\mathbf{x_1}', \ldots, \mathbf{x_n}' \in R^d$

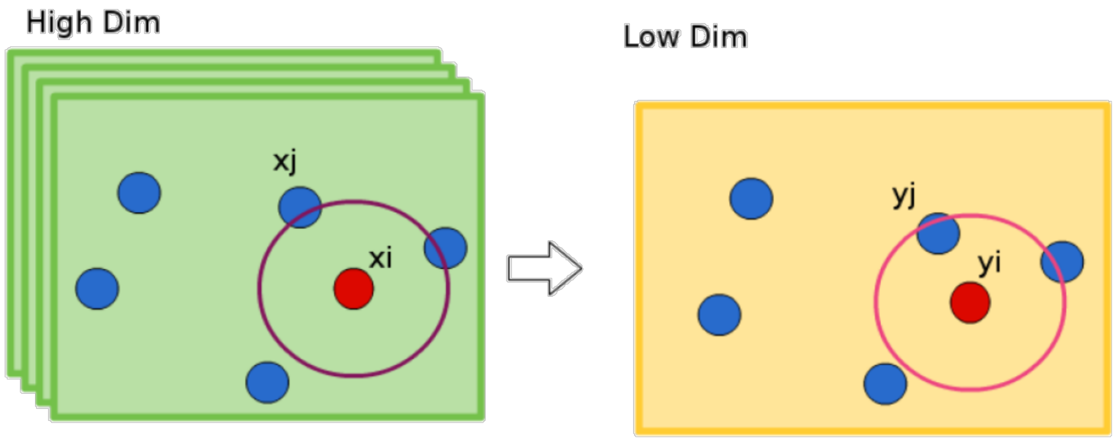
$$\min_{\mathbf{x_1}', \ldots, \mathbf{x_n}'} \sum_i \left\| \mathbf{x_i}' - \sum_{j \in N(i)} w_{ij} \mathbf{x_j}' \right\|^2$$

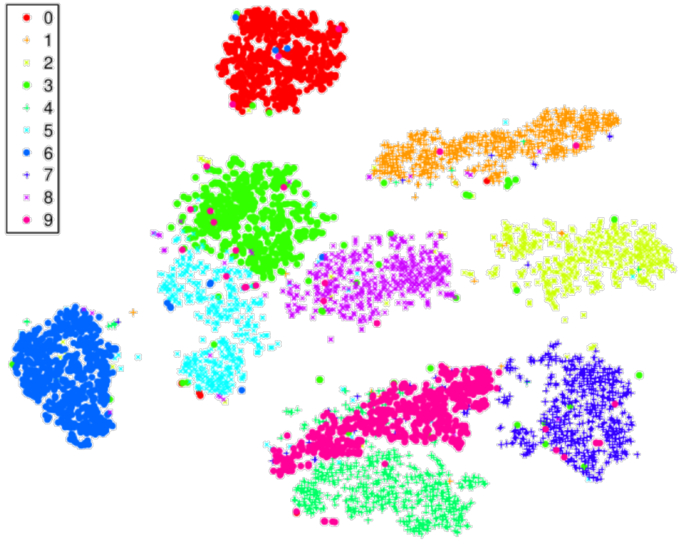


$x_i' = Y_i$

t-Distributed Stochastic Neighbor Embedding

Measure pairwise similarities between high-dimensional and low-dimensonal objects





- Similarity of datapoints in High Dimension

$$p_{ij} = \frac{exp(-||x_i - x_j||^2/2\sigma^2)}{\sum_{k \neq l} exp(-||x_l - x_k||^2/2\sigma^2)}$$

- Similarity of datapoints in Low Dimension

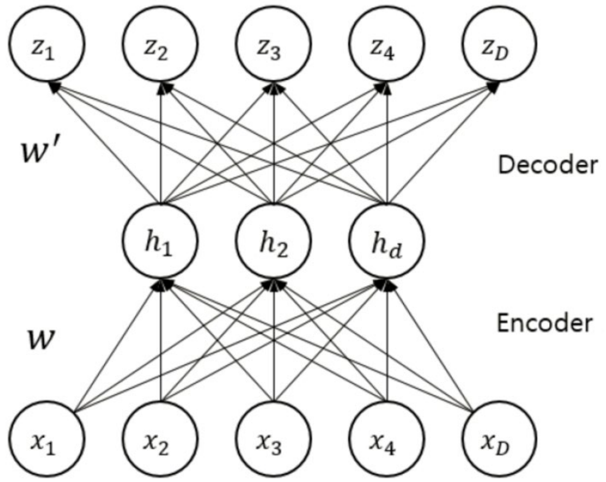
$$q_{ij} = \frac{(1 + ||y_i - y_j||^2)^{-1}}{\sum_{k \neq l}(1 + ||y_k - y_l||^2)^{-1}}$$

Heavy tail Student t-distribution
William Sealy Gosset

$$C = \sum_i KL(P_i||Q_i) = \sum_i \sum_j p_{j|i} log \frac{p_{j|i}}{q_{j|i}}$$

Autoencoder

- An autoencoder is a feedforward neural network trained to reproduce its input at the output layer



- Encoder:

$$\mathbf{h} = f_\theta(\mathbf{x}) = a(\mathbf{W}\mathbf{x} + \mathbf{b})$$

- Decoder:

$$\mathbf{z} = g_{\theta'}(\mathbf{h}) = a(\mathbf{W}'\mathbf{h} + \mathbf{b}')$$

$$\theta, \theta' : \{\mathbf{W}, \mathbf{b}\}, \{\mathbf{W}', \mathbf{b}'\}$$

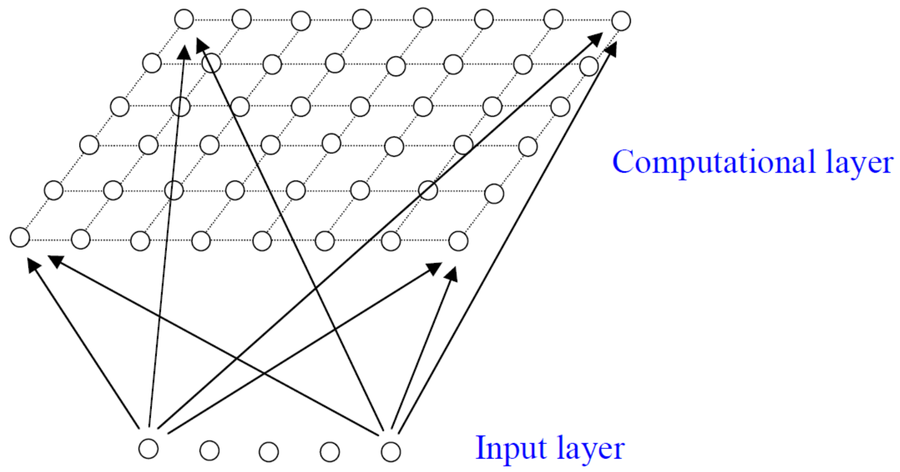
- Parameter estimation
- Back-propagation

$$\theta^*, \theta'^* = \underset{\theta,\theta'}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} L\left(\mathbf{x}^{(i)}, \mathbf{z}^{(i)}\right)$$

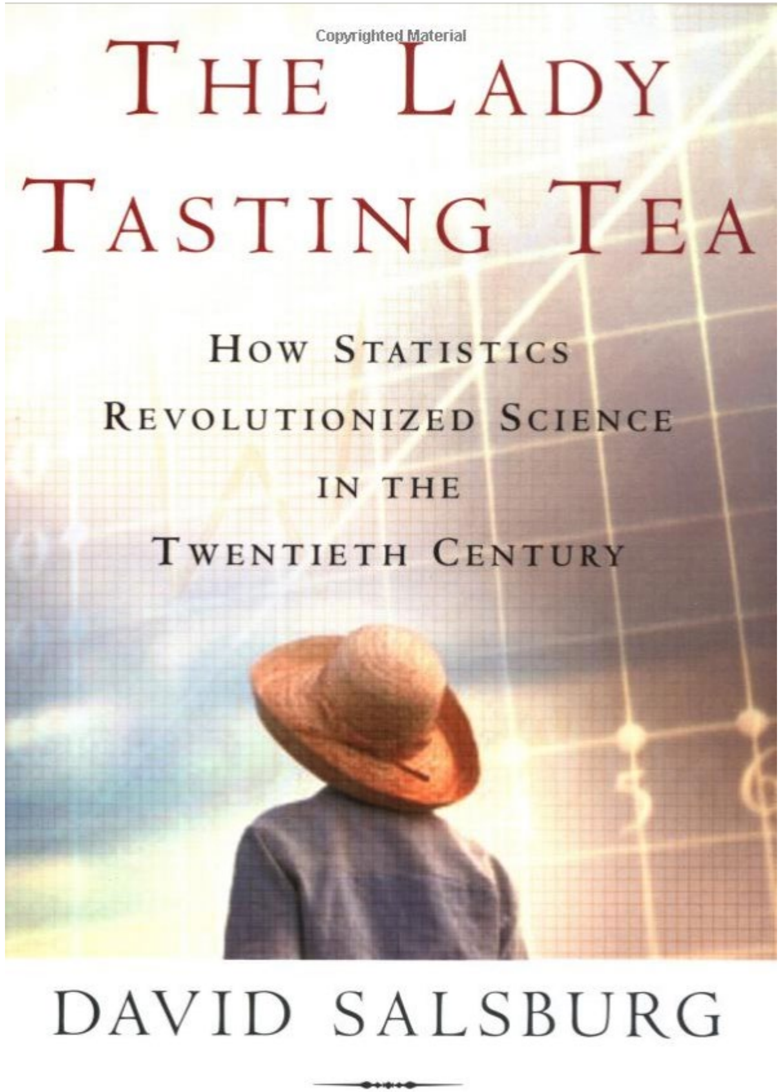
$$= \underset{\theta,\theta'}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} L\left(\mathbf{x}^{(i)}, g_{\theta'}(f_\theta(\mathbf{x}^{(i)}))\right)$$

The Self-Organizing Map Algorithm

1. ***Initialization*** – Choose random values for the initial weight vectors $\mathbf{w}_j$.
2. ***Sampling*** – Draw a sample training input vector $\mathbf{x}$ from the input space.
3. ***Matching*** – Find the winning neuron $I(\mathbf{x})$ that has weight vector closest to the input vector, i.e. the minimum value of $d_j(\mathbf{x}) = \sum_{i=1}^{D}(x_i - w_{ji})^2$.
4. ***Updating*** – Apply the weight update equation $\Delta w_{ji} = \eta(t)\ T_{j,I(\mathbf{x})}(t)\ (x_i - w_{ji})$ where $T_{j,I(\mathbf{x})}(t)$ is a Gaussian neighbourhood and $\eta(t)$ is the learning rate.
5. ***Continuation*** – keep returning to step 2 until the feature map stops changing.



History of Statistics



Today:
Topological Data Analysis (TDA)

Slides ack: Afra Zomorodian, Ryan Lewis,
Gunnar Carlsson, Samir Chowdhury

Topology and Data Analysis

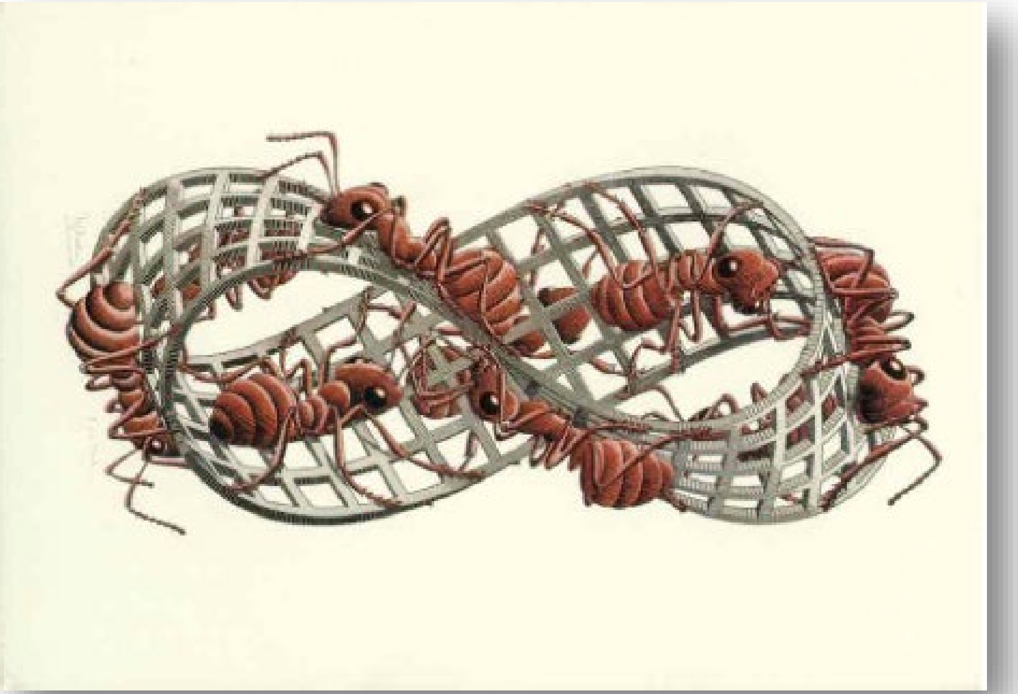
Topology — Computational Topology

↓

Homology — **Persistent Homology**

↓

Topological Data Analysis Applications



Computational Topology and Data Analysis

Herbert Edelsbrunner

Gunnar Carlsson

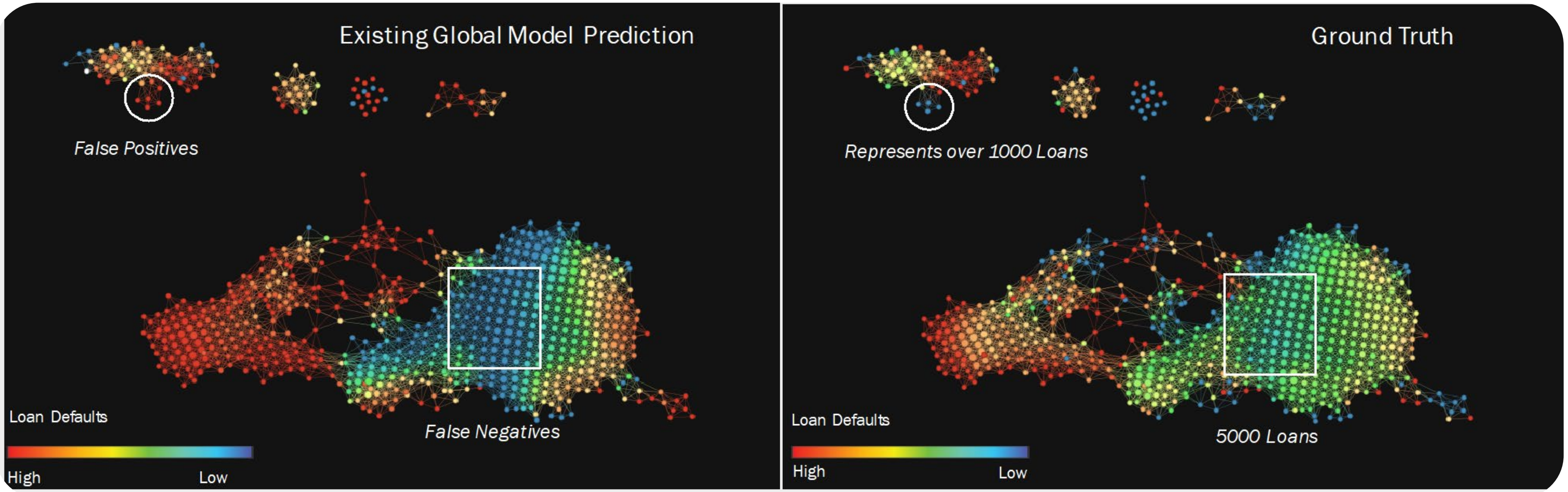
Afra Zomorodian

Frederic Chazal

1990 -- 2021

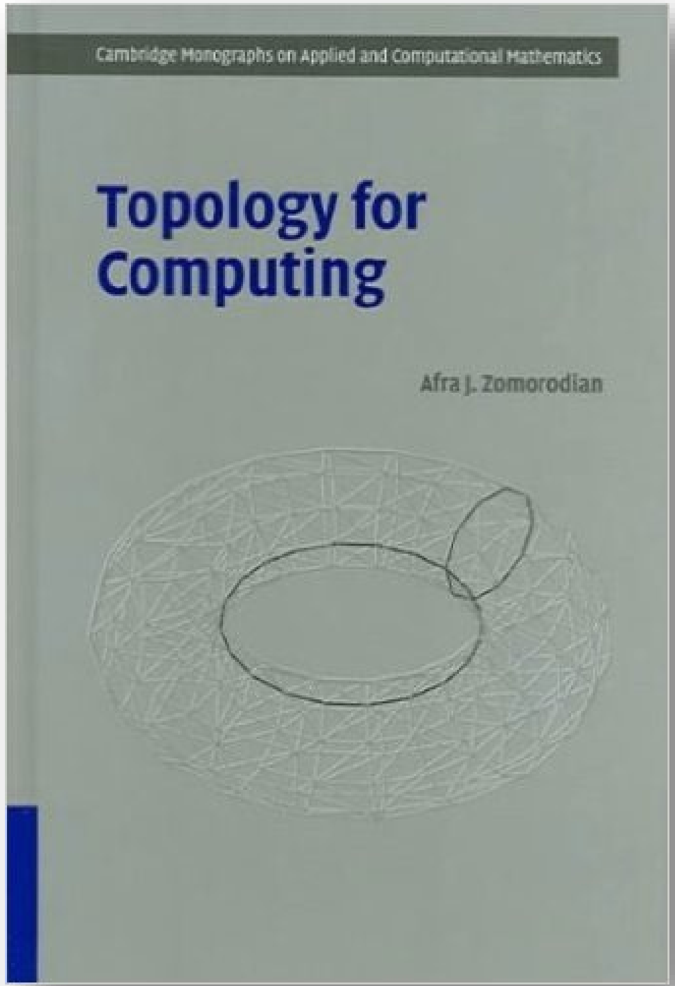
Robert Ghrist

Some Companies

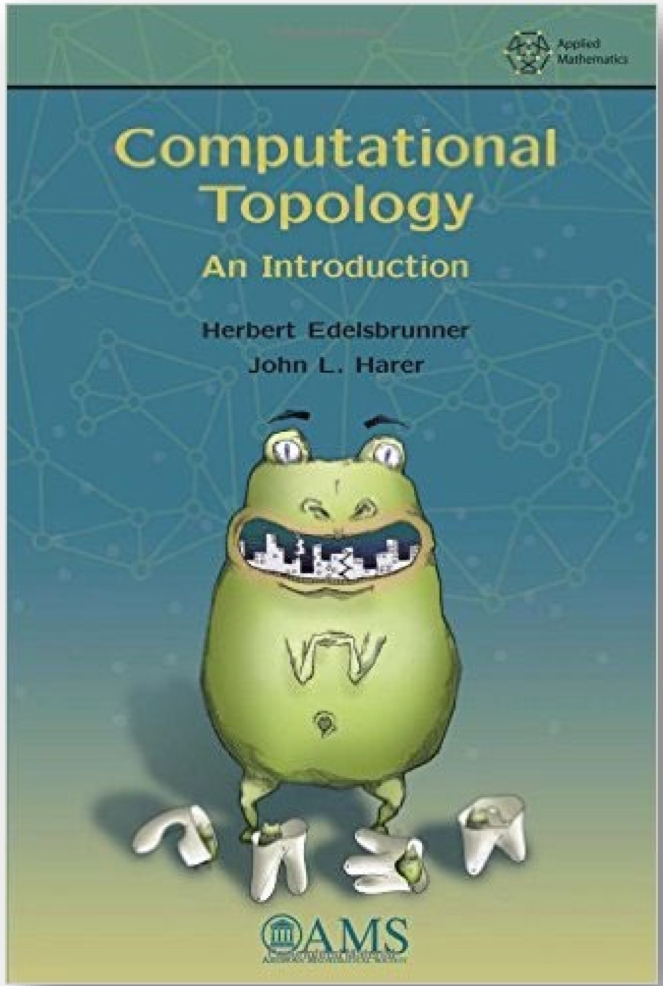


AYASDI

Some Textbooks



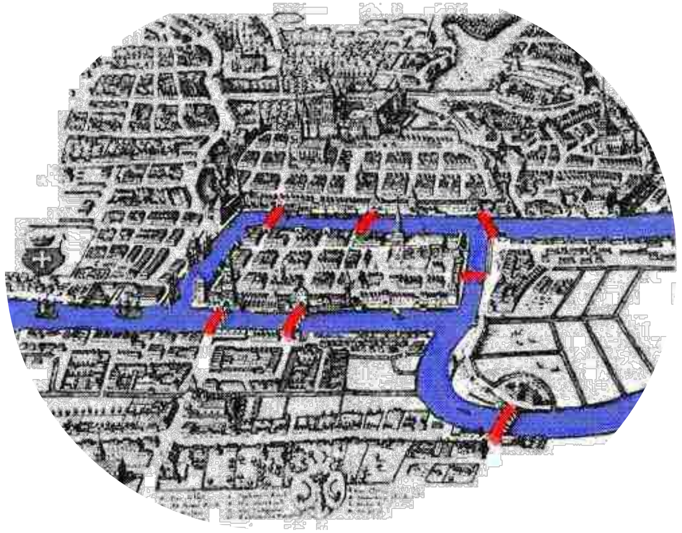
2001



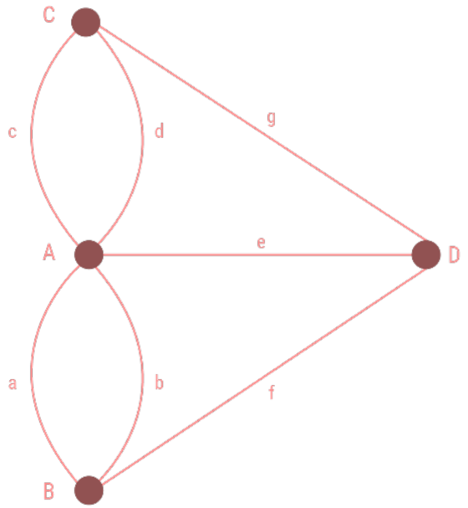
2010

Topology Studies Connectivity

… and obstructions to connectivity

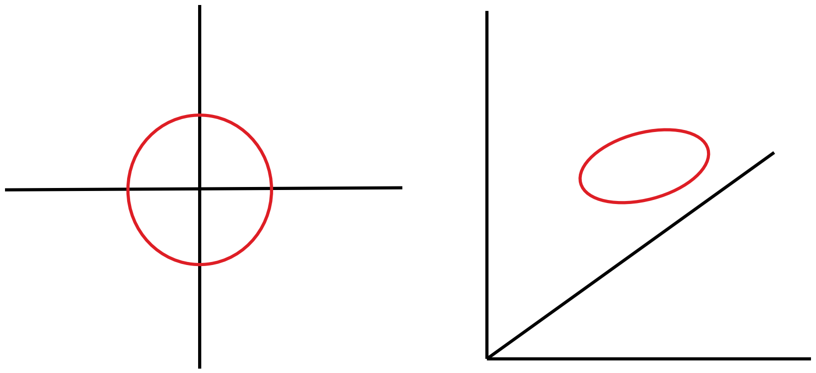


The bridges of
Köningsberg (Kalingrad)



Leonhard Euler
1736

Topological (Connectivity) Invariances



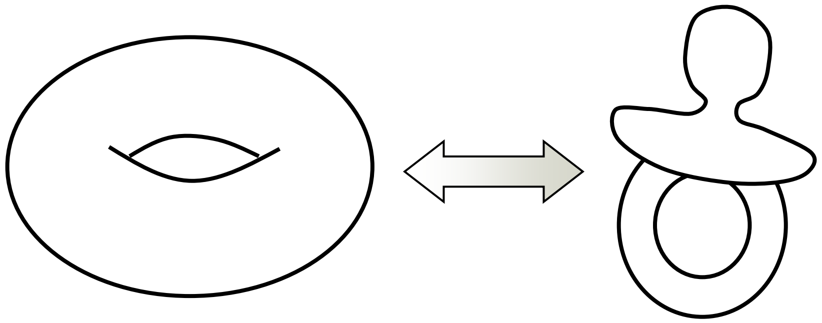
Coordinate-free

Deformation-invariant

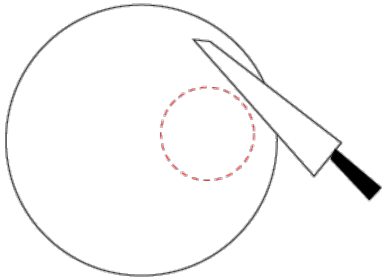
Gunnar Carlsson: Topology does not take distances too seriously

2-Manifolds: Ordinary Surfaces

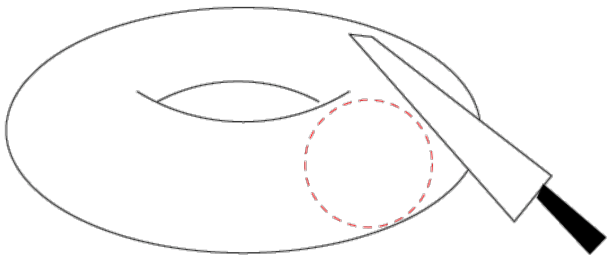
- We are allowed to stretch and shrink
  - Homeomorphism: 1-1, onto, bi-continuous



- But we care about cutting, puncturing, stitching, gluing …



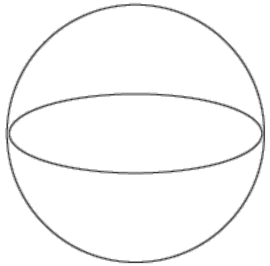
(a) No matter where we cut the sphere, we get two pieces



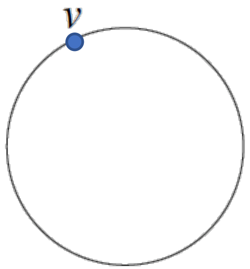
(b) If we're careful, we can cut the torus and still leave it in one piece.

- Note: connectivity information is indexed by dimension

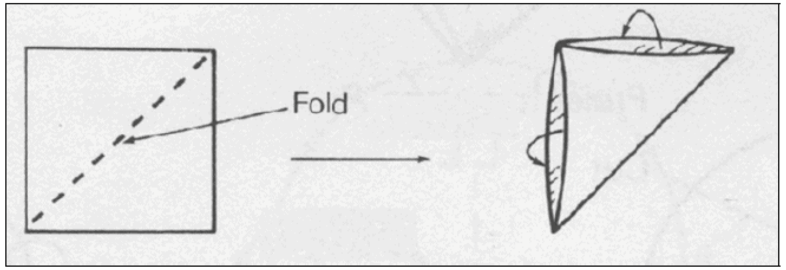
2-Manifold Zoo



(a) $\{x \in \mathbb{R}^3 \mid |x| = 1\}$

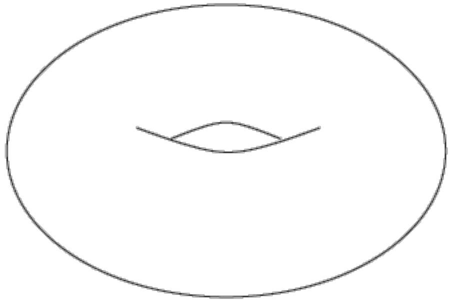


(b) Identify boundary to v

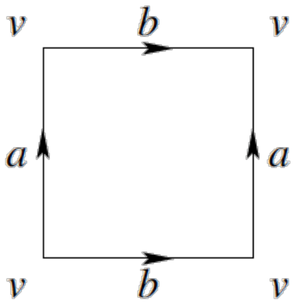


(c) Instructions for a flat sphere

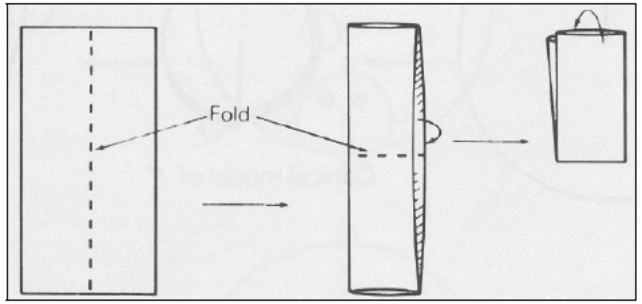
Sphere



(a) Donut surface



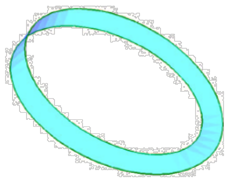
(b) Diagram



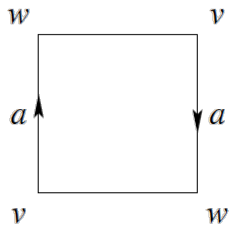
(c) Instructions for a flat torus

Torus

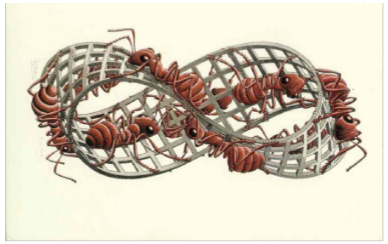
More Exotic Animals



(a) Embedded

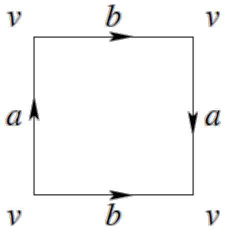


(b) Diagram



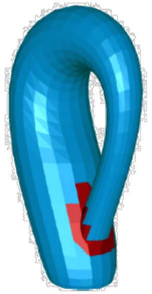
(c) Escher's *Möbius Strip II* (on its side)

Möbius strip

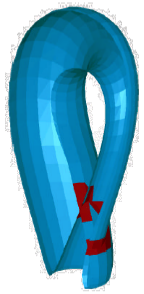


(a) Diagram

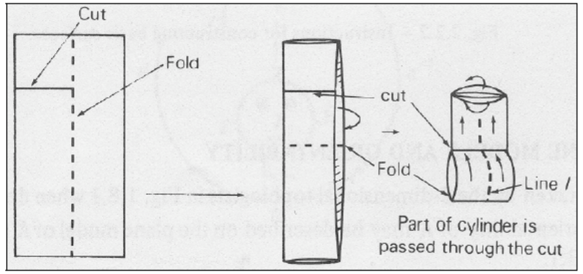
Cross cap + disk



(b) An immersion

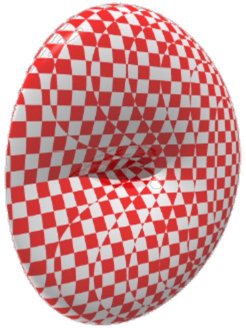


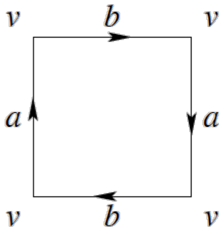
(c) Cut in half (a Möbius strip)



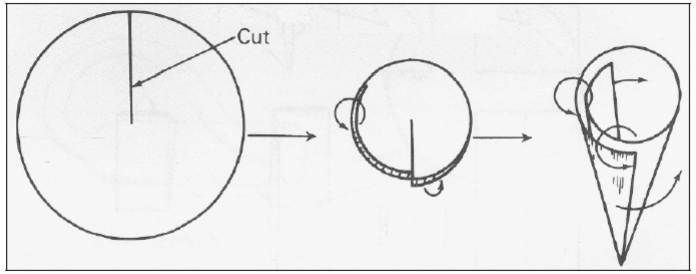
(d) Instructions for a flat $\mathbb{K}^2$

Klein bottle





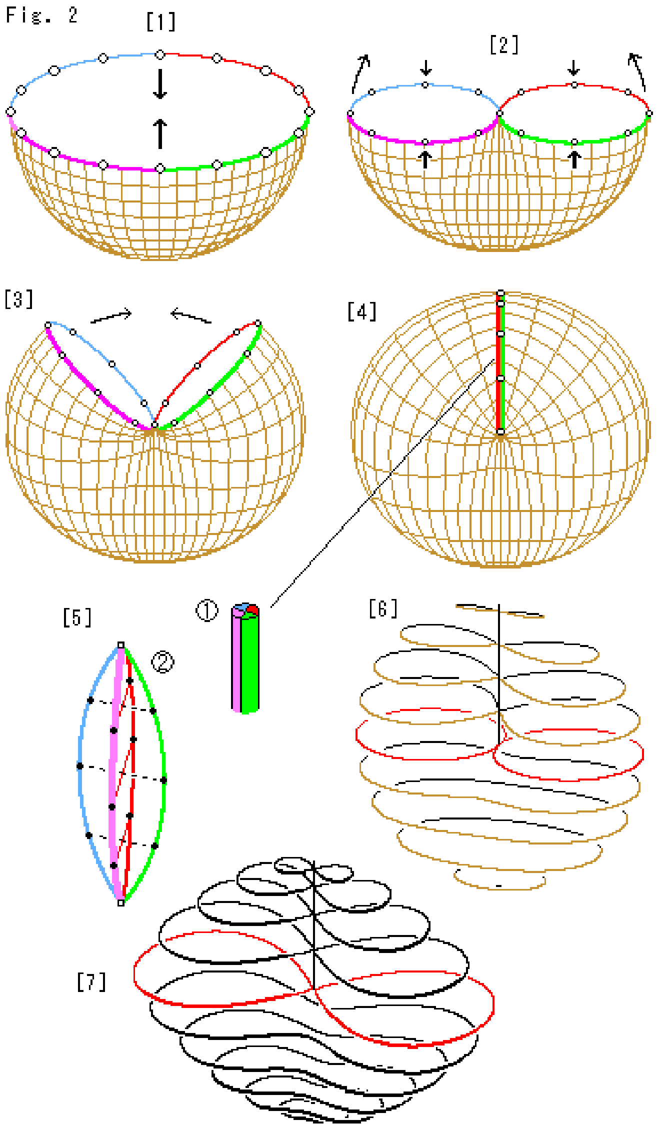
(a) Diagram



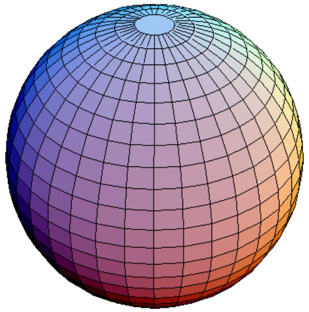
(b) Instructions for a flat $\mathbb{R}\mathrm{P}^2$

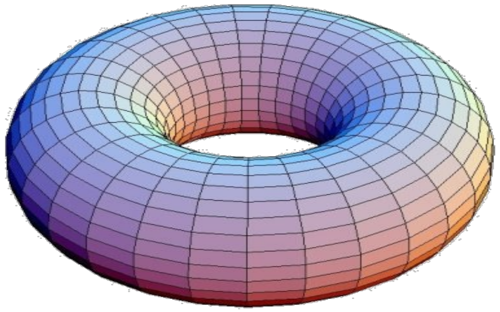
Projective plane

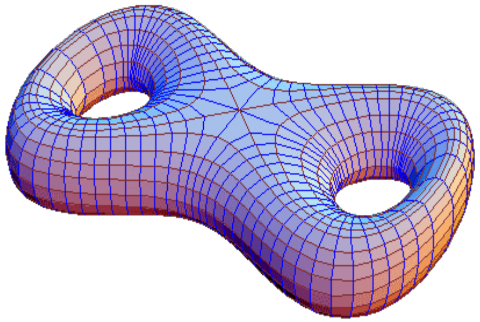
Projective Plane Detail

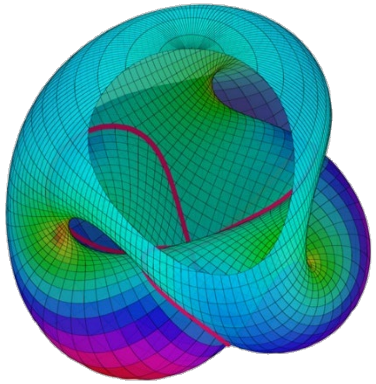


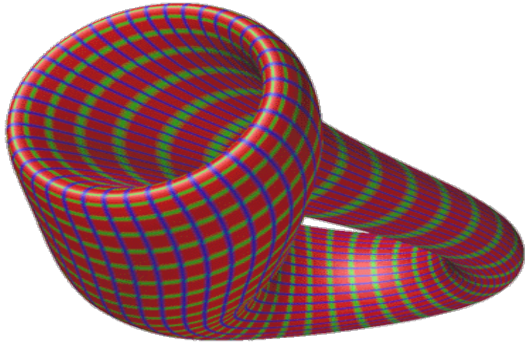
Topology Aims to Classify



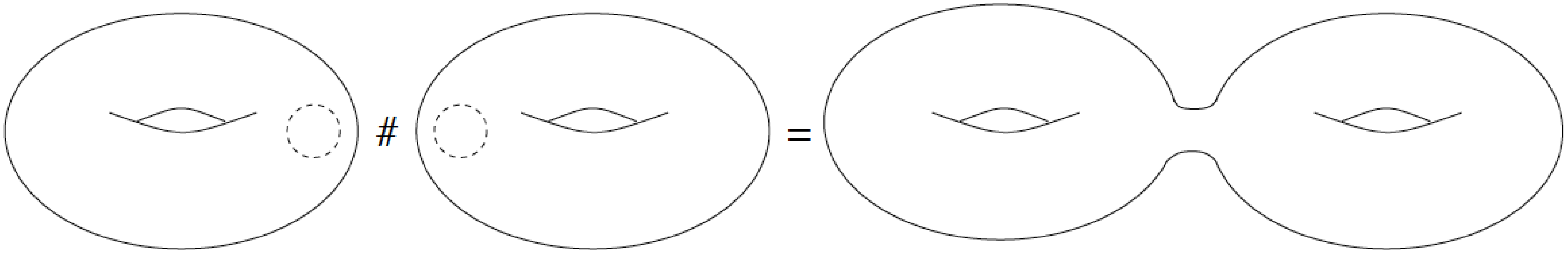


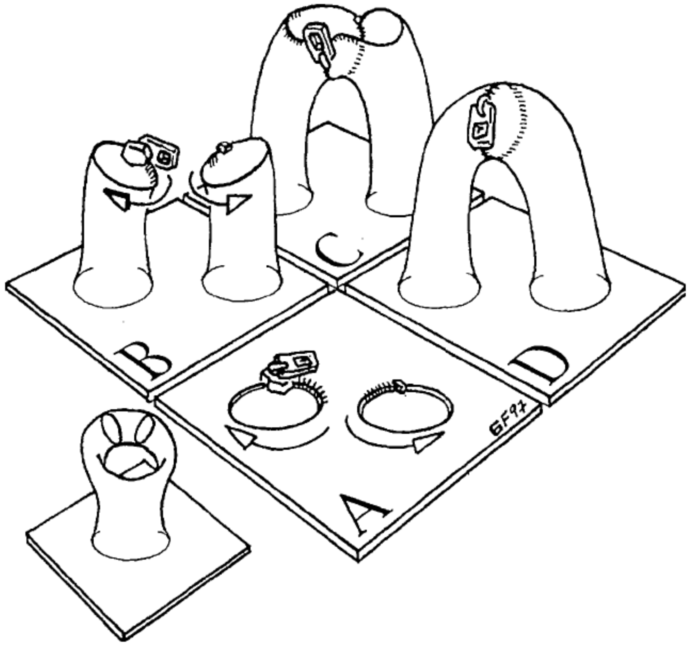




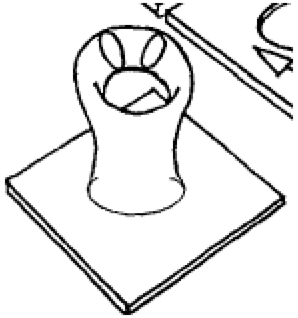


Connected Sums



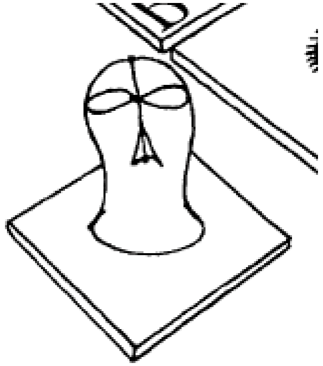


- **Classification Theorem of 2-Manifolds (1860):** Every closed connected compact surface is a connected sum of a sphere with a number of tori and projective planes (sphere + handles + cross cups)



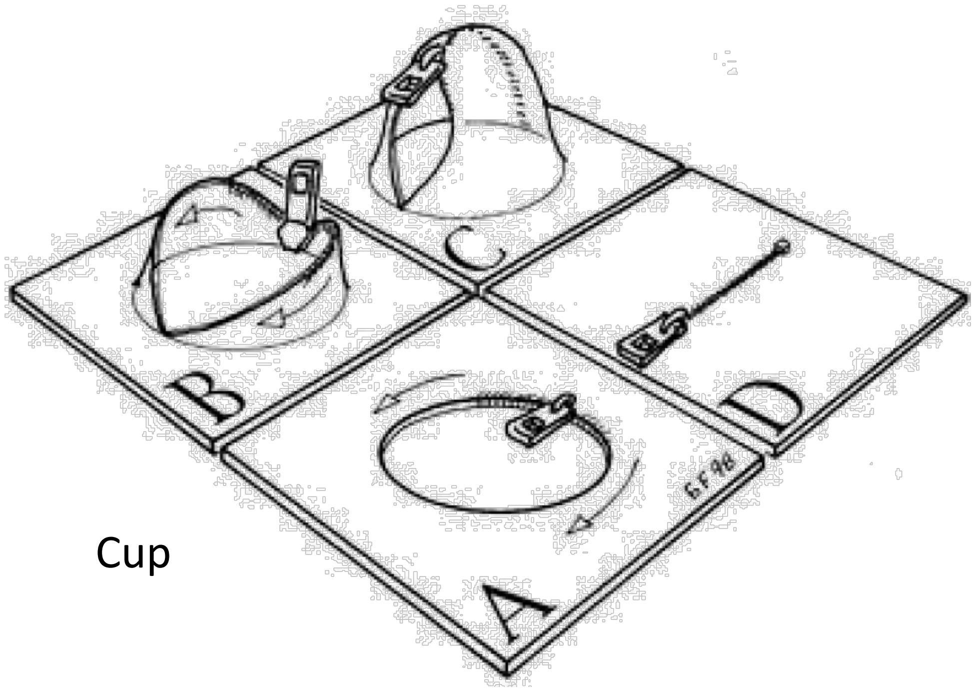
Handle

Klein bottle = sphere + 2 cross cups
= sphere + cross handle



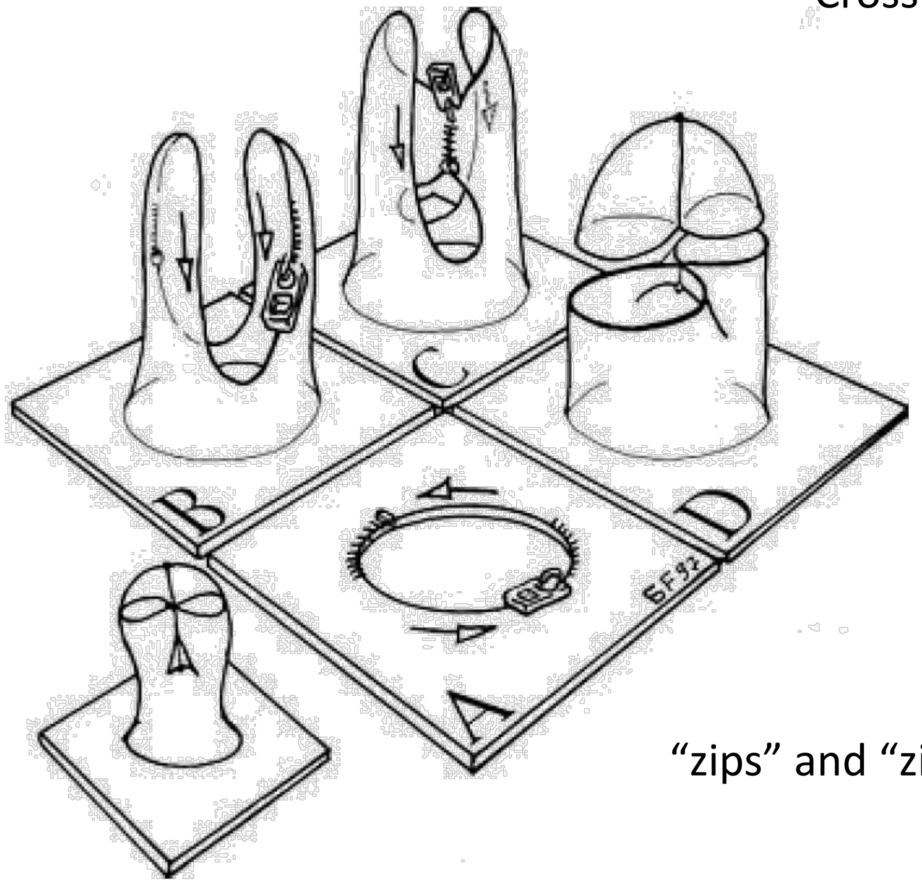
Cross cup

J. Conway's ZIP Proof (Zero Irrelevancy Proof)



Cup

Cross cup



"zips" and "zippers"

[Francis and Weeks, AMM, 1999
https://www.maths.ed.ac.uk/~v1ranick/papers/francisweeks.pdf]

Algebraic Topology

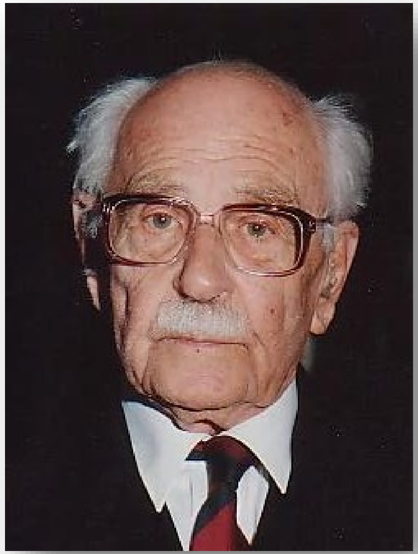
Many other branches: Point set topology, differential topology, combinatorial topology

Henri Poincaré
1854-1912

Luitzen Egbertus Jan Brouwer
1881-1966

Emmy Noether
1882-1935

Heinz Hopf
1894-1971

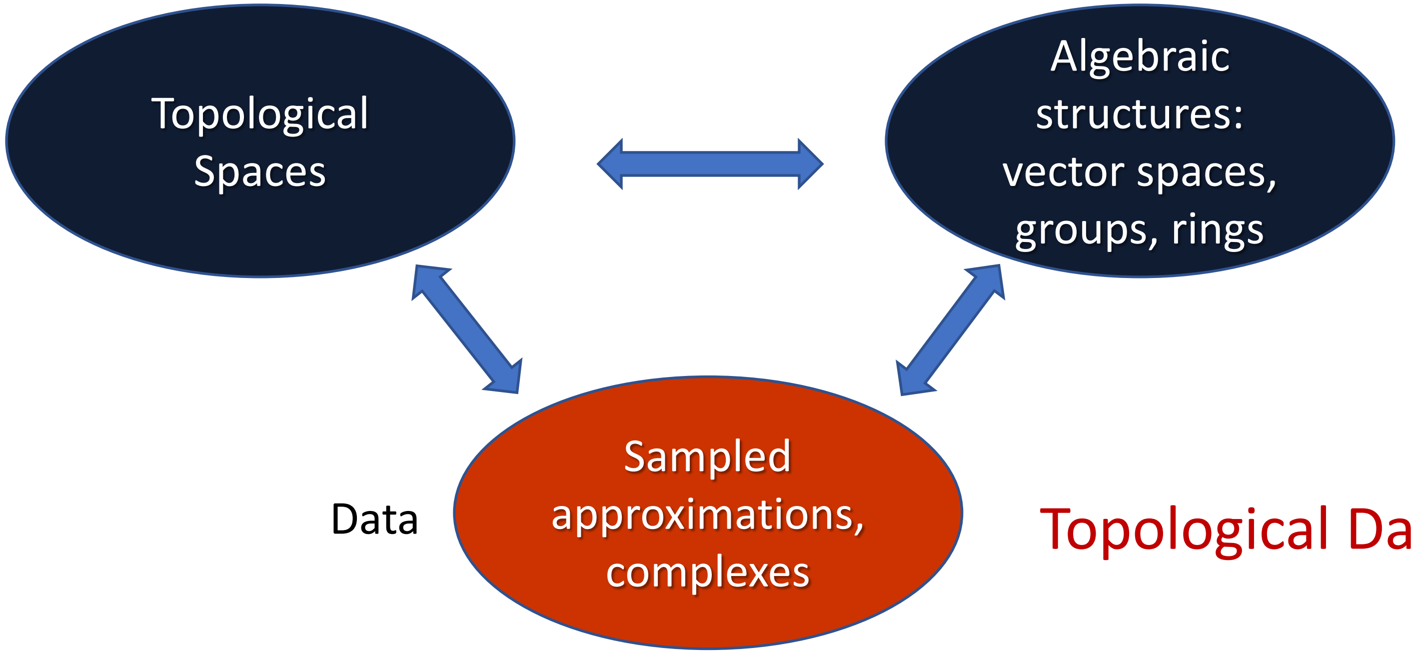


Leopold Vietoris
1892-2002

Algebraic Topology and Topology Inference

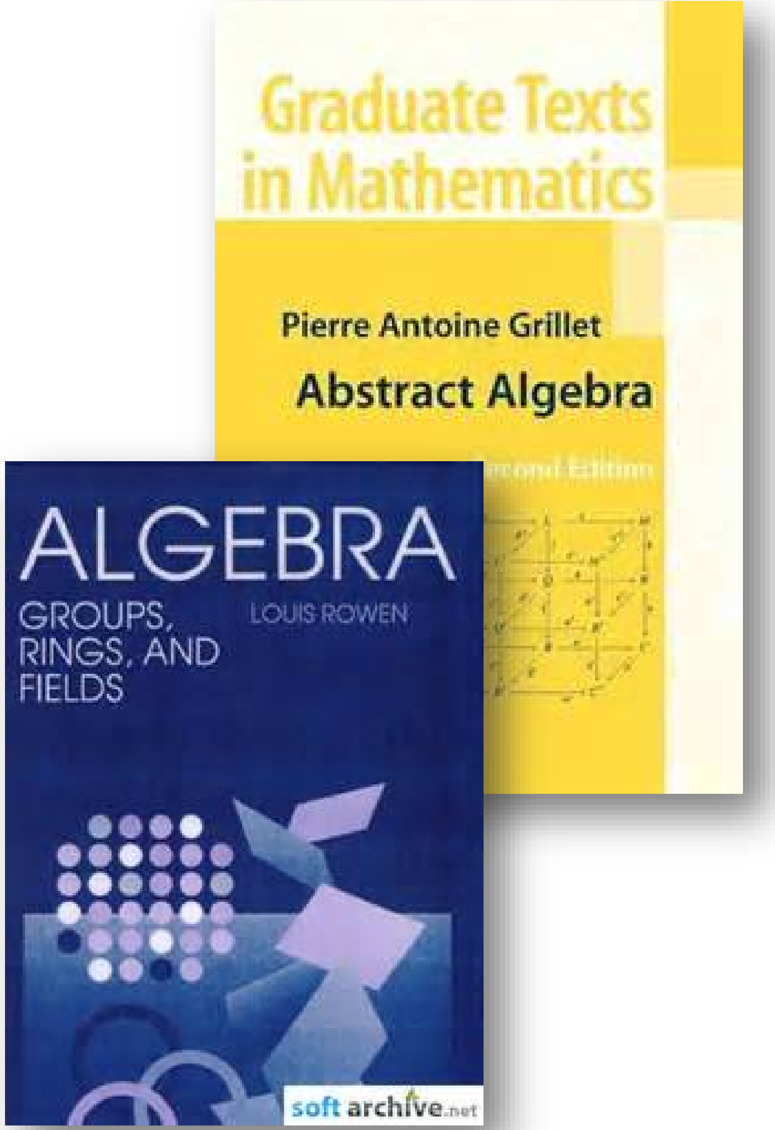
- Unlike geometry, topology studies the global structure of spaces

- Getting to this structure via algebra   algebraic topology



Topological Data Analysis (TDA)

From Data to Algebraic Objects

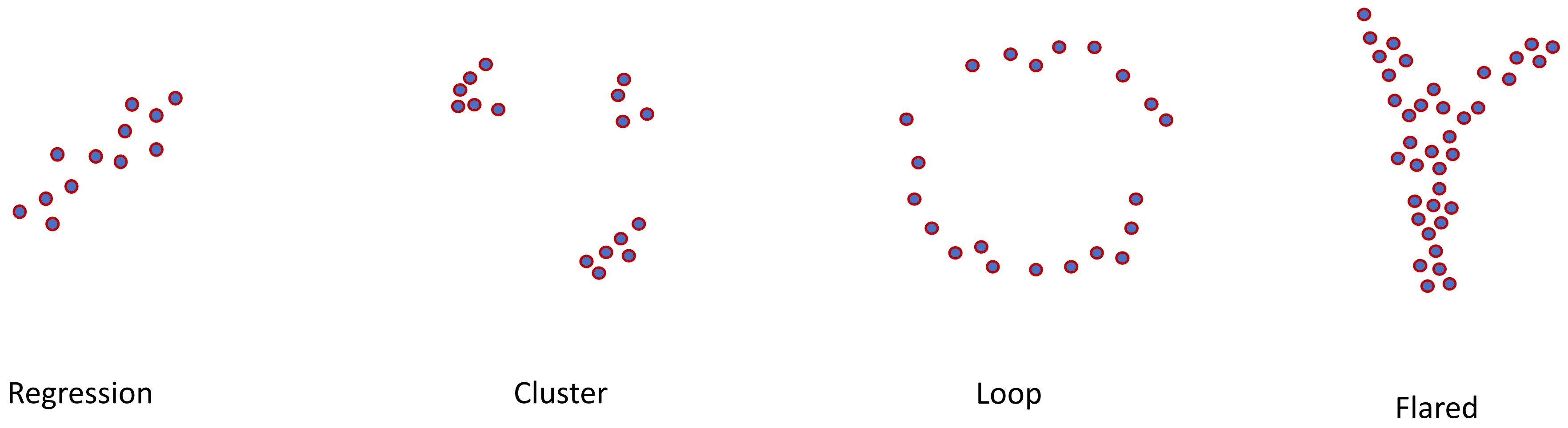


Algebraic entities as data descriptors

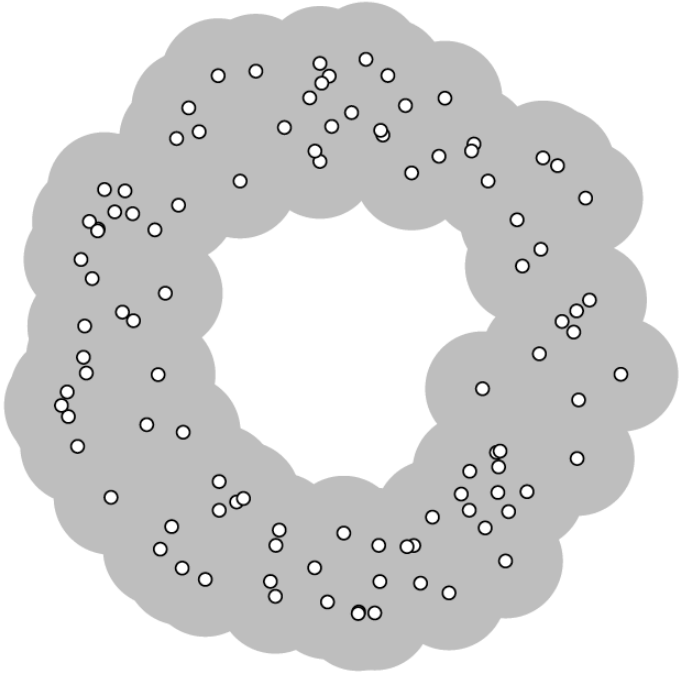
Studying
Sampled Spaces

The "Shape of Data"

- TDA Studies Sampled Spaces



Recovering “Shape” from Sampled Data



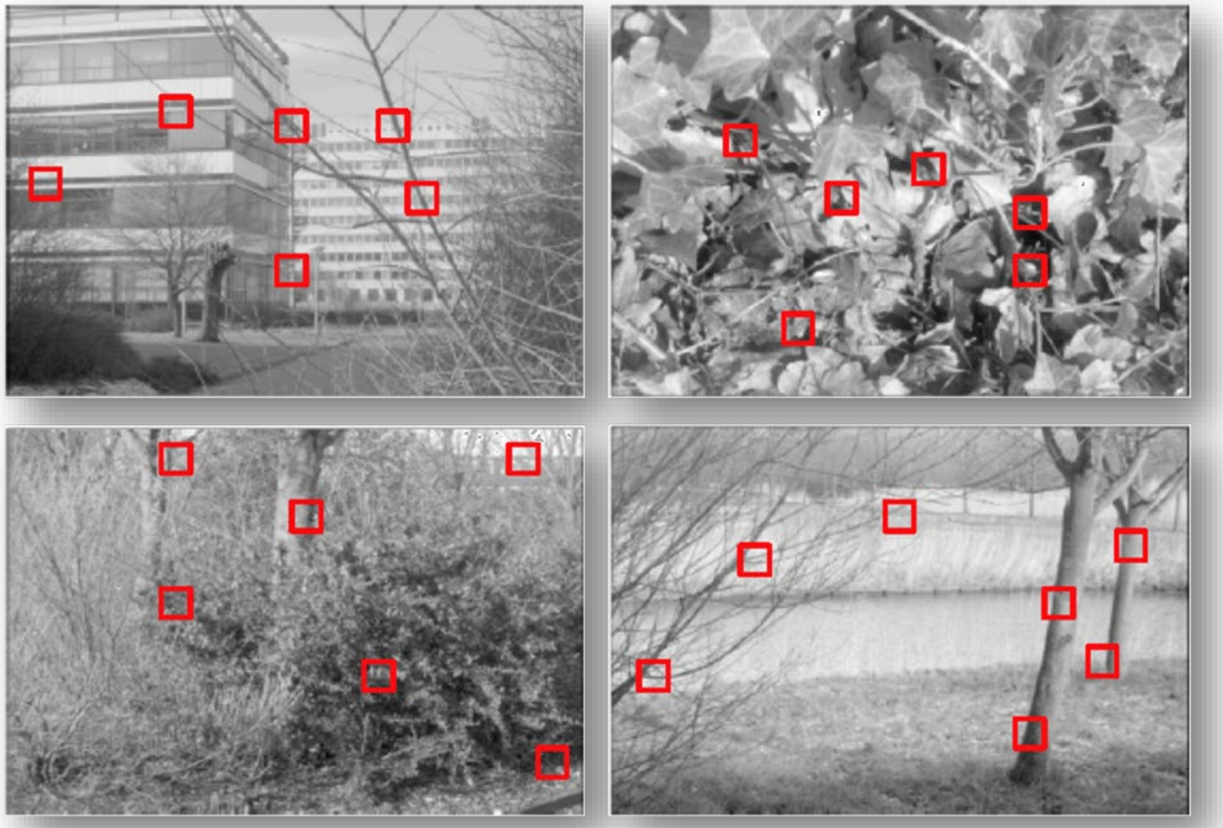
What is this?

What does it look like ?

1. Set of points in $\mathbb{R}^2$
2. Looks like an annulus.

Aim: recover the topology of the underlying space from which the data was sampled

Example: The Space of Natural Images



Example: The Space of Natural Images

(Carlsson, Ishkanov, de Silva, Zomorodian IJCVC 2008)

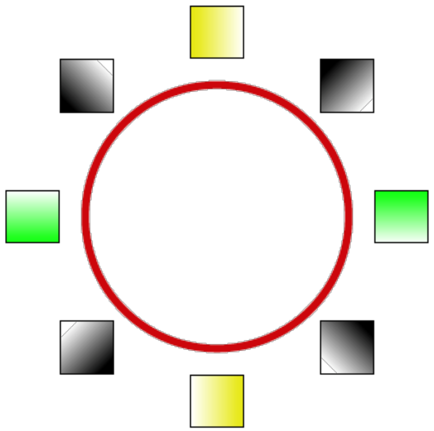
- Lee-Mumford-Pedersen (2003) investigated whether a statistically significant difference exists between natural and random images
- Natural images form a “subspace” of all images. Dimension of ambient space e.g. 640 x 480 = 307 200
- This space of natural images should have:
  - high dimension: there are many different images
  - even higher co-dimension: random images look nothing like natural ones
- Data is a collection of black-and-white images used in cognitive science research

Natural 3 x 3 Patches

- Instead of studying entire images, we consider the distribution of 3 x 3 pixel patches
- Most of these are roughly constant in natural images -- they drown out any real structure
- L.M.P. chose 8,500,000 patches with high contrast
- Each 3 x 3-patch is considered a vector in R^9
- Normalize brightness: $R^9 \to R^8$
- Normalize contrast: $R^8 \to S^7$

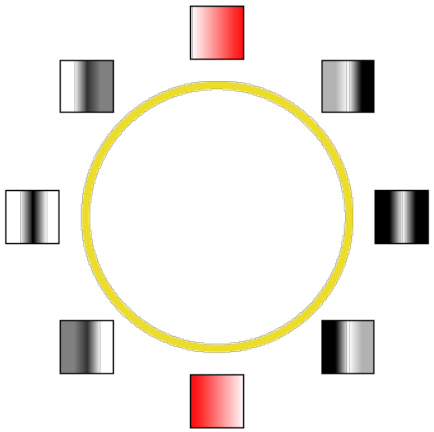
High-Density Areas

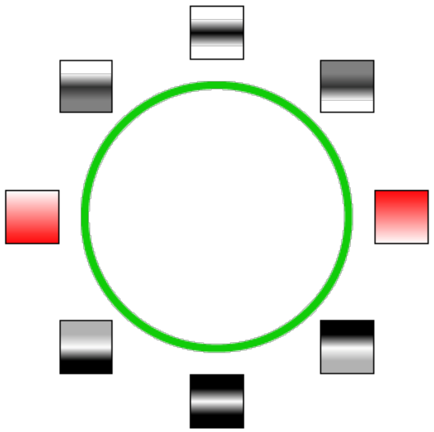
edges



highest density

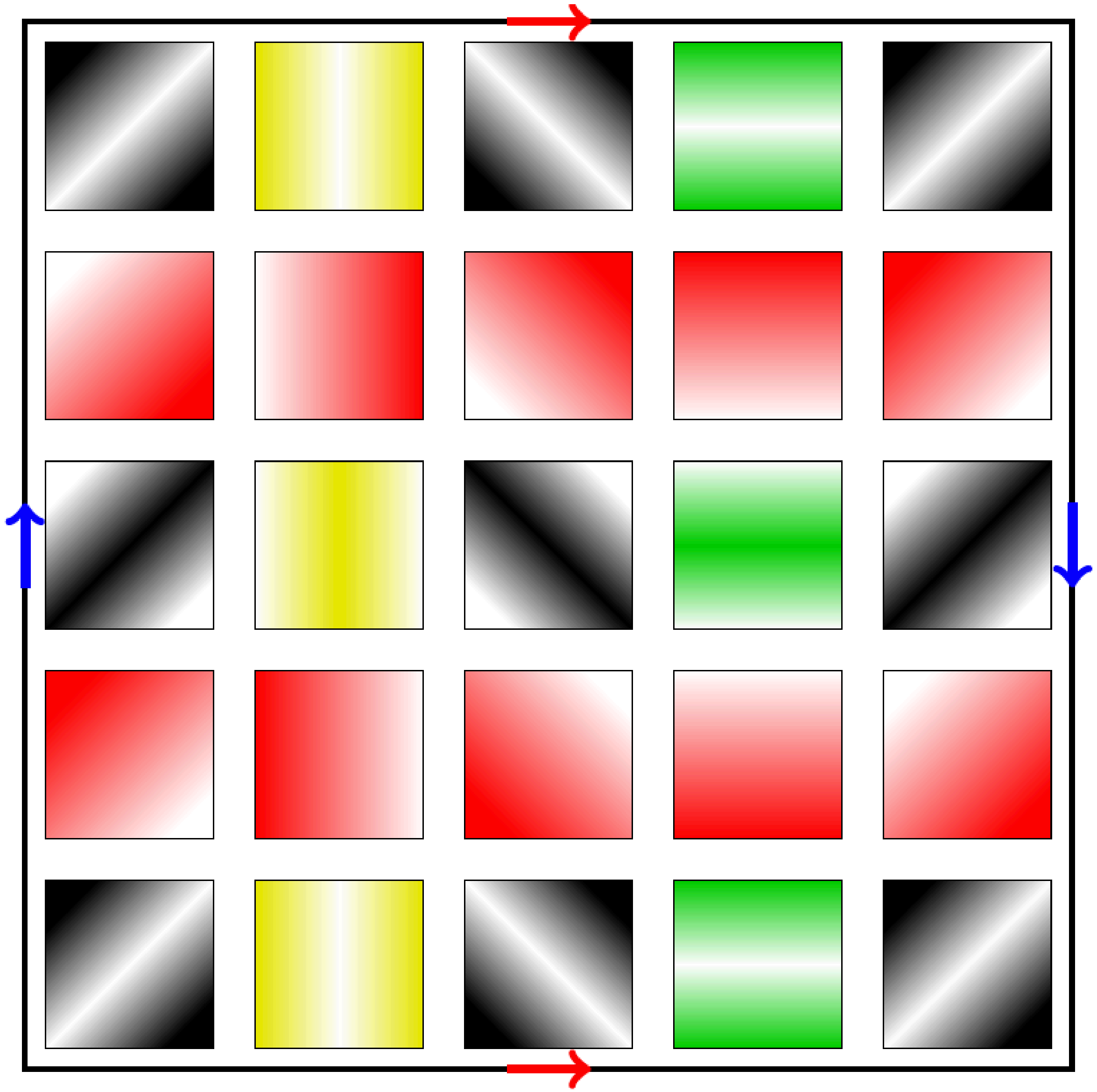
ridges

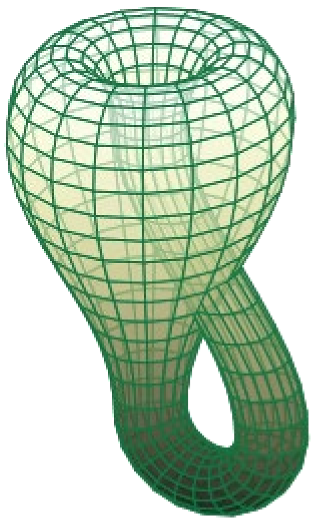




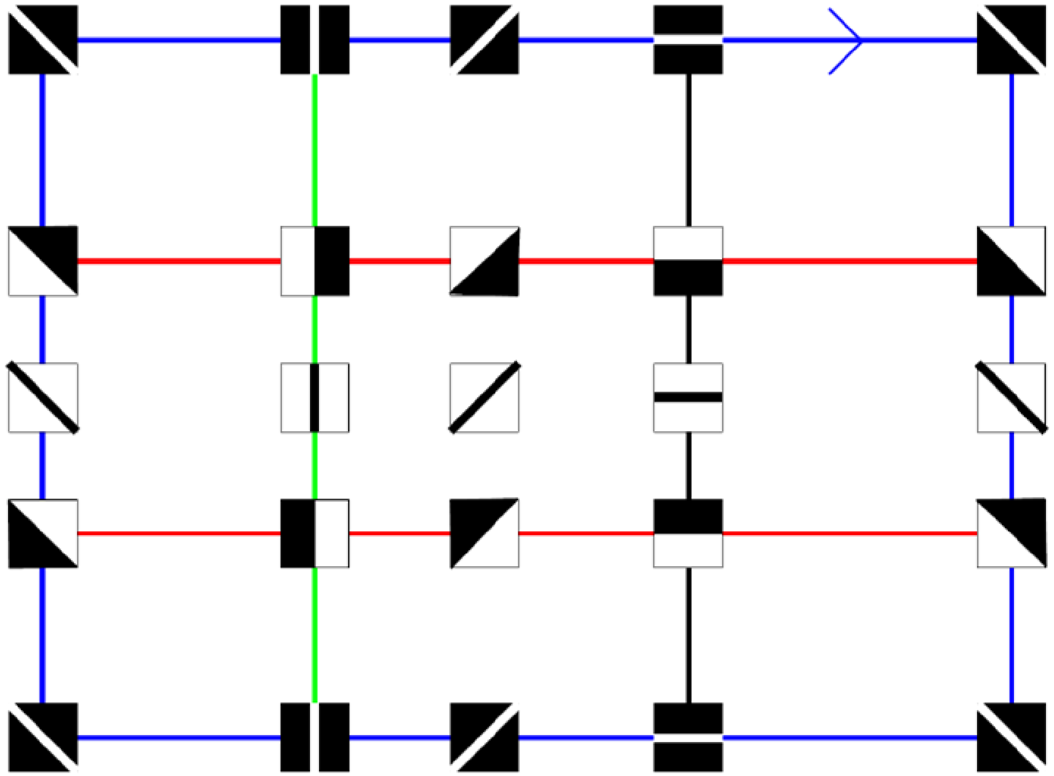
next highest density

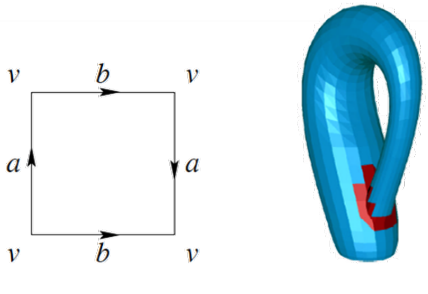
Klein Bottle of Pixel Patches



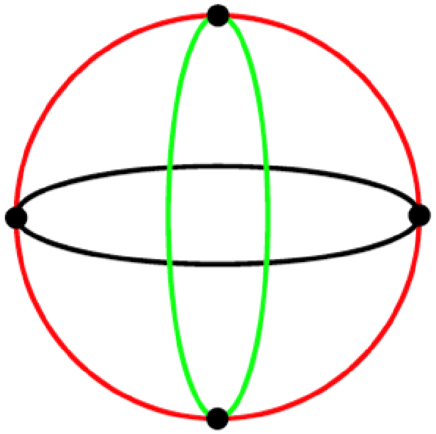


Klein Bottle Structure





(a) Diagram (b) An immersion



$(\beta_1 = 5)$

(source: [Carlsson, Ishkhanov, de Silva, Zomorodian 2008])

Applications of the Analysis

- An efficient way to parametrize image patches.

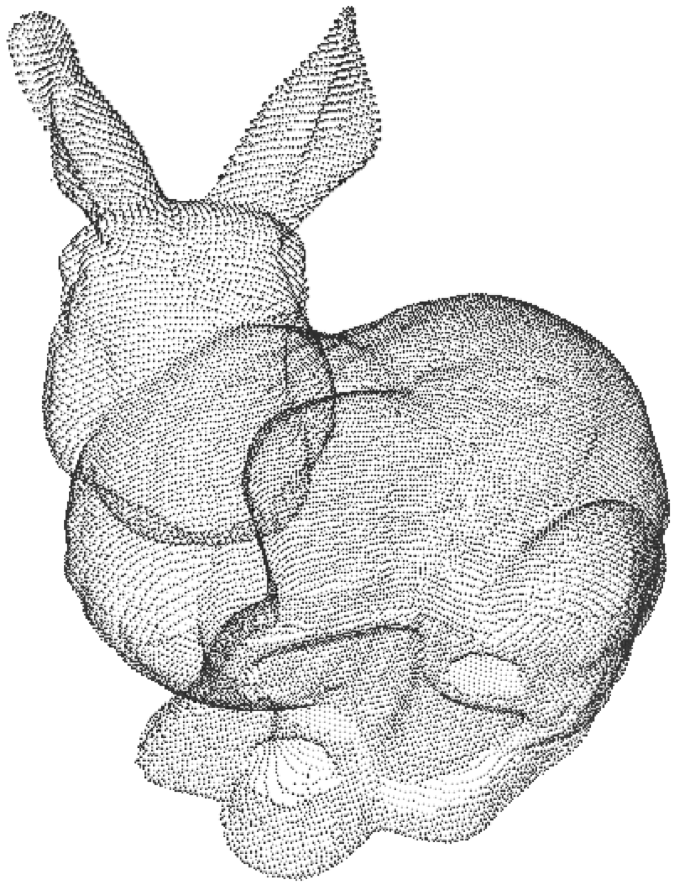
- Image compression: a 3 x 3-cluster may be described using 4 values:
  - 2x : Position of its projection onto the Klein bottle
  - 1x : Original brightness
  - 1x : Original contrast

- Texture analysis: textures yield distributions of occurring patches on the Klein bottle. Rotating the texture corresponds to translating the distribution.

- Deep nets: regularization of initial layers

Simplicial Complexes: Combinatorial Topology

Point Clouds Have No Higher-D Topology

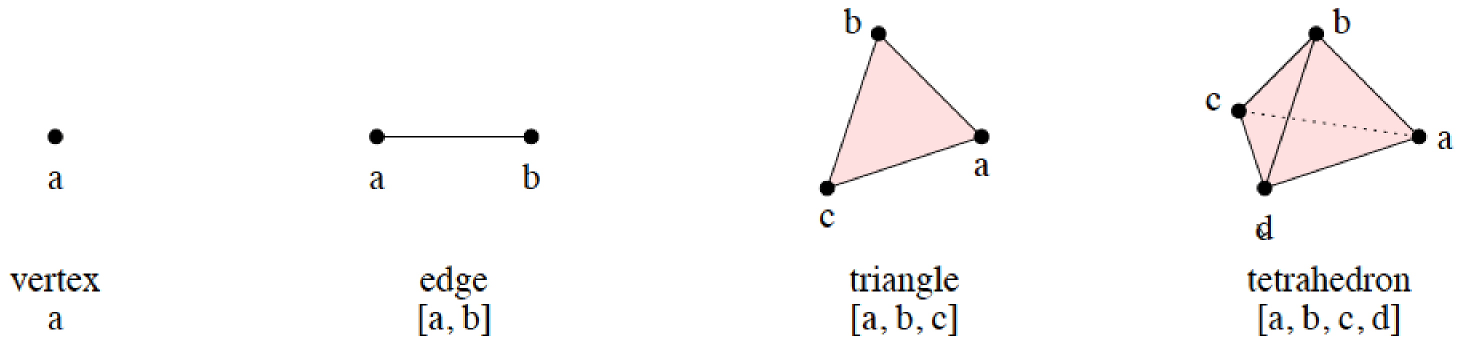


Connect the points into simplicial complexes

Simplicial homology

A Simplex

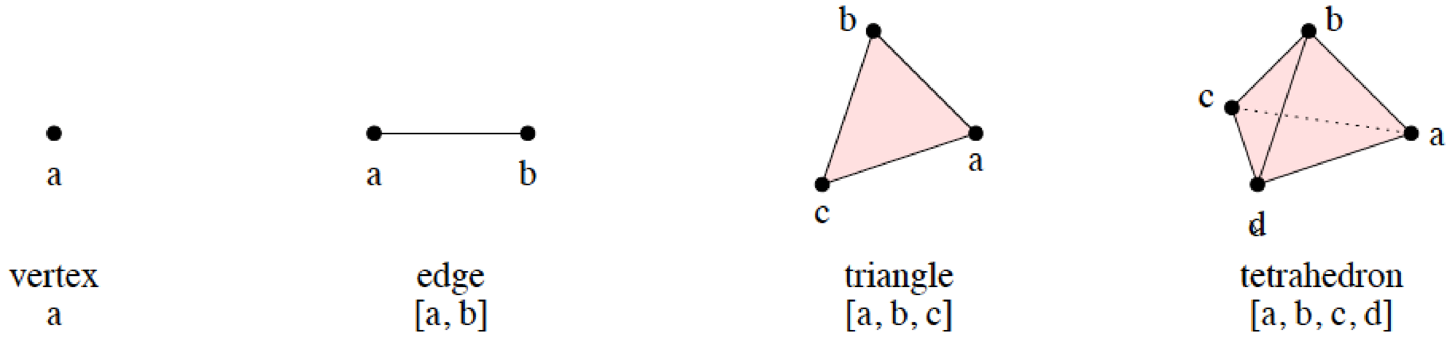
- A k-simplex is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the vertices of the simplex.
- A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$.



d=0 *d*=1 *d*=2 *d*=3

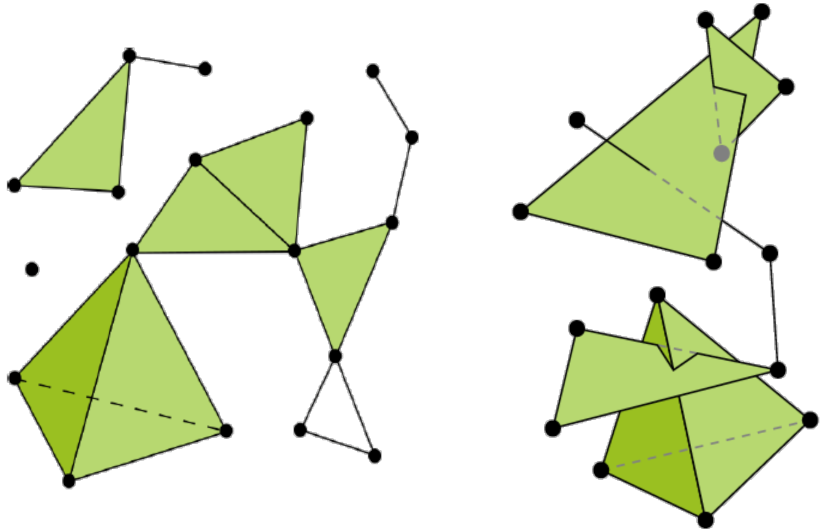
Faces / Subsimplices

- σ: a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$.
- τ defined by $T \subseteq S$ is a face of σ
- σ is its coface.
- $\sigma \geq \tau$ and $\tau \leq \sigma$.
- $\sigma \leq \sigma$ and $\sigma \geq \sigma$.



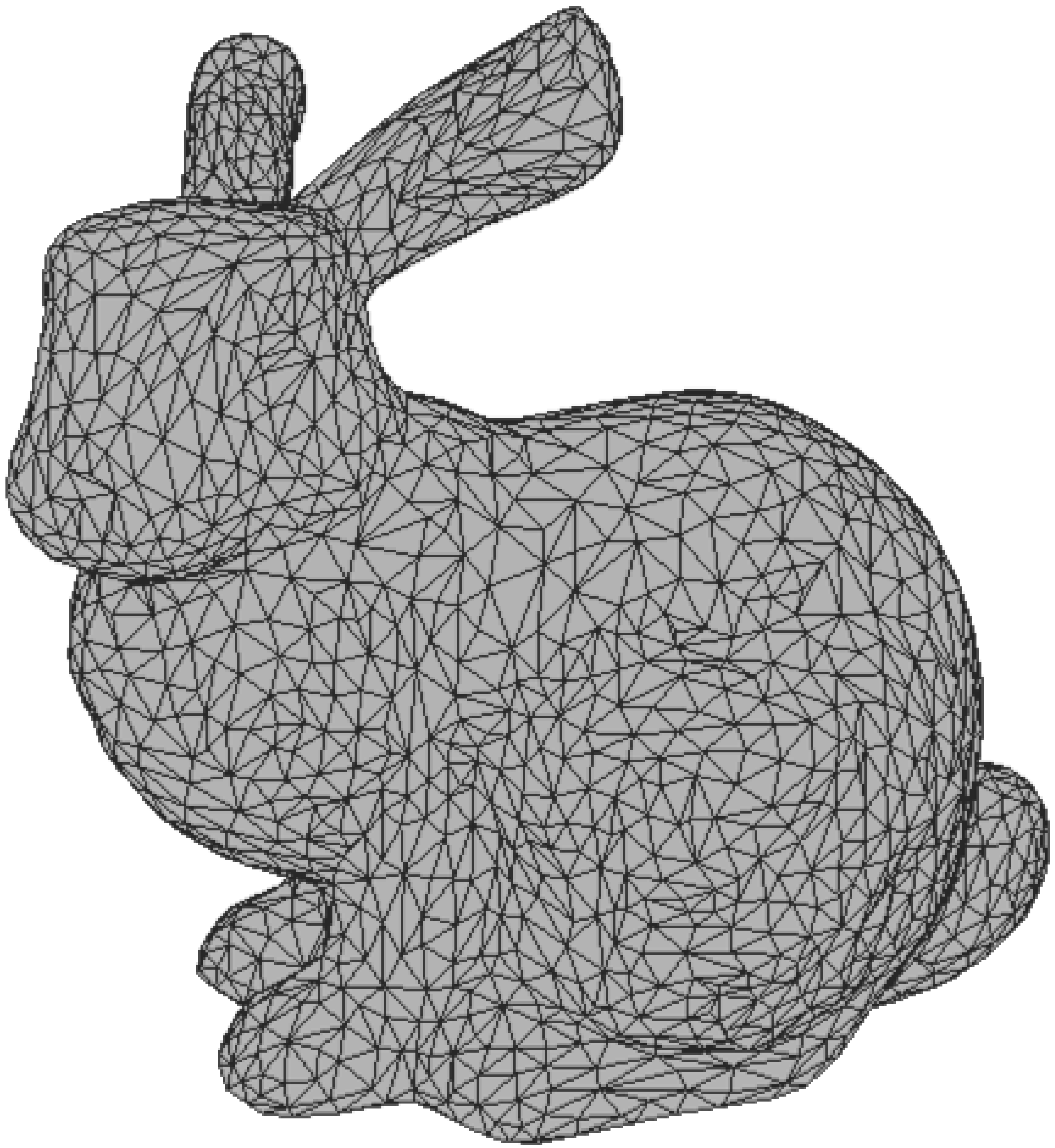
Simplicial Complexes

- A simplicial complex K is a finite set of simplices such that
  1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$,
  2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ or $\sigma \cap \sigma' = \emptyset$.
- The dimension of K is $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$.
- The vertices of K are the zero-simplices in K.
- A simplex is principal if it has no proper coface in K.



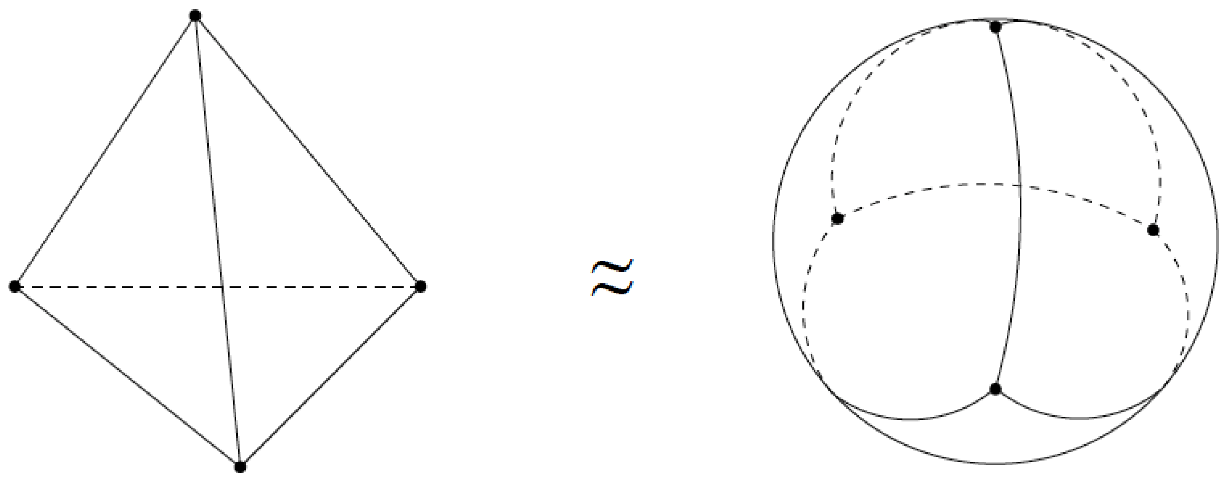
(left) an example (right) a non example

Meshes are 2-D Complexes



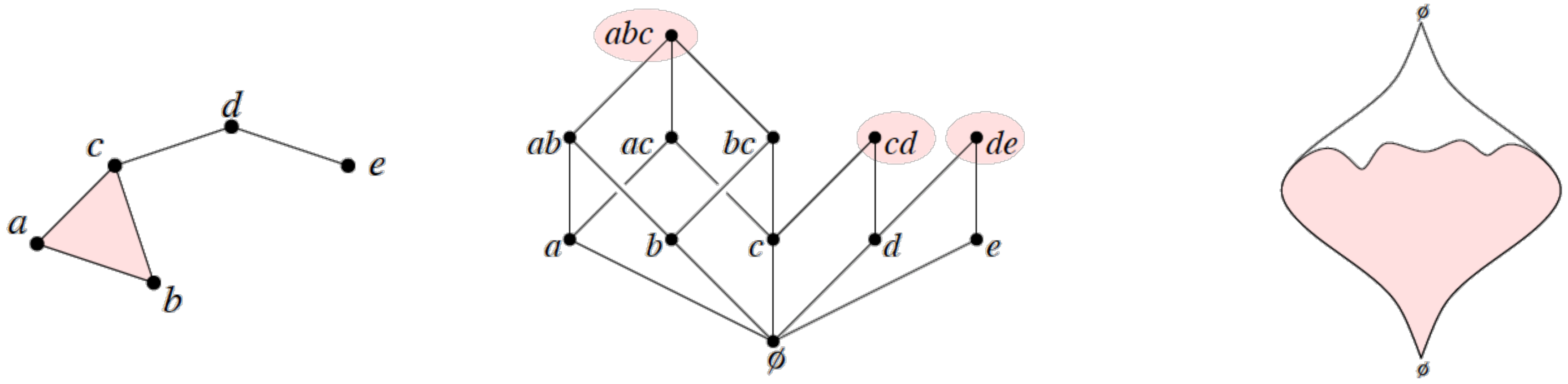
Continuous to Discrete Link: Triangulations

- The underlying space $|K|$ of a simplicial complex K is $|K| = \cup_{\sigma \in K} \sigma$.
- $|K|$ is a topological space.
- A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.



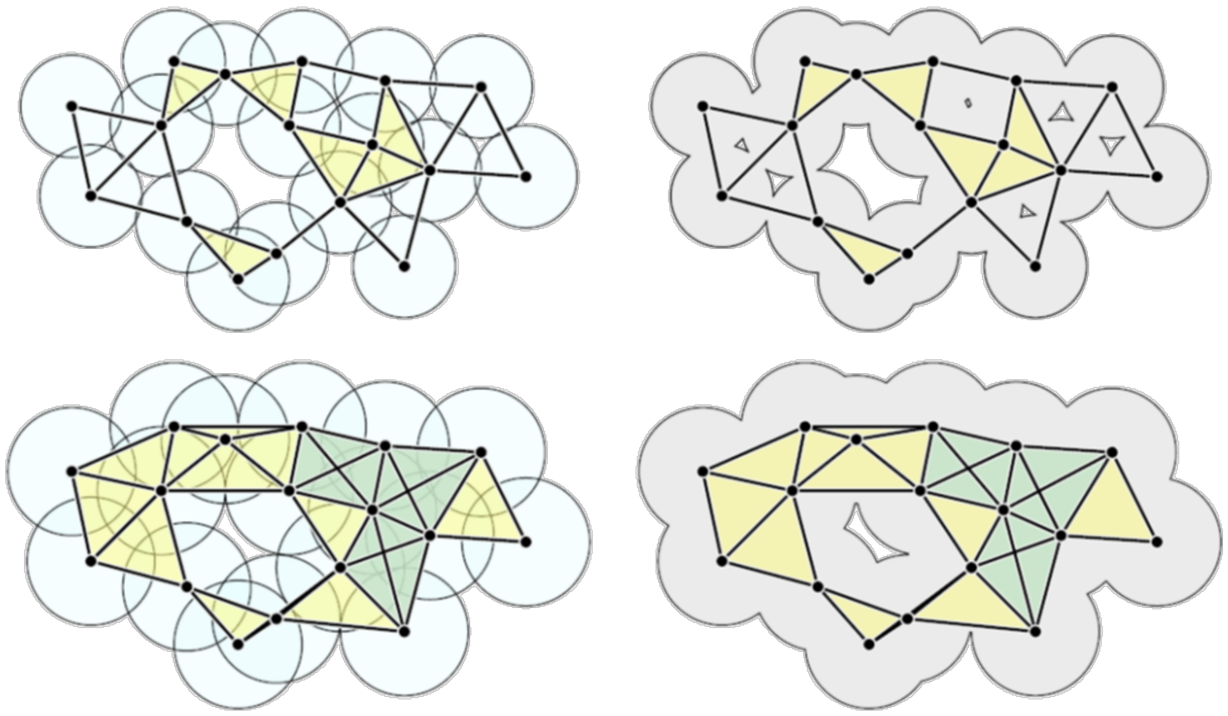
Abstract Simplicial Complexes

- An abstract simplicial complex is a set K, together with a collection $\mathcal{S}$ of subsets of K called (abstract) simplices such that:
  1. For all $v \in K$, $\{v\} \in \mathcal{S}$. We call the sets $\{v\}$ the vertices of K.
  2. If $\tau \subseteq \sigma \in \mathcal{S}$, then $\tau \in \mathcal{S}$.
- We call $\mathcal{S}$ the complex.



Natural partial order structure

The Nerve of a Finite Cover



The nerve of $\mathcal{U}$ is the simplicial complex $K(U)$ defined by

$$\sigma = [U_{i_0}, \ldots, U_{i_k}] \in K(U) \quad \Leftrightarrow \quad \cap_{i=1}^{k} U_{i_j} \neq \emptyset$$

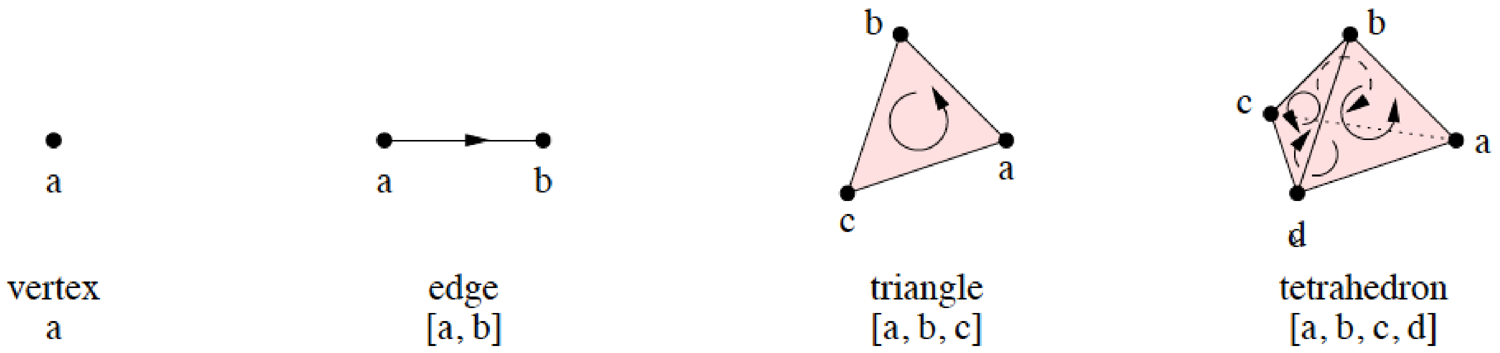
Orientability

- An orientation of a k-simplex $\sigma \in K$, $\sigma = \{v_0, v_1, \ldots, v_k\}, v_i \in K$ is an equivalence class of orderings of the vertices of σ, where

$$(v_0, v_1, \ldots, v_k) \sim (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(k)})$$

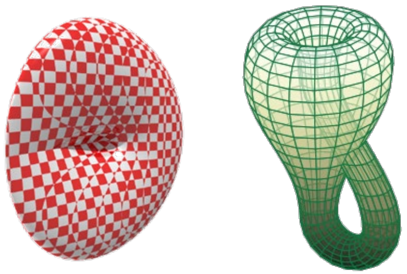
  are equivalent orderings if the parity of the permutation τ is even.

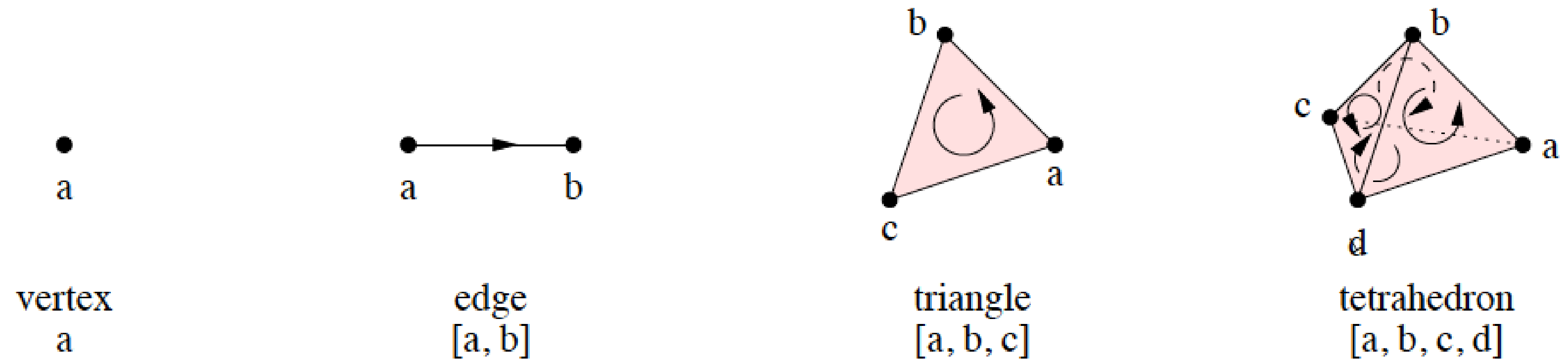
- We denote an oriented simplex, a simplex with an equivalence class of orderings, by $[\sigma]$.



Orientability

- Two k-simplices sharing a $(k-1)$-face σ are consistently oriented if they induce different orientations on σ.
- A triangulable d-manifold is orientable if all d-simplices can be oriented consistently.
- Otherwise, the d-manifold is non-orientable





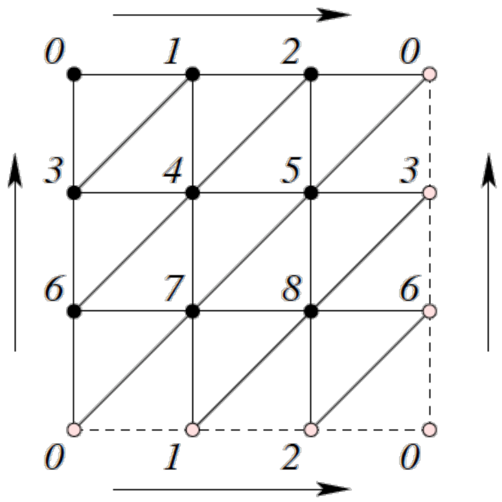
Euler Characteristic: A Topological Invariant

- K a simplicial complex with s_k k-simplices.
- The Euler characteristic $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}.$$

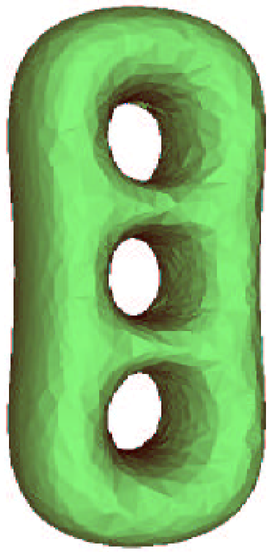
- $v - e + f = 1$ (Graph Theory)
- Invariant for $|K|$
- Any triangulation gives the same answer!
- Intrinsic property

More on Euler



2-Manifold	χ
Sphere $\mathbb{S}^2$	2
Torus $\mathbb{T}^2$	0
Klein bottle $\mathbb{K}^2$	0
Projective plane $\mathbb{R}\mathrm{P}^2$	1

- (Theorem) For compact surfaces $\mathbb{M}_1, \mathbb{M}_2$, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.
- $\chi(g\mathbb{T}^2) = 2 - 2g$
- $\chi(g\mathbb{R}\mathrm{P}^2) = 2 - g$
- The connected sum of g tori is called a surface with genus g.



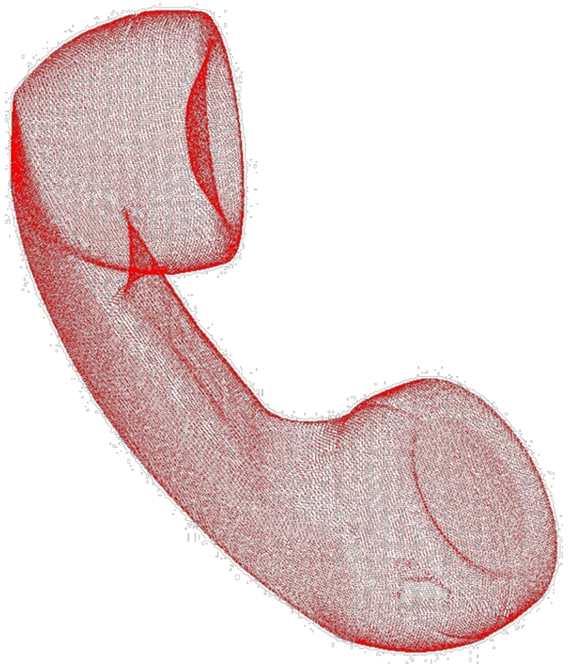
$\#$ denotes connected sum

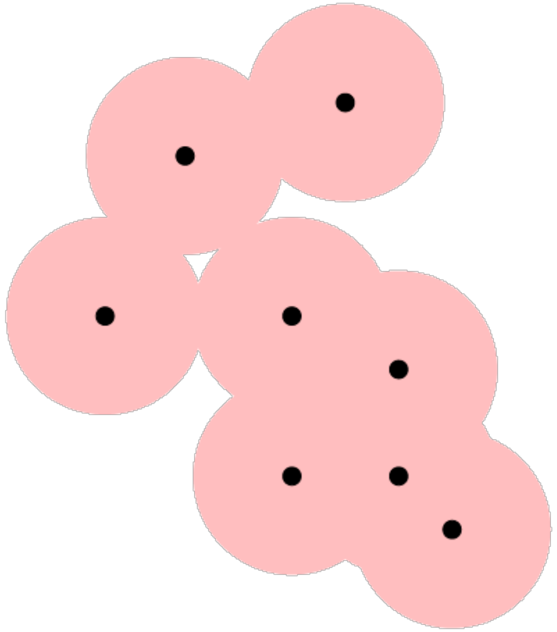
Topological Classification via Invariants

- (Theorem) Closed compact surfaces $\mathbb{M}_1$ and $\mathbb{M}_2$ are homeomorphic, $\mathbb{M}_1 \approx \mathbb{M}_2$ iff
  1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ and
  2. either both surfaces are orientable or both are non-orientable.
- "iff" so full answer. We're done!
- Higher dimensions?

Useful Complexes on Point Clouds

ϵ-Balls





- ϵ-ball: $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$.
- Open sets and topology
- Manifold is $\tilde{\mathbb{M}} = \bigcup_{m_i \in M} B_\epsilon(m_i)$

A Model Space

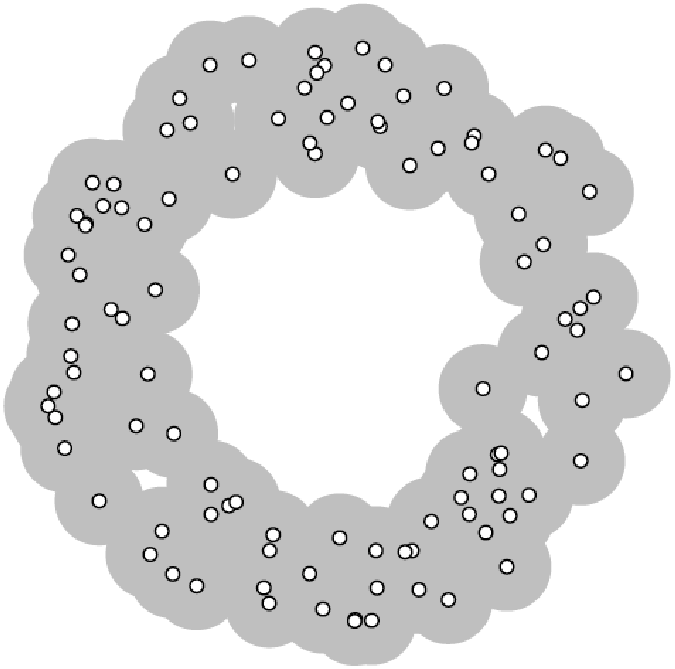
For a dataset X we study the topology of the *union of balls*

$$M_\epsilon = \bigcup_{x \in X} B_\epsilon(x)$$

Two Issues:

Scale: No natural choice of ϵ!

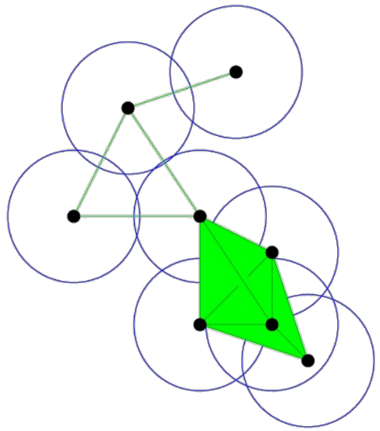
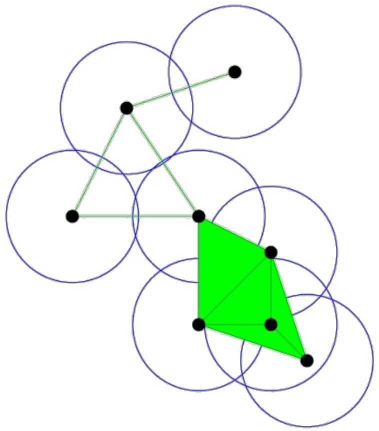
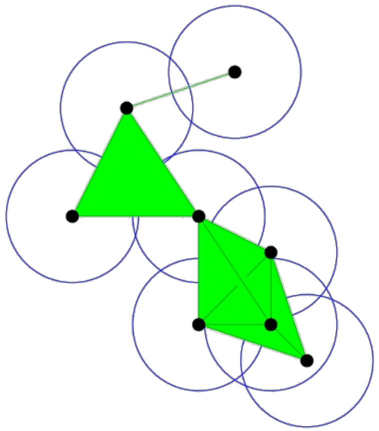
Conception: How to encode M_ϵ on computer?



Complex Zoo

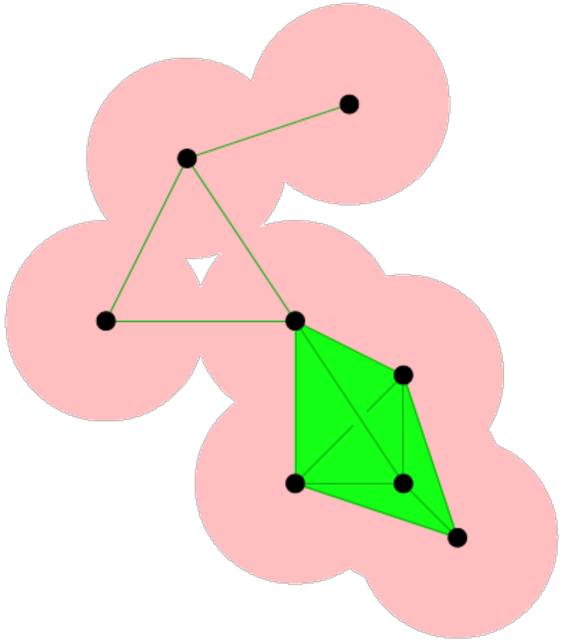
Must choose which simplices to introduce

Čech | Alpha | Rips

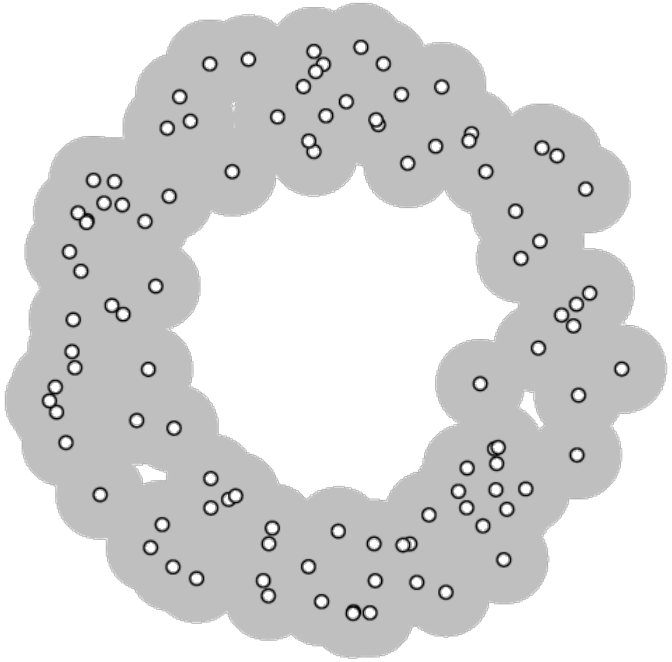
Combinatorial complexes provide discrete representations of the underlying space

Čech Complex



- $C_\epsilon(M) = \left\{\operatorname{conv} T \mid T \subseteq M, \bigcap_{m_i \in T} B_\epsilon(m_i) \neq \emptyset\right\}.$
- $\sum_{k=0}^{m} \binom{m}{k} = 2^{m+1} - 1$
- $C_\epsilon(M) \simeq \tilde{\mathbb{M}}$

Čech Complex

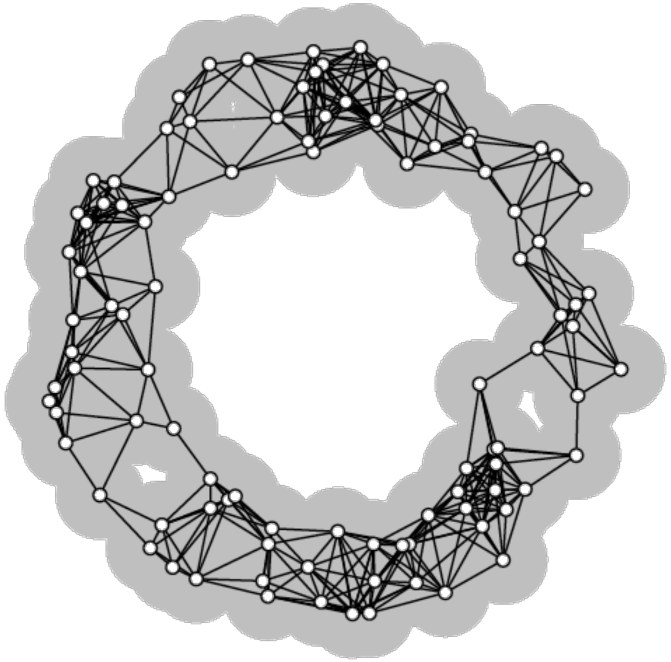


$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

Čech Complex



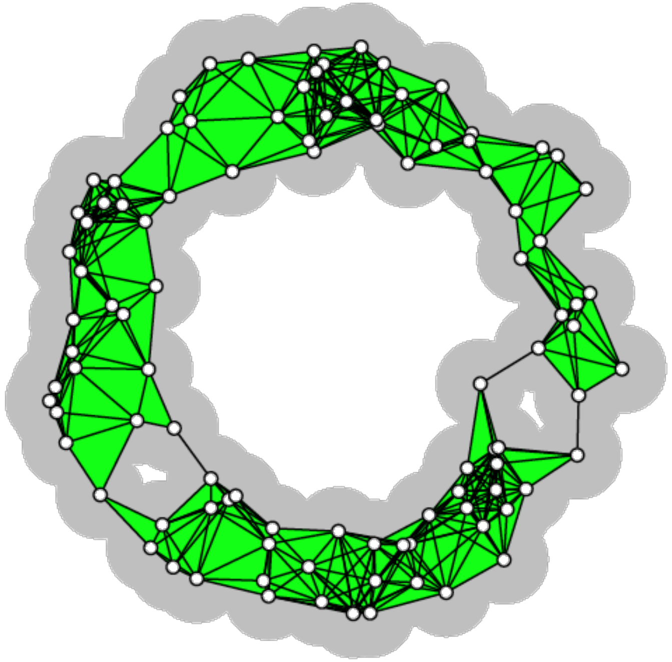
$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

Pairwise intersections as *edges* (1-cells)

Čech Complex



$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

Points as *vertices* (0-cells)

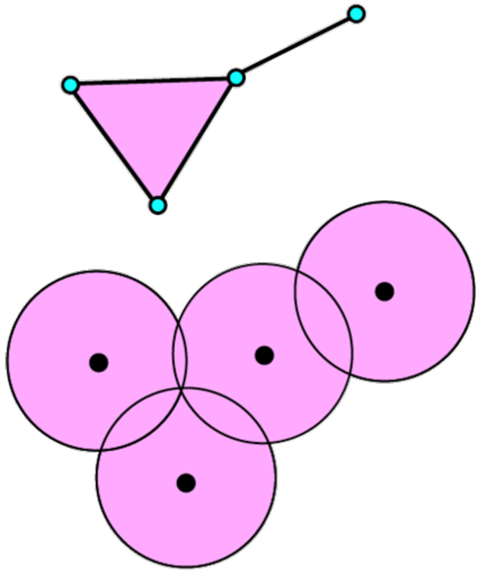
Pairwise intersections as *edges* (1-cells)

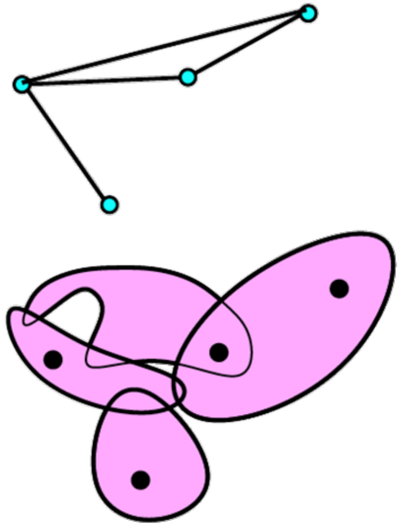
Threeway intersections as *triangles* (2-cells)

k-way intersections as *(k+1)-cells*

Can be hard to compute …

General Čech Complex = Nerve

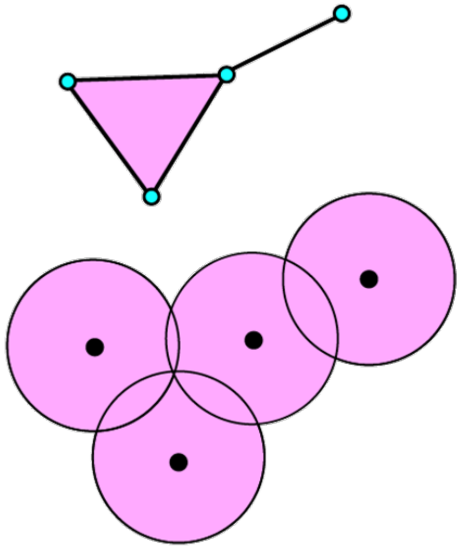


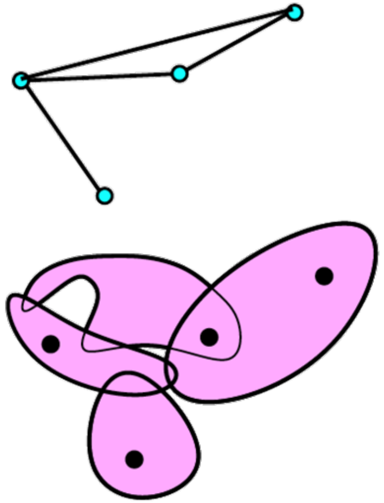


- Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of a topological space X by open sets: $X = \cup_{i \in I} U_i$.
- The Cěch complex $C(\mathcal{U})$ associated to the covering $\mathcal{U}$ is the simplicial complex defined by:
  - the vertex set of $C(\mathcal{U})$ is the set of the open sets U_i
  - $[U_{i_0}, \cdots, U_{i_k}]$ is a k-simplex in $C(\mathcal{U})$ iff $\cap_{j=0}^{k} U_{i_j} \neq \emptyset$.

General Čech Complex

Lemma (Nerve Lemma, Leray '45)
$\check{C}_\epsilon$ is topologically equivalent to M_ϵ.





Nerve theorem (Leray): If all the intersections between opens in $\mathcal{U}$ are either empty or contractible then $C(\mathcal{U})$ and $X = \cup_{i \in I} U_i$ are homotopy equivalent.

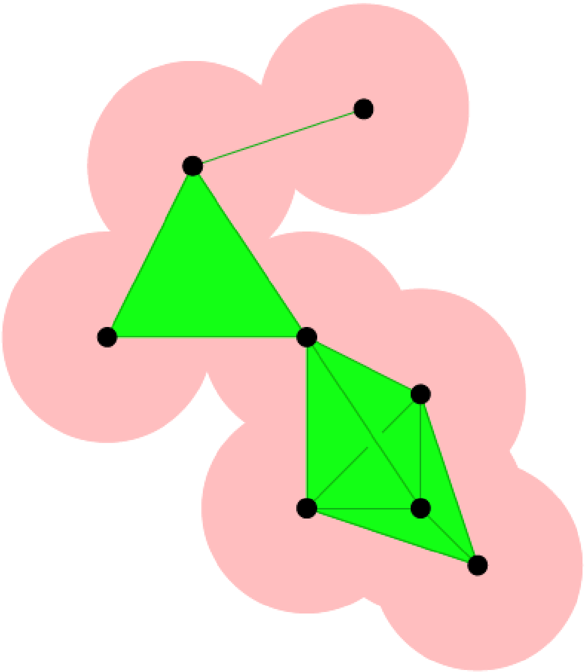
$\Rightarrow$ The combinatorics of the covering (a simplicial complex) carries the topology of the space.

Warning: even when the open sets are euclidean balls, the computation of the Cěch complex is a very difficult task!

Rips-Vietoris Complex

The “poor man’s” alternative
to the Čech

This is a common complex
For computations



- $R_\epsilon(M) = \{\operatorname{conv} T \mid T \subseteq M,\ d(m_i, m_j) < \epsilon, m_i, m_j \in T\}$.
- Still $O\left(\binom{m}{k}\right)$ for the kth skeleton
- Need $(k+1)$st skeleton for computing H_k

Rips vs. Čech

Let $L = \{p_0, \cdots p_n\}$ be a (finite) point cloud (in a metric space).

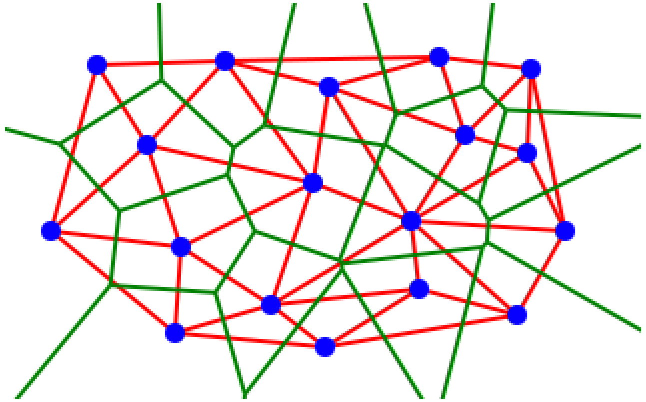
The Rips complex $\mathcal{R}^\alpha(L)$: for $p_0, \cdots p_k \in L$,

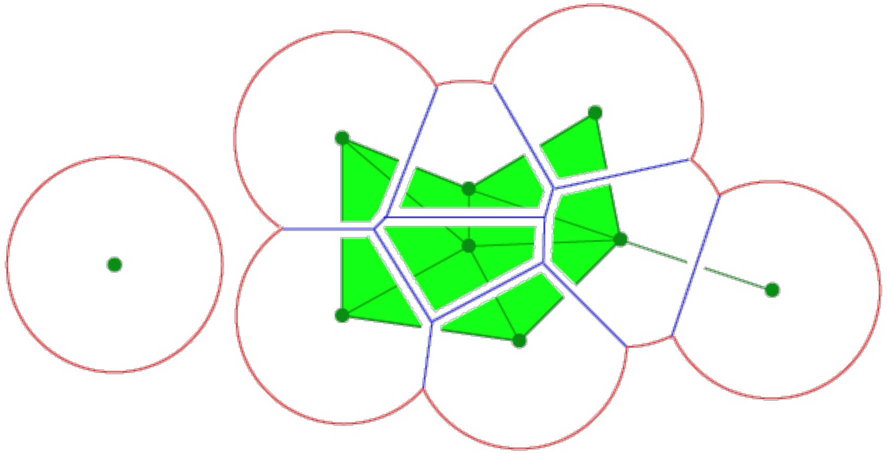
$$\sigma = [p_0 p_1 \cdots p_k] \in \mathcal{R}^\alpha(L) \;\text{ iff }\; \forall i, j \in \{0, \cdots k\},\; d(p_i, p_j) \leq \alpha$$

- Easy to compute and fully determined by its 1-skeleton
- Rips-Čech interleaving: for any $\alpha > 0$,

$$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L) \subseteq \mathcal{R}^{2\alpha}(L) \subseteq \cdots$$

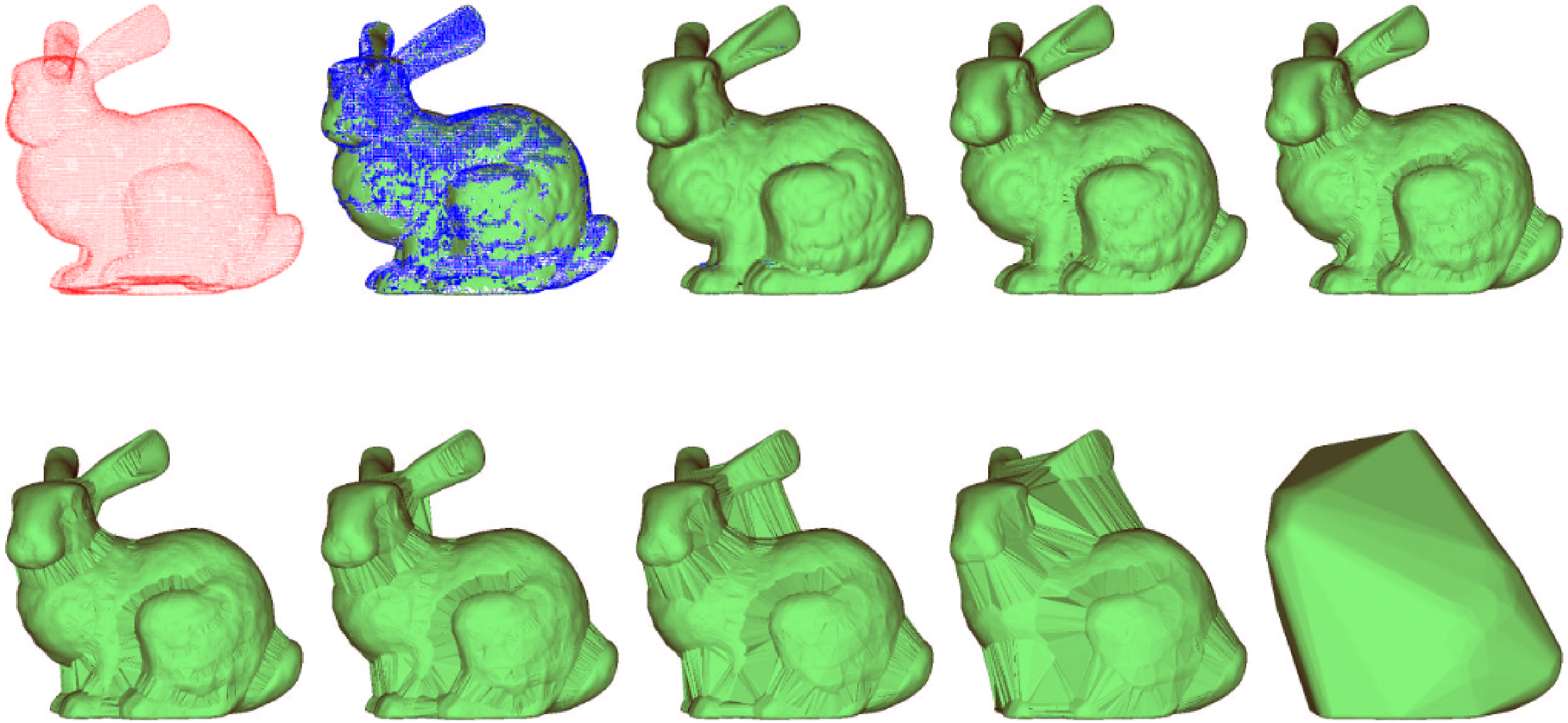
Alpha Complex





- $V(m_i) = \{x \in \mathbb{R}^3 \mid d(x, m_i) \leq d(x, m_j) \, \forall m_j \in M\}$
- $\hat{V}(m_i) = B_\epsilon(m_i) \cap V(m_i)$
- $A_\epsilon = \left\{ \mathrm{conv}\, T \mid T \subseteq M, \bigcap_{m_i \in T} \hat{V}(m_i) \neq \emptyset \right\}$
- $A_\epsilon(M) \simeq \tilde{\mathbb{M}}$, $A_\epsilon \subseteq D$, the Delaunay complex
- $O(n \log n + n^{\lceil d/2 \rceil})$

Alpha Complexes on the Stanford Bunny



- 34,834 points, 1,026,111 complexes

Algebraic Structures: Group Theory

Groups

- A group $\langle G, * \rangle$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:
  (a) $*$ is associative.
  (b) G has an identity e element for $*$ such that $e * x = x * e = x$ for all $x \in G$.
  (c) any element a has an inverse a' with respect to the operation $*$, i.e. $\forall a \in G, \exists a' \in G$ such that $a' * a = a * a' = e$.
- If G is finite, the order of G is $|G|$.
- We often omit the operation and refer to G as the group.
- $\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{R}, \cdot \rangle$, $\langle \mathbb{R}, + \rangle$, are all groups.
- A group G is abelian if its binary operation $*$ is commutative.

Subgroups

- Let $\langle G, *\rangle$ be a group and $S \subseteq G$. If S is closed under $*$, then $*$ is the induced operation on S from G.
- A subset $H \subseteq G$ of group $\langle G, *\rangle$ is a subgroup of G if H is a group and is closed under $*$. The subgroup consisting of the identity element of G, $\{e\}$ is the trivial subgroup of G. All other subgroups are nontrivial.
- (Theorem)$H \subseteq G$ of a group $\langle G, *\rangle$ is a subgroup of G iff:
  1. H is closed under $*$,
  2. the identity e of G is in H,
  3. for all $a \in H$, $a^{-1} \in H$.
- Example: subgroups of $\mathbb{Z}_4$

Cosets

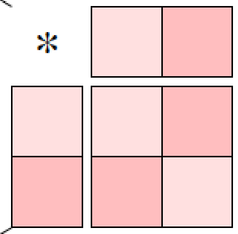
- Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by: $a \sim_L b$ iff $a^{-1}b \in H$. Let $\sim_R$ be defined by: $a \sim_R b$ iff $ab^{-1} \in H$. Then $\sim_L$ and $\sim_R$ are both equivalence relations on G.
- Let H be a subgroup of group G. For $a \in G$, the subset $aH = \{ah \mid h \in H\}$ of G is the left coset of H containing a, and $Ha = \{ha \mid h \in H\}$ is the right coset of H containing a.
- If left and right cosets match, the subgroup is normal.
- All subgroups H of an abelian group G are normal, as $ah = ha, \forall a \in G, h \in H$
- $\{0, 2\}$ is a subgroup of $\mathbb{Z}_4$. It is normal. The coset of 1 is $1 + \{0, 2\} = \{1, 3\}$. That's all folks!

Factor / Quotient Groups

- Let H be a normal subgroup of group G.
- Left coset multiplication is well-defined by the equation $(aH)(bH) = (ab)H$
- The cosets of H form a group G/H under left multiplication
- G/H is the factor group (or quotient group) of G modulo H.
- The elements in the same coset of H are congruent modulo H.

Example

$\mathbb{Z}_6$	0	2	4	1	3	5
0	0	2	4	1	3	5
2	2	4	0	3	5	1
4	4	0	2	5	1	3
1	1	3	5	2	4	0
3	3	5	1	4	0	2
5	5	1	3	0	2	4

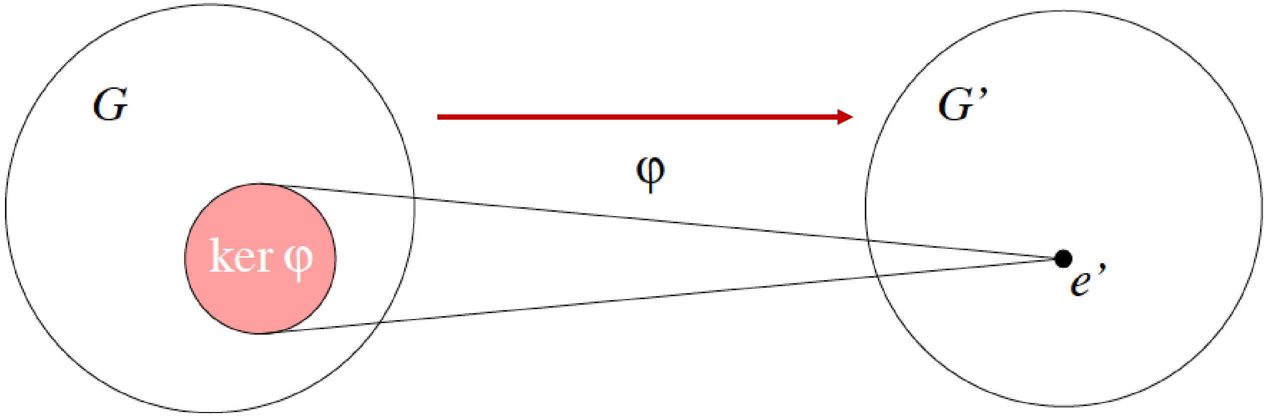


- $\{0, 2, 4\}$ is a normal subgroup
- Cosets $\{0, 2, 4\}, \{1, 3, 5\}$
- $\mathbb{Z}_6/\{0, 2, 4\} \cong \mathbb{Z}_2$

$\mathbb{Z}_m$ integers modulo *m*

Homomorphisms

- A map φ of a group G into a group G' is a *homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.
- If e is the identity in G, then $\varphi(e)$ is the identity e' in G'.
- If $a \in G$, then $\varphi(a^{-1}) = \varphi(a)^{-1}$.
- If H is a subgroup of G, then $\varphi(H)$ is a subgroup of G'.
- If K' is a subgroup of G', then $\varphi^{-1}(K')$ is a subgroup of G.
- The normal subgroup $\ker\varphi = \varphi^{-1}(\{e'\}) \subseteq G$, is the kernel of φ.



Finitely Generated Abelian Groups

- Let $G_1, G_2, \ldots, G_n$ be groups.
- The set is $\prod_{i=1}^{n} G_i$ (Cartesian product)
- Binary operation:
  $(a_1, a_2, \ldots, a_n) \times (b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$.
- Then $\langle \prod_{i=1}^{n} G_i, \times \rangle$ is a group.
- We call it the direct product of the groups G_i.
- Sometimes called direct sum with $\oplus$.

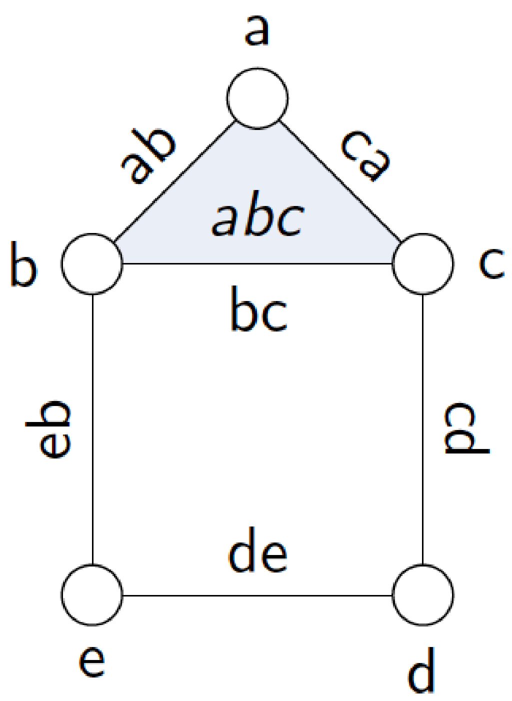
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

  where m_i divides m_{i+1} for $i = 1, \ldots, r-1$.
- The direct product is unique: the number of factors of $\mathbb{Z}$ is unique and the cyclic group orders m_i are unique.
- Free: basis, rank, vector space
- Torsion: module

Algebraic Topology: Homology

Topology of Simplicial Complexes



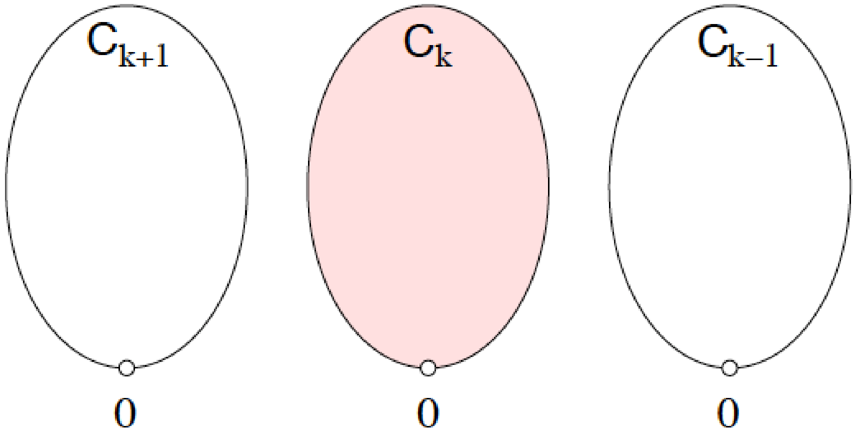
A simplicial complex is a collection of sim-plices

- Each simplex has a dimension.
- Collection is closed under subset relation.
- Simplices of dimension d have $d + 1$ vertices
- Each simplex represented by an ordered list of vertices

Chain Groups

Other coefficient fields/rings also OK

- Simplicial complex K
- k-chain: $c = \sum_i n_i[\sigma_i], n_i \in \mathbb{Z}, \sigma_i \in K$ (like a path)
- $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ have different orientations.
- The kth chain group C_k of K is the free abelian group on its set of oriented k-simplices
- rank C_k =?



We take linear combinations of simplices of a given dimension *k*

Boundary Operator

- The boundary operator $\partial_k : \mathsf{C}_k \to \mathsf{C}_{k-1}$ is a homomorphism defined linearly on a chain c by its action on any simplex $\sigma = [v_0, v_1, \ldots, v_k] \in c$,

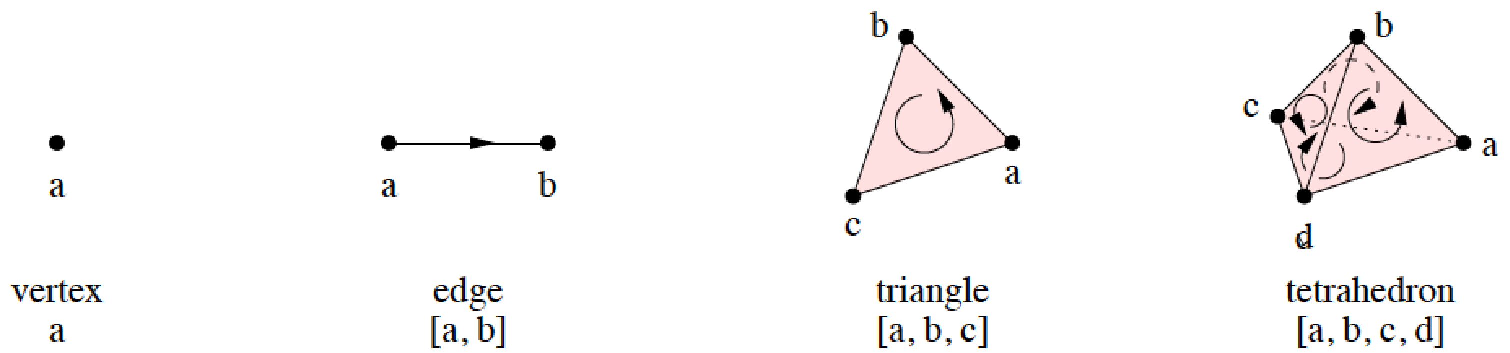
$$\partial_k \sigma = \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v_i}, \ldots, v_k],$$

where $\hat{v_i}$ indicates that v_i is deleted from the sequence.

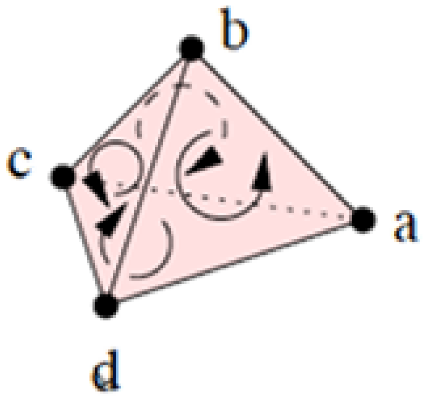
- $\partial_1 [a, b] = b - a$.
- $\partial_2 [a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b]$.
- $\partial_3 [a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c]$.

Boundary Examples

- $\partial_1[a, b] = b - a.$
- $\partial_2[a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b].$
- $\partial_3[a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c].$
- $\partial_1\partial_2[a, b, c] = [c] - [b] - [c] + [a] + [b] - [a] = 0.$



Boundary Theorem



- (Theorem) $\partial_{k-1}\partial_k = 0$, for all k.
- Proof:

$\partial_{k-1}\partial_k[v_0, v_1, \ldots, v_k] =$

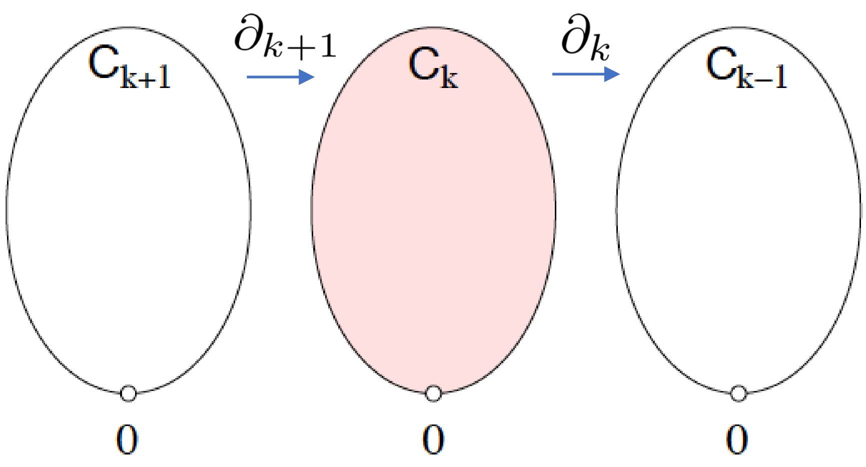
$$\begin{aligned} &= \partial_{k-1}\sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k] \\ &= \sum_{j<i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k] \\ &\quad + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k] \\ &= 0, \end{aligned}$$

as switching i and j in the second sum negates the first sum.

Chain Complex

- The boundary operator connects the chain groups into a chain complex C_*:

$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots .$$

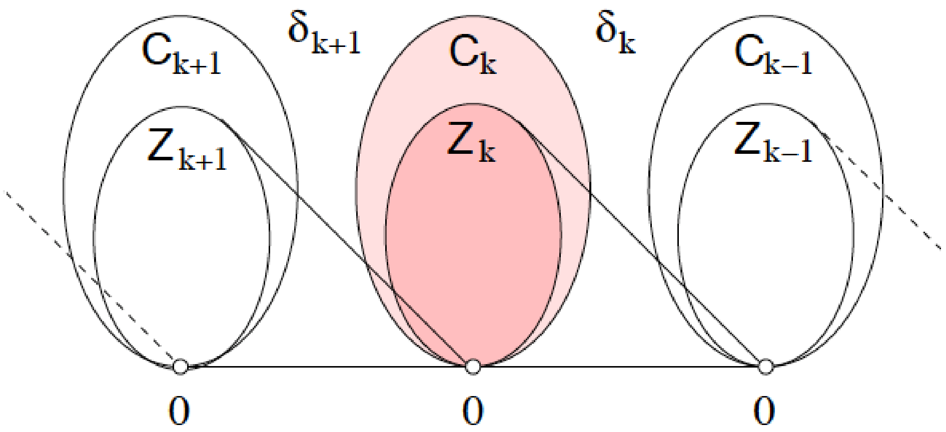


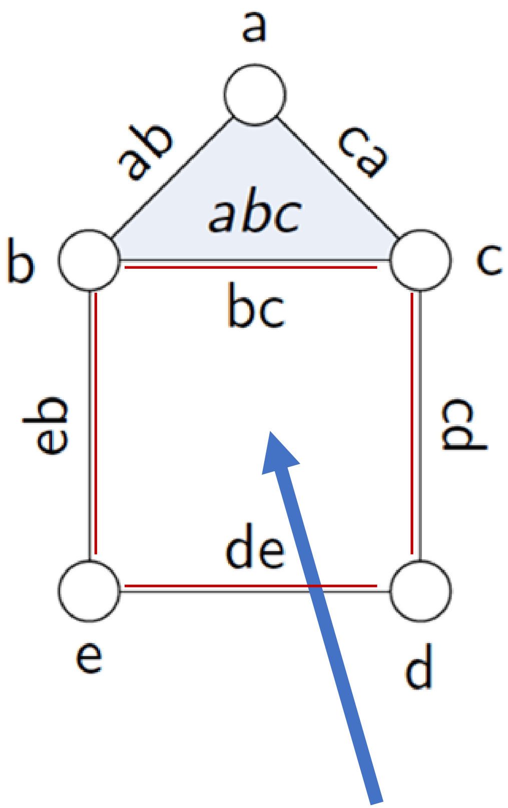
Cycle Group

- Let c be a k-chain
- If it has no boundary, it is a k-cycle (zycle?)
- $\partial_k c = \emptyset$, so $c \in \ker \partial_k$
- The kth cycle group is

$$\mathsf{Z}_k = \ker \partial_k = \{c \in \mathsf{C}_k \mid \partial_k c = \emptyset\}.$$

$$\partial = \delta$$





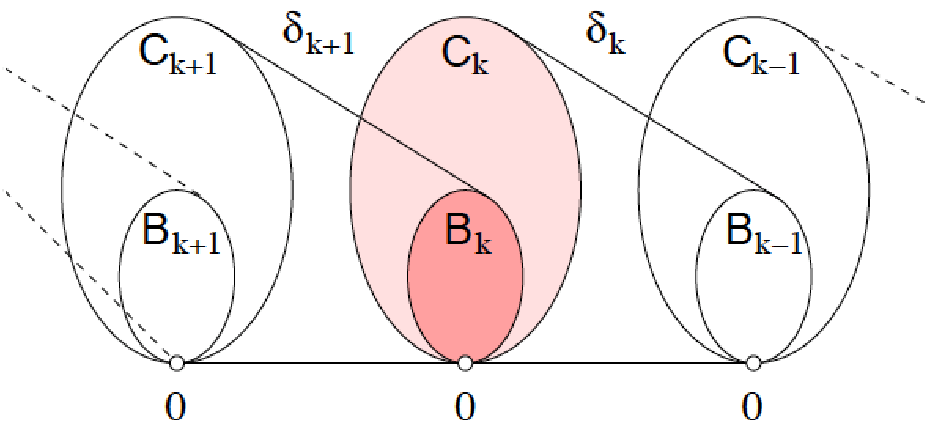
$$\partial(bc + cd + de + eb)$$
$$= c - b + d - c + e - d + b - e = 0.$$

Boundary Group

- Let b be a k-chain
- If b is a boundary of something, it is a k-boundary.
- The kth boundary group is

$$\mathsf{B}_k = \operatorname{im} \partial_{k+1} = \{c \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1} : c = \partial_{k+1} d\}.$$

$$\partial = \delta$$



Boundaries are Cycles!

- Let b be a k-boundary.
- Then, $\exists c \in \mathbf{C}_{k+1}$, such that $b = \partial_{k+1} c$.
- What is the boundary of b?

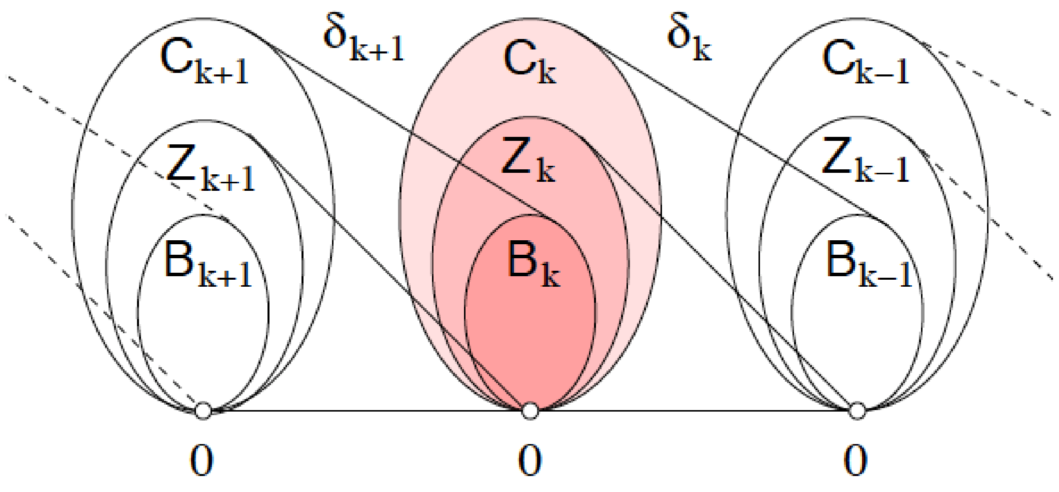
$$\partial_k b = \partial_k \partial_{k+1} c = \emptyset,$$

by the boundary theorem.

- That is, every boundary is a cycle!

- $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$

$$\partial = \delta$$



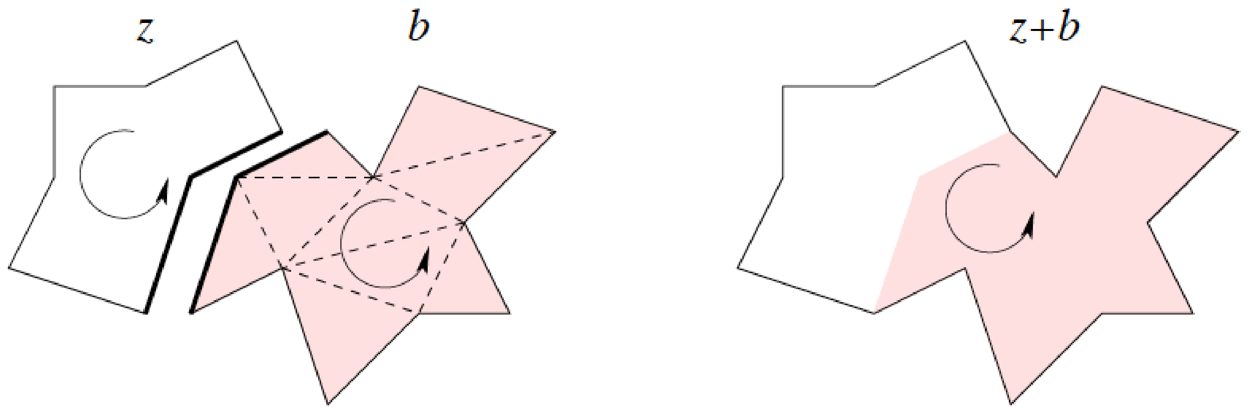
Nesting behavior!

Equivalent Cycles

- z is a k-cycle
- b is a k-boundary
- We would like to have $z + b$ be equivalent to z

Cosets!

- That is, if $z_1 - z_2 = b$ where b is a boundary, then $z_1 \sim z_2$
- Any boundary would do!

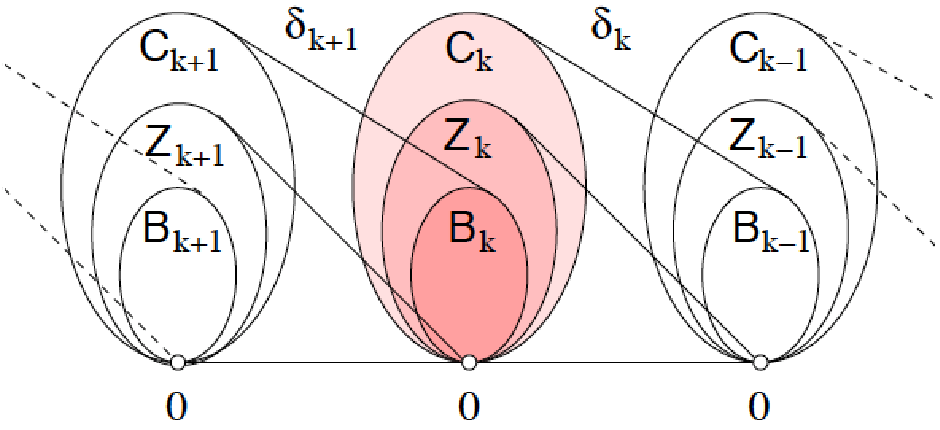


Simplicial Homology

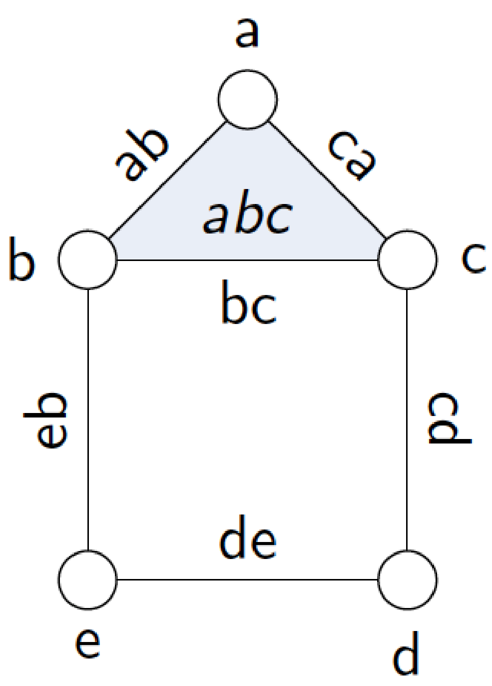
- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

- If $z_1 = z_2 + \mathsf{B}_k$, $z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are homologous
- $z_1 \sim z_2$.



To Repeat



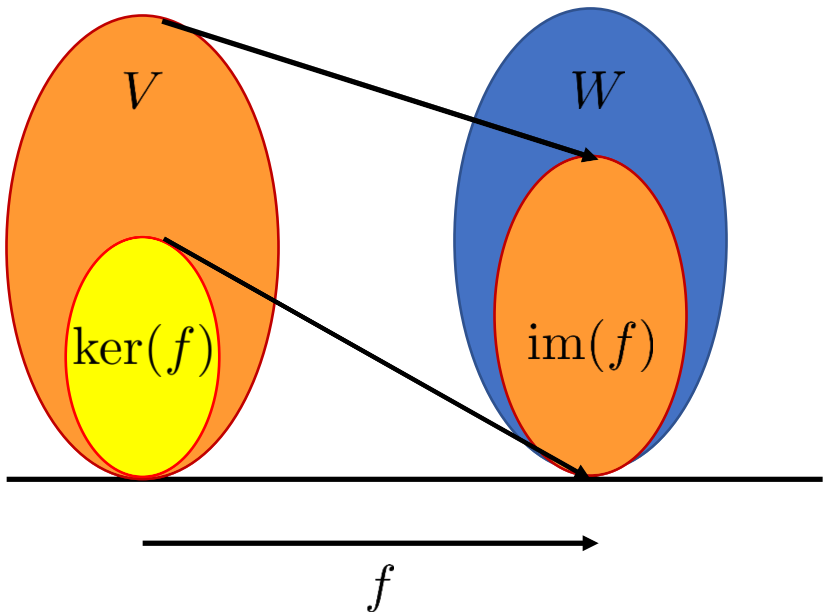
In other words..

- The kernel (null space) of ∂_k is the vector space of cycles in dimension k.
- The image of ∂_k is the subspace of boundary cycles in dimension $k-1$.

Homology of a space X is the quotient:

$$H_k(X) = \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$$

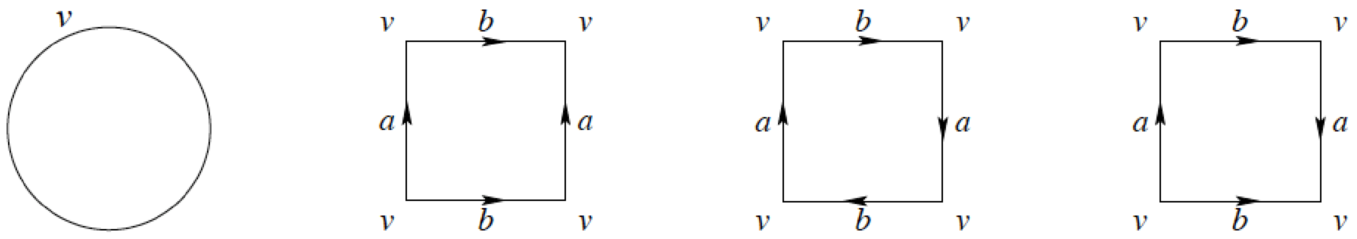
In Vector Spaces



$$V \approx \ker(f) \oplus \mathrm{im}(f)$$

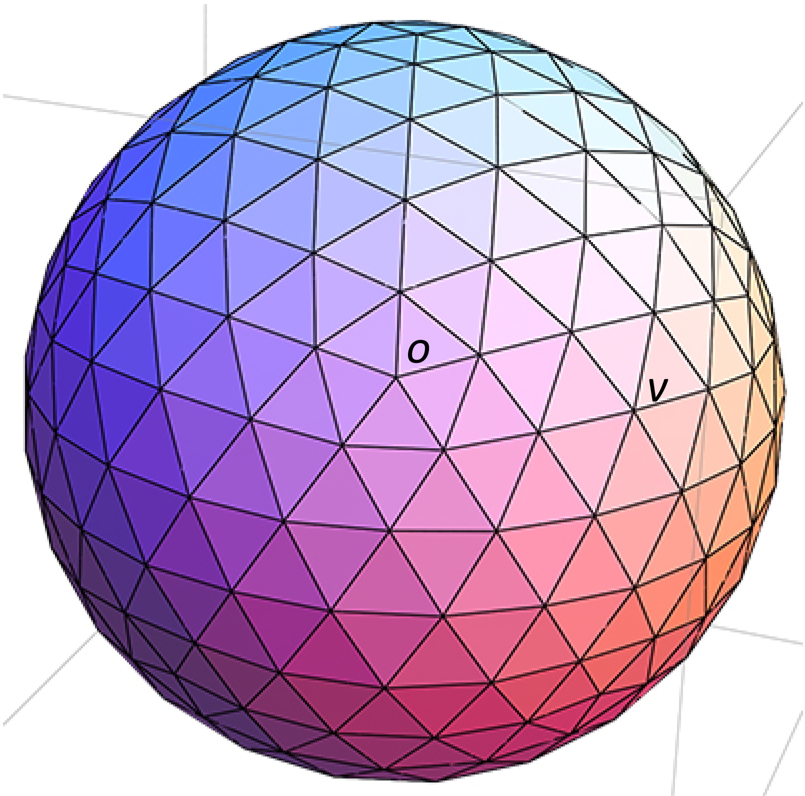
Homology of 2-Manifolds

2-manifold	H_0	H_1	H_2
sphere	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$
torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$



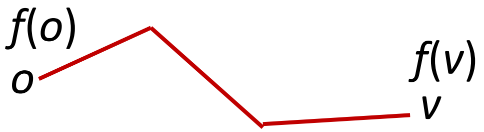
(a) Sphere (b) Torus (c) Projective plane (d) Klein bottle

Why is $H_0(S^2) = Z$?

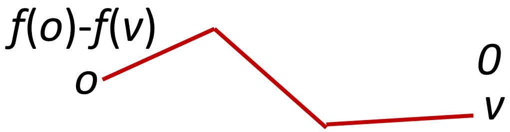


A 0-chain is an assignment of integers f(v) to each mesh vertex v.

The mesh is connected, so all pairs of vertices are homologous.



So we can pick a specific vertex o and use paths from o to each other v to "cancel" f at v.



Homology Groups

- Homology groups are finitely generated abelian.
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

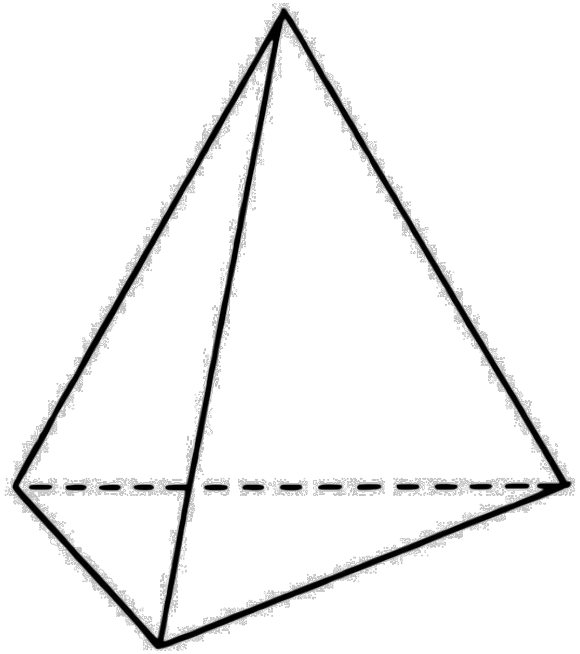
$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \boxed{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},}$$

- The kth Betti number β_k of a simplicial complex K is $\beta_k = \beta(\mathsf{H}_k)$, the rank of the free part of H_k.
- Torsion coefficients

- Alexander Duality:
  - β_0 measures the number of components of the complex.
  - β_1 is the rank of a basis for the tunnels.
  - β_2 counts the number of voids in the complex.

Betti Numbers β_i

- Ranks of the free part of homology groups H_i
- β_0 counts the number of connected components
- β_1 counts the number of independent loops
- β_2 counts the number of independent voids
- …



Topology is fundamentally a tool for classification

Invariance of Homology Groups

- (Hauptvermutung) Any two triangulations of a topological space have a common refinement (Poincaré 1904)
  - True for polyhedra of dimension ≤ 2 (Papakyriakopoulos 1943)
  - True for 3-manifolds (Moïse 1953)
  - False in dimensions ≥ 6 (Milnor 1961)
  - False for manifolds of dimension ≥ 5 (Kirby and Siebenmann 1969)
- Singular homology

Euler Revisited

- Let K be a simplicial complex and $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$. The Euler characteristic $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}.$$

- We have new language!
- Let C_* be the chain complex on K
- $\operatorname{rank}(\mathsf{C}_i) = |\{\sigma \in K \mid \dim \sigma = i\}| \;\; (= n_i = z_i + b_{i-1})$
- $\chi(K) = \chi(\mathsf{C}_*) = \sum_i (-1)^i \operatorname{rank}(\mathsf{C}_i)$.

$$\sum_i (-1)^i (z_i + b_{i-1}) = \sum_i (-1)^i (z_i - b_i)$$

Euler – Poincaré Formula

- Homology functors H_*
- $\mathsf{H}_*(\mathsf{C}_*)$ is a chain complex:

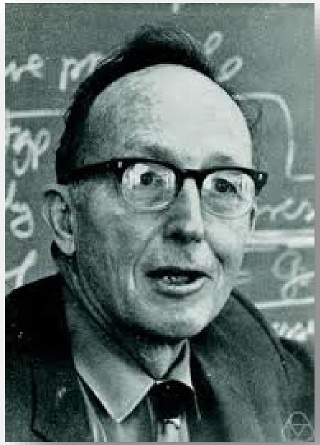
$$\ldots \to \mathsf{H}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{H}_k \xrightarrow{\partial_k} \mathsf{H}_{k-1} \to \ldots .$$

- What is its Euler characteristic?
- (Theorem) $\chi(K) = \chi(\mathsf{C}_*) = \chi(\mathsf{H}_*(\mathsf{C}_*))$.
- $\sum_i (-1)^i s_i = \sum_i (-1)^i \operatorname{rank}(\mathsf{H}_i) = \sum_i (-1)^i \beta_i$
- Sphere: $2 = 1 - 0 + 1$
- Torus: $0 = 1 - 2 + 1$

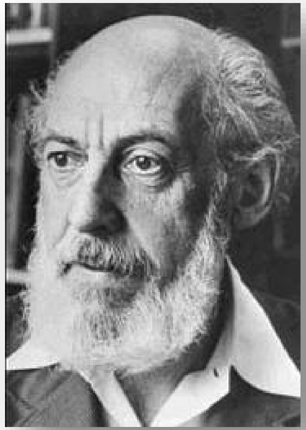
$$\sum_i (-1)^i (z_i + b_{i-1}) = \sum_i (-1)^i (z_i - b_i)$$

Homological Algebra: Functors and Categories

Herni Cartan
1904-2008



Saunders MacLane
1909-2005



Samuel Eilenberg
1913 -1998

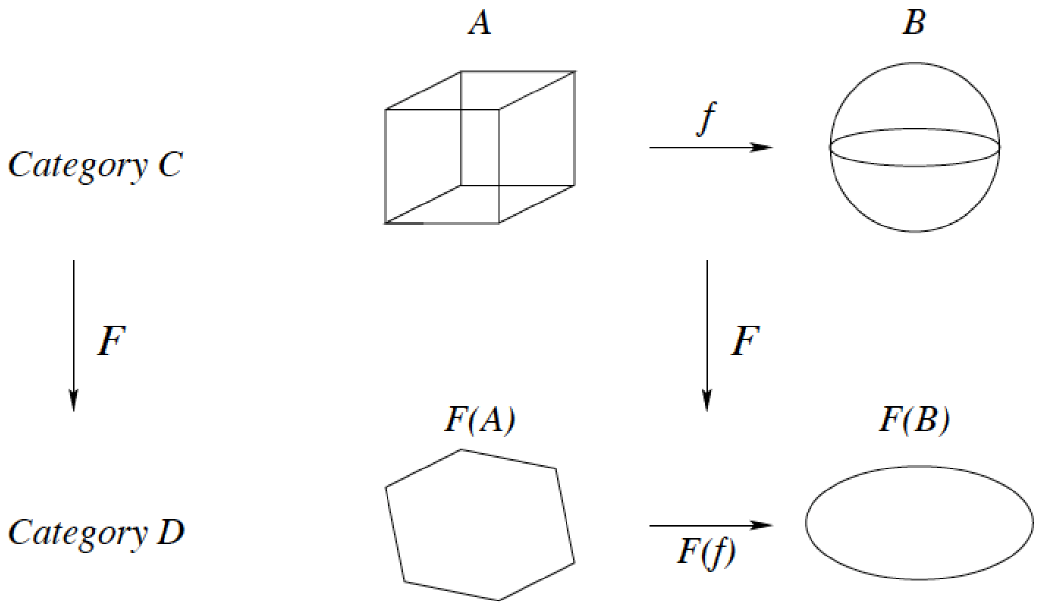
Categories

- A collection $\mathrm{Ob}(\mathcal{C})$ of objects
- Sets $\mathrm{Mor}(X, Y)$ of morphisms for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$
- An identity morphism $1 = 1_X \in \mathrm{Mor}(X, X)$ for each X.
- a composition of morphisms function $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, satisfying $f \circ 1 = 1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.
- A category $\mathcal{C}$

Example Categories

category	morphisms
sets	arbitrary functions
groups	homomorphisms
topological spaces	continuous maps
topological spaces	homotopy classes of maps

Functors



- $X \in \mathcal{C}, F(X) \in \mathcal{D},$
- $f \in \text{Mor}(X, Y), F(f) \in \text{Mor}(F(X), F(Y))$
- $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$
- F is a (covariant) functor

Functoriality, Commutative Diagrams

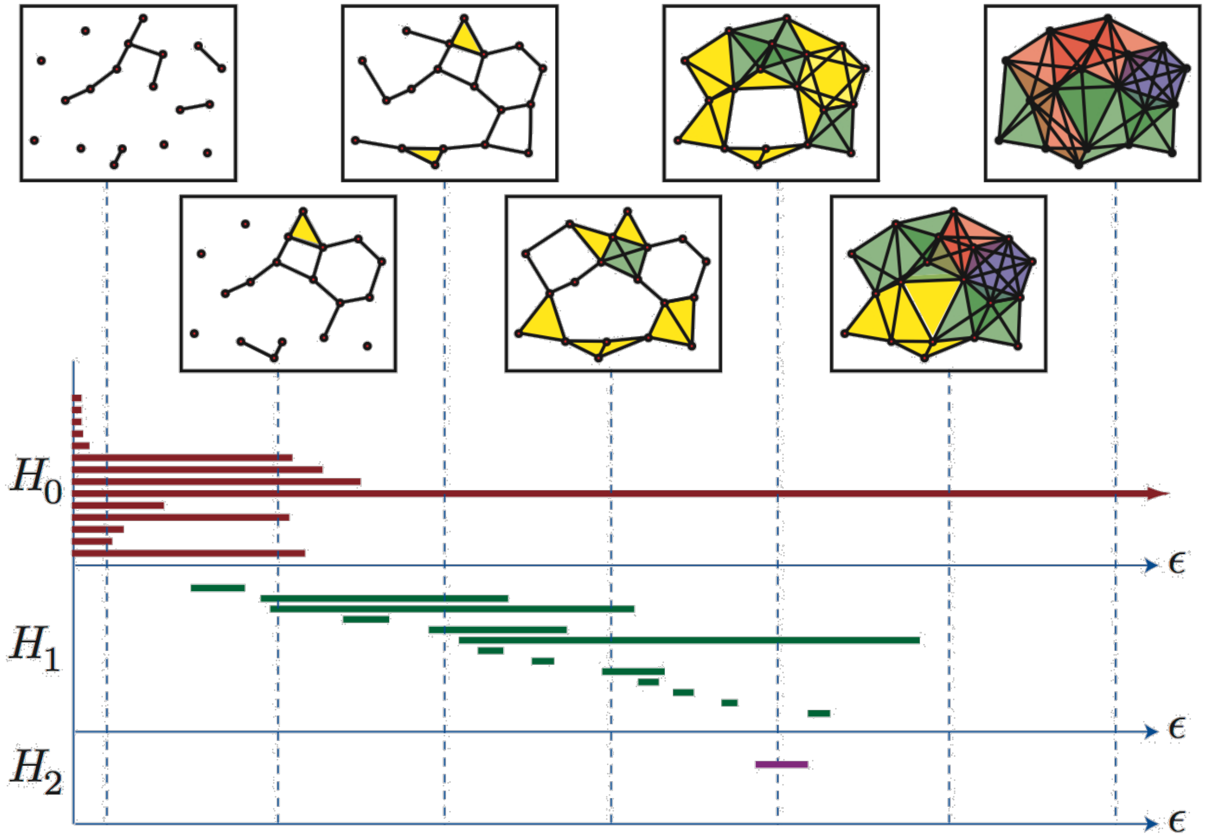
transformation of input $\Rightarrow$ transformation of output

Specifically, this is a commutative diagram:

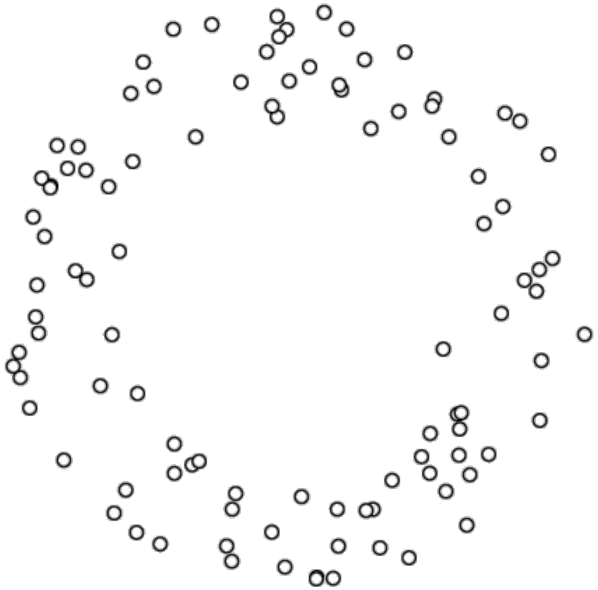
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle H_*}\downarrow & & \downarrow{\scriptstyle H_*} \\ H_*(X) & \xrightarrow{H_*(f)} & H_*(Y) \end{array}$$

Moral: Invariants are not artifacts of arbitrary choices!

Persistent Homology



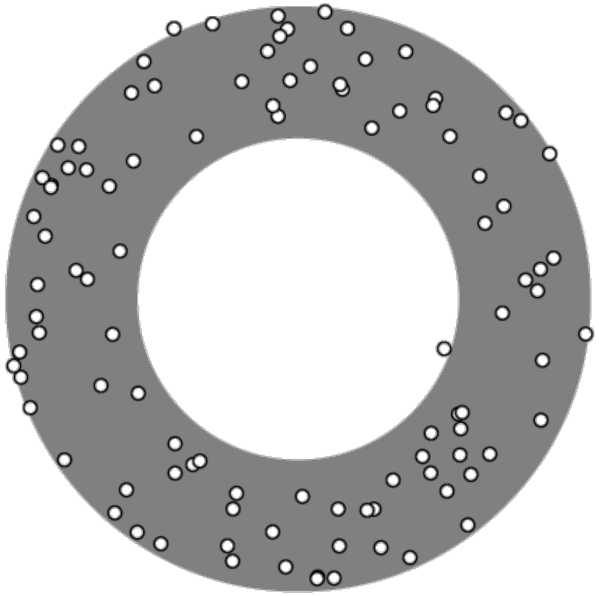
Sampled Data Has “Shape”



2-dimensional

Approximates annulus

Sampled Data Has “Shape”



2-dimensional

Approximates annulus

Topological features of annulus:

1 component ($\beta_0 = 1$)

1 loop ($\beta_1 = 1$)

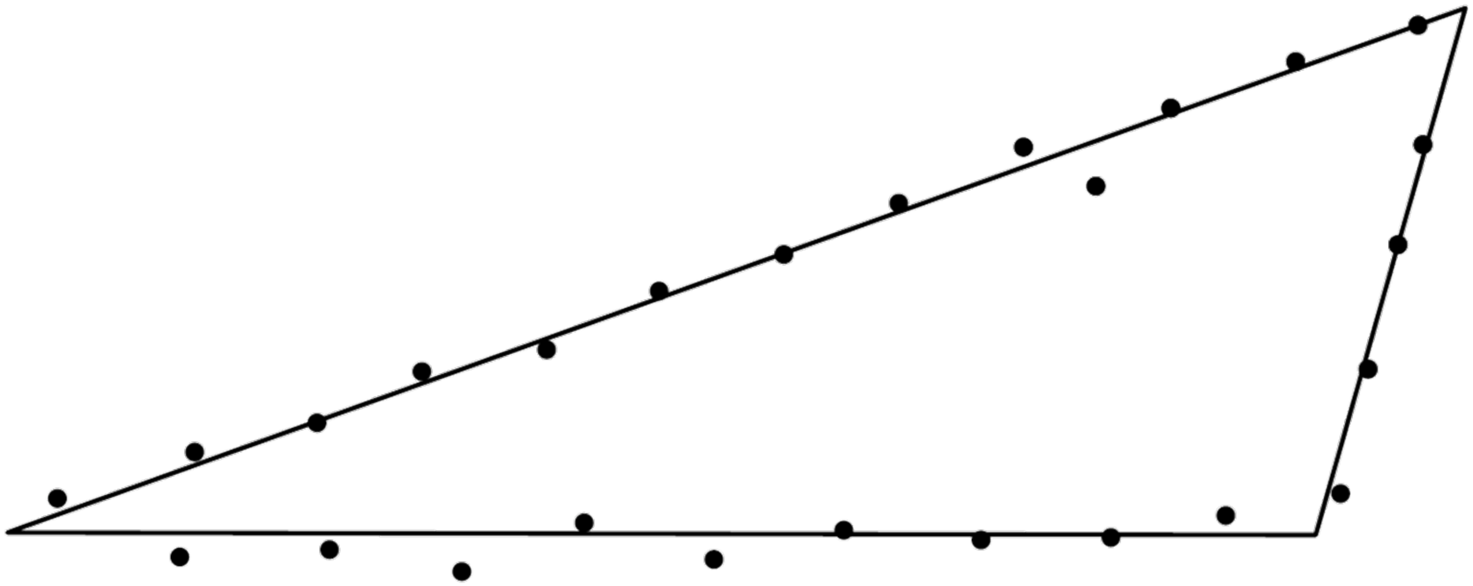
Goal: Recover *topology* of annulus from point cloud

We do so by building various complexes on the point cloud

Filtrations

How to Choose ϵ?

- How to determine the topology of the underlying space from a point cloud approximation?



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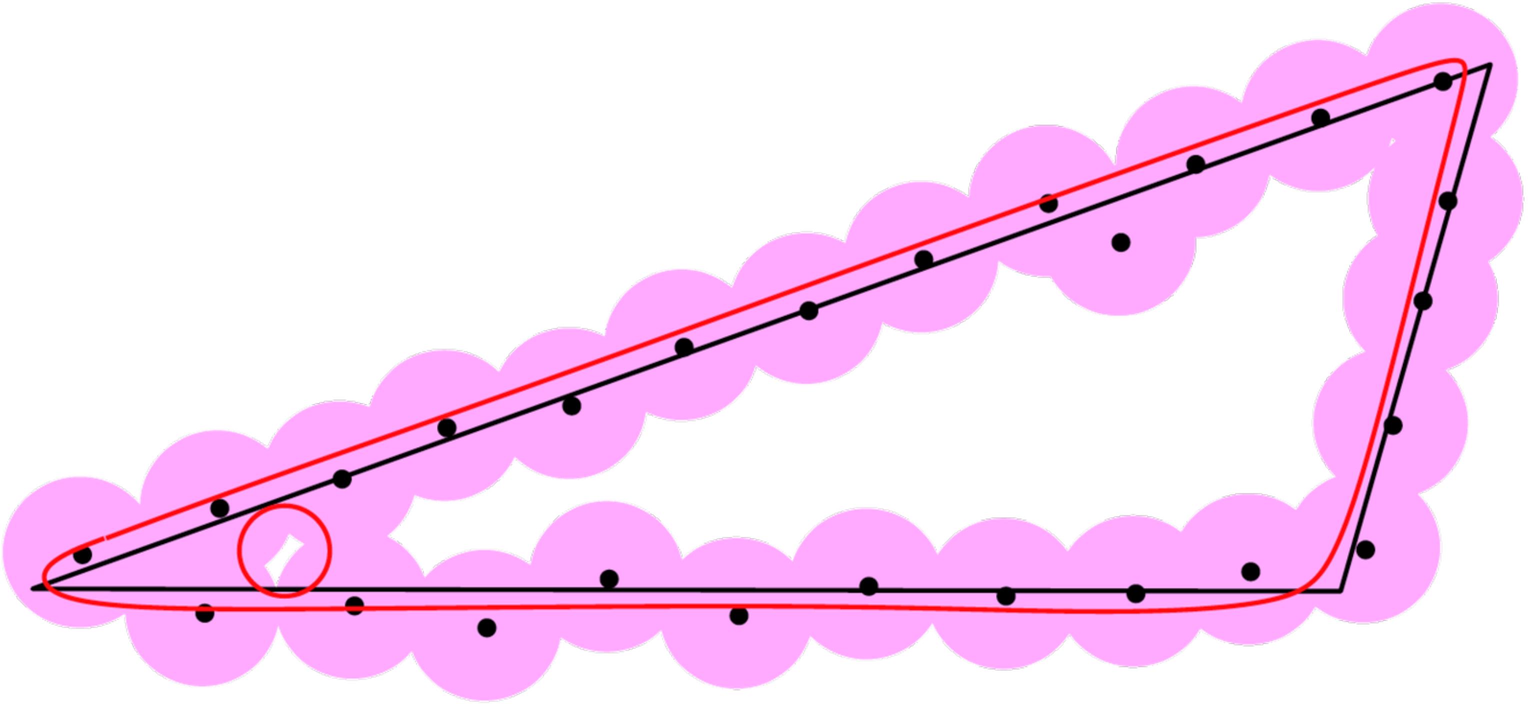

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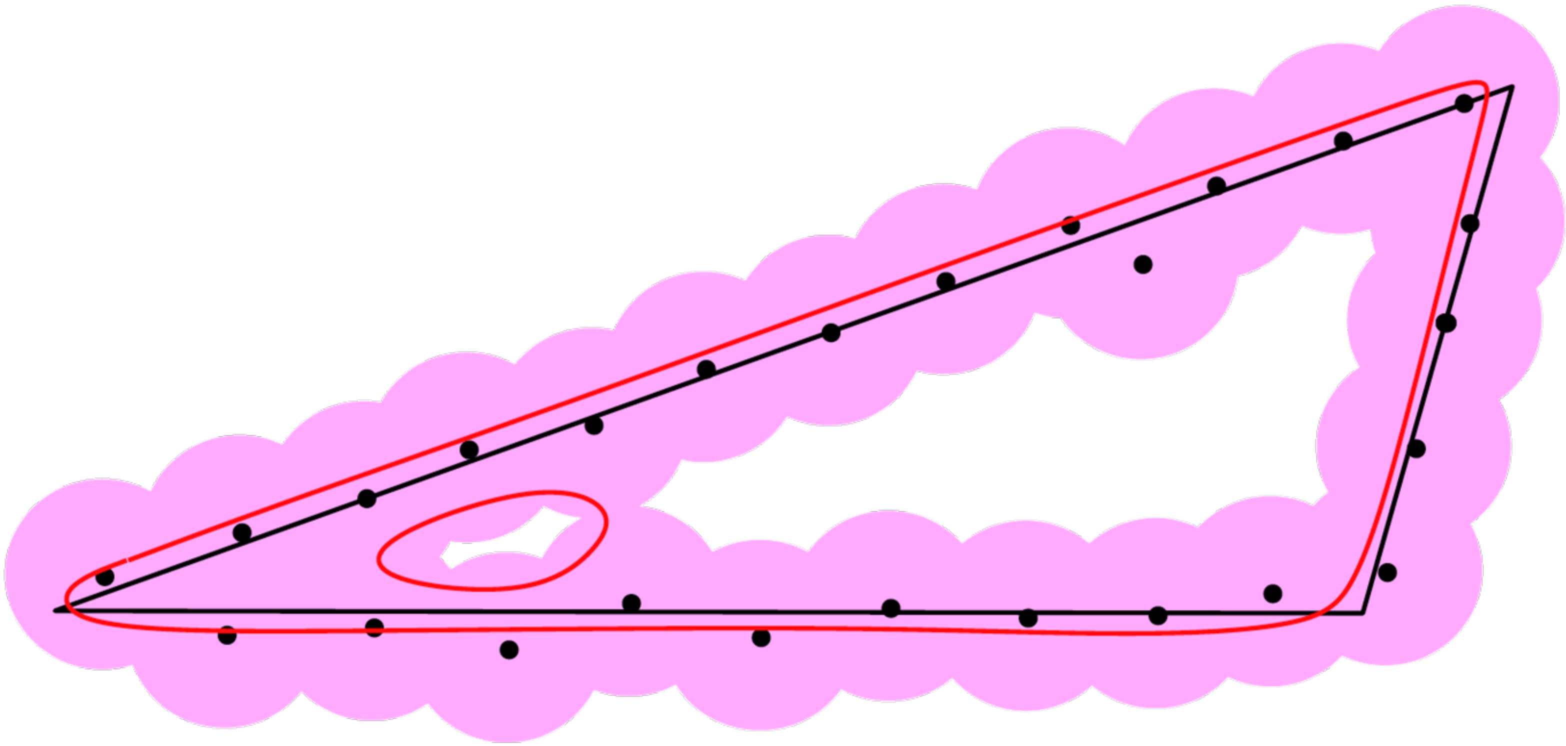

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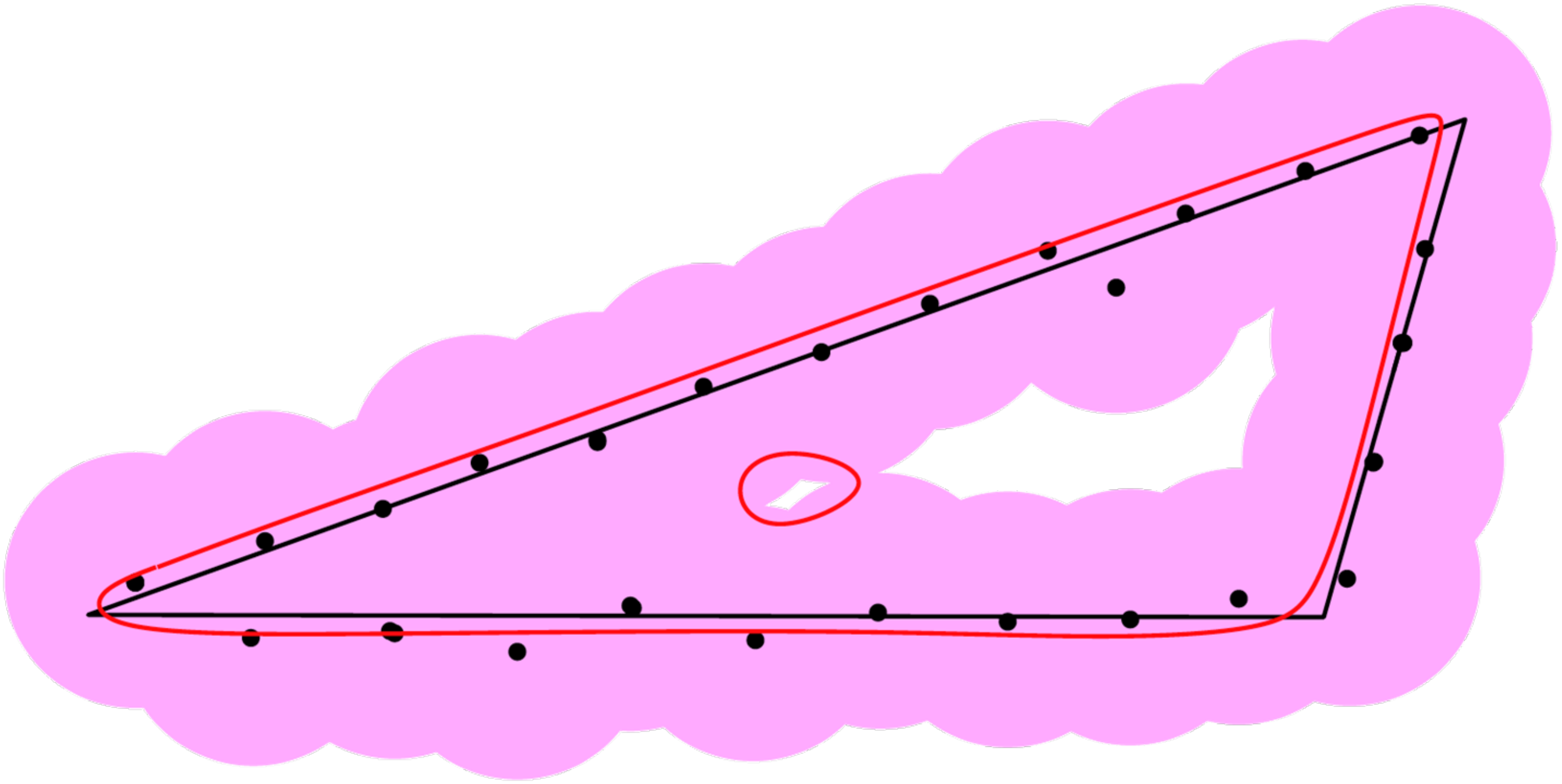
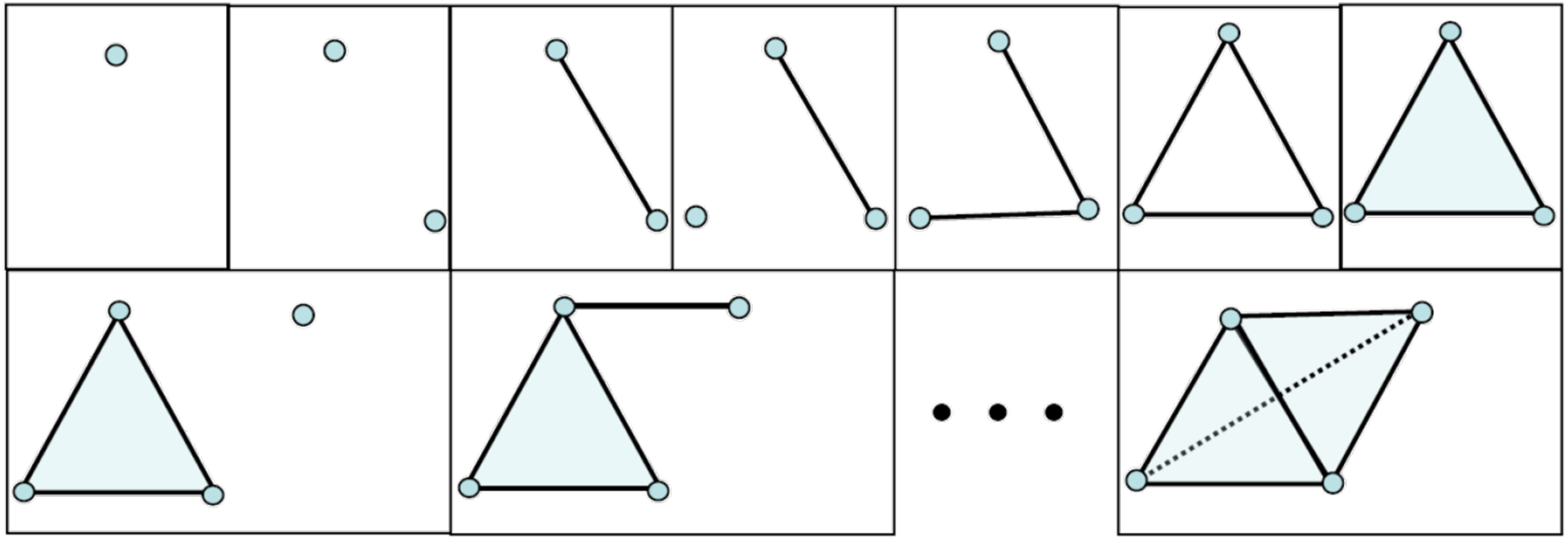


Filtrations



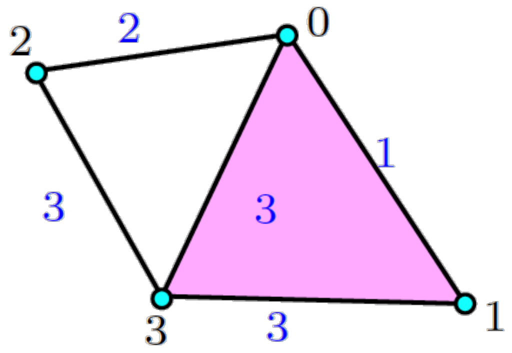
A filtration of a (finite) simplicial complex K is a sequence of subcomplexes such that

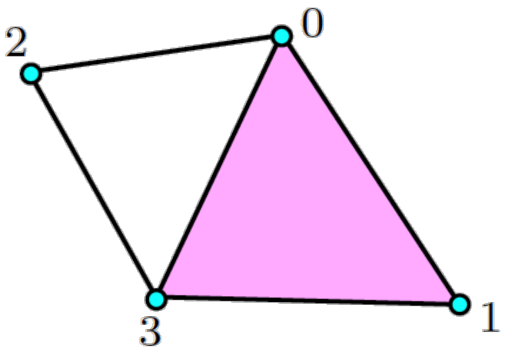
i) $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$,

ii) $K^{i+1} = K^i \cup \sigma^{i+1}$ where σ^{i+1} is a simplex of K.

Sub-simplices of a simplex must be added before the simplex!

The Sub-Level Set Filtration

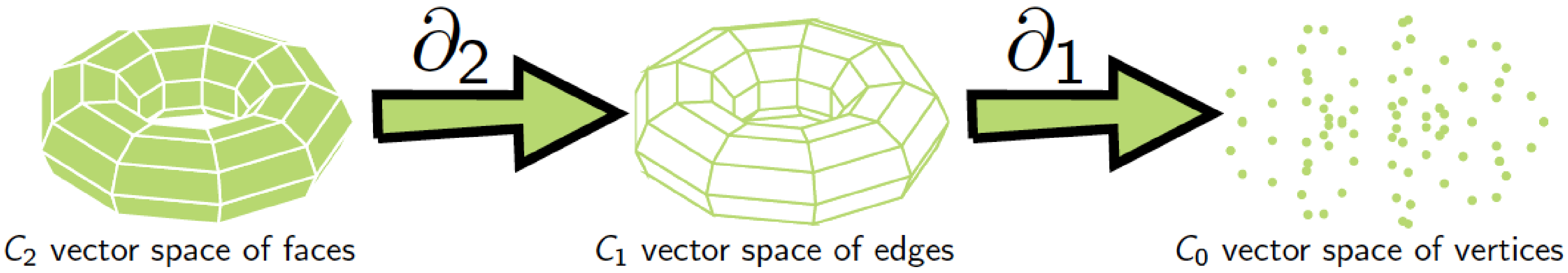




- f a real valued function defined on the vertices of K
- For $\sigma = [v_0, \cdots, v_k] \in K$, $f(\sigma) = \max_{i=0,\cdots,k} f(v_i)$
- The simplices of K are ordered according increasing f values (and dimension in case of equal values on different simplices).

Persistent Homology: Do not choose an ϵ!

Standard Homology



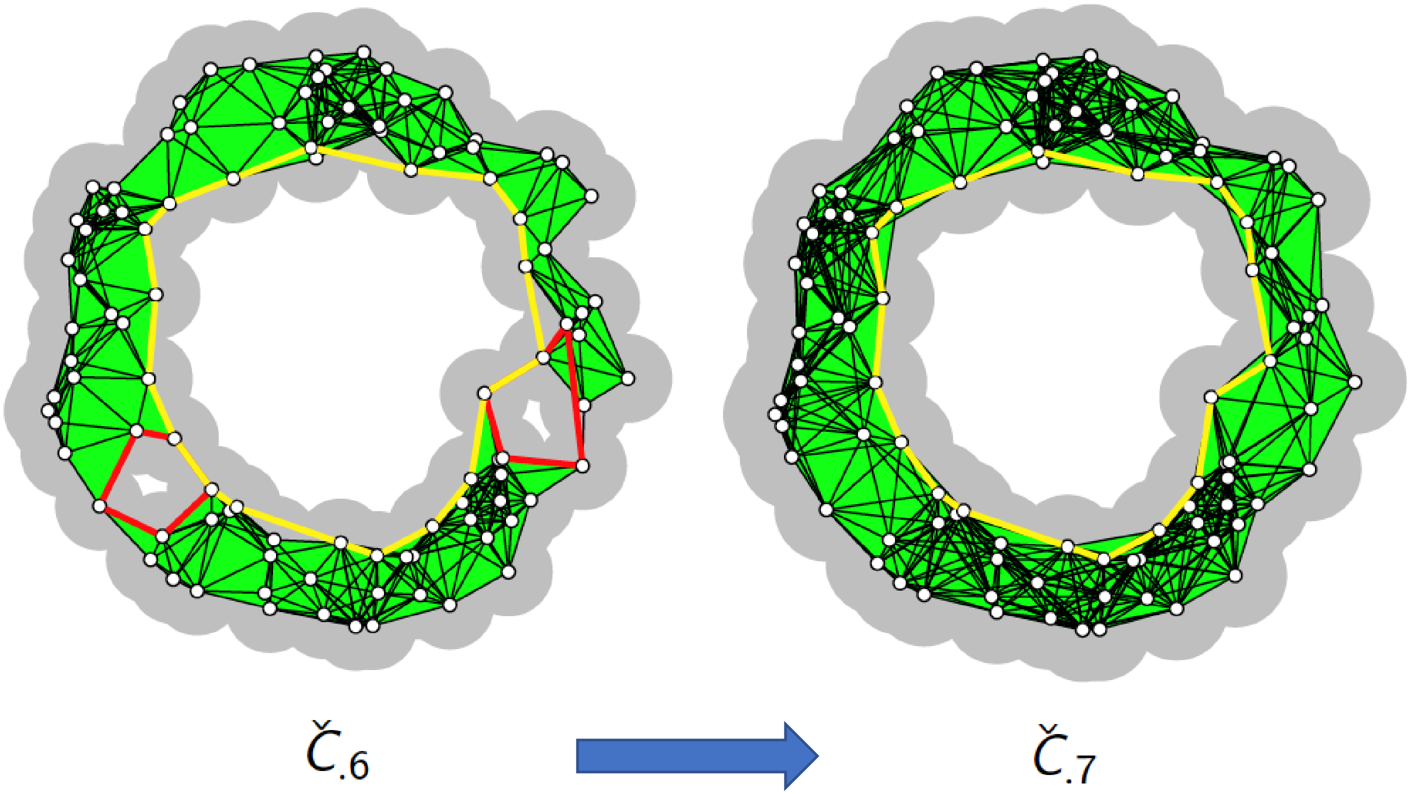
Take the linear extension of the boundary operator:

$$\partial_d([v_0, \ldots v_d]) = \sum_{i=0}^{d} (-1)^i [v_0 \ldots, \hat{v}_i, \ldots v_d]$$

Fact: $\partial_{d-1} \circ \partial_d \equiv 0 \Rightarrow \mathrm{Im}\, \partial_d \subseteq \ker \partial_{d-1}$

Definition: $H_d(K) = \ker \partial_d / \mathrm{Im}\, \partial_{d+1}$

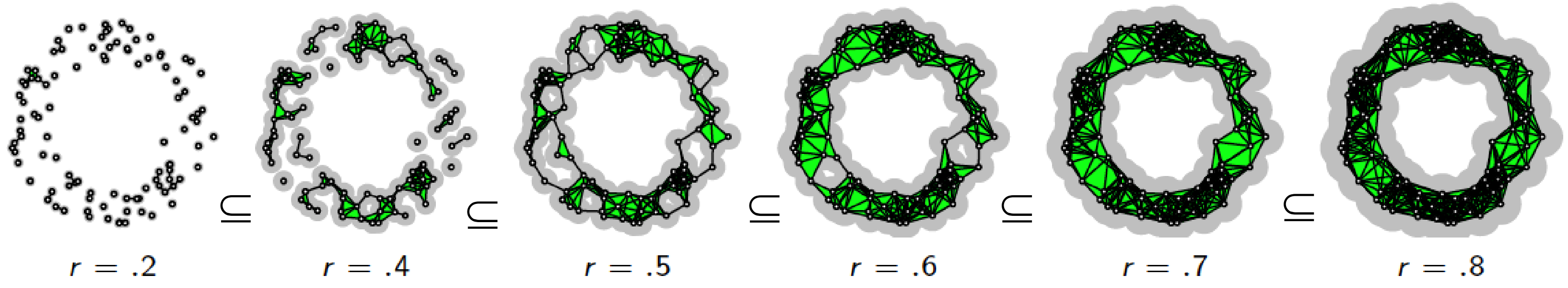
We Can Track Topological Features in a Filtration



Functorality allows us to systematically track holes over time!

The inclusion map among the complexes translates to a homomorphism between the homology groups

Persistent Homology is Functorial Homology

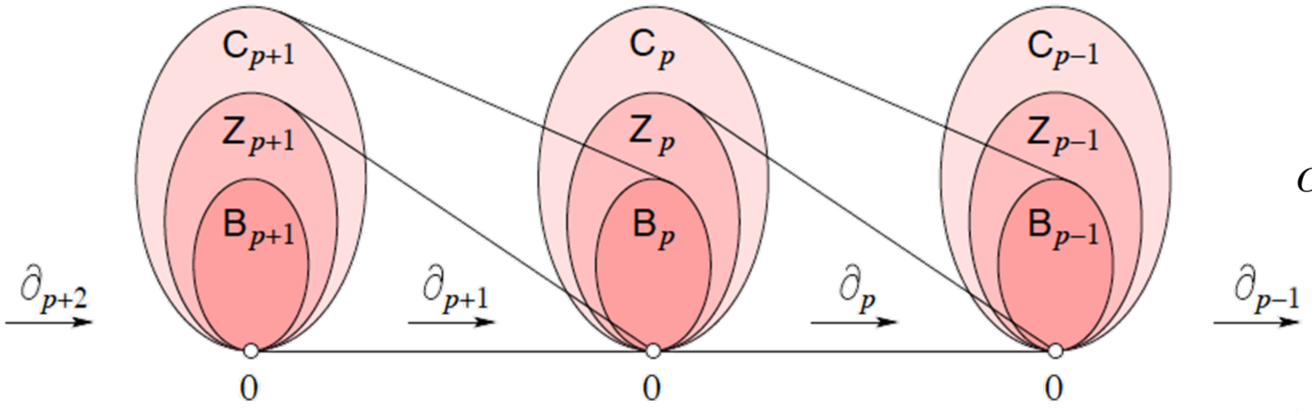


$$H_d(\check{C}_*) = \bigoplus_{\epsilon} H_d(\check{C}_\epsilon)$$

Homology of the entire filtration

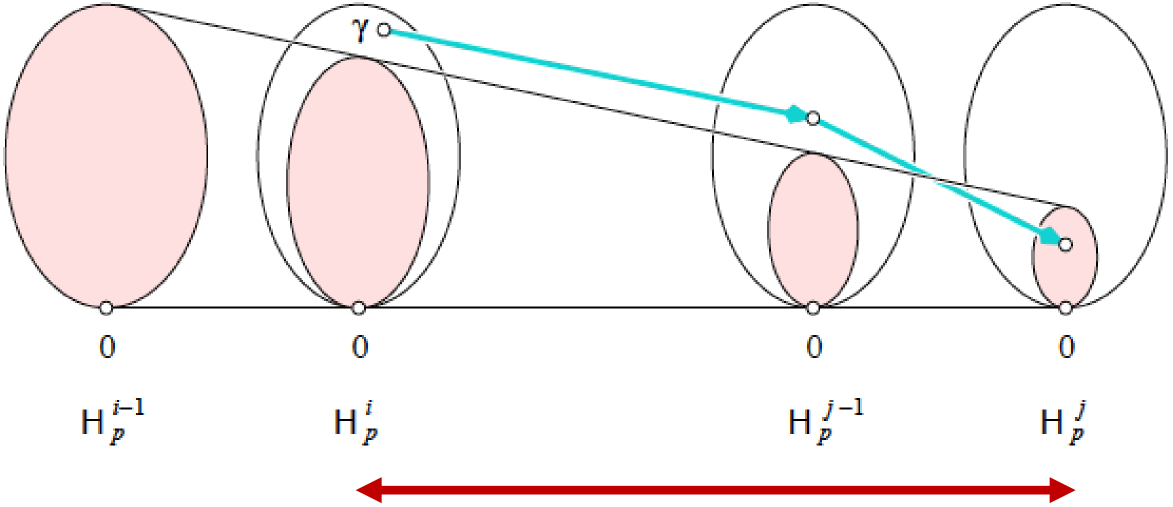
Homomorphisms at the homology level allow us to track homology classes – i.e., topological features

Persistence of Homology Classes



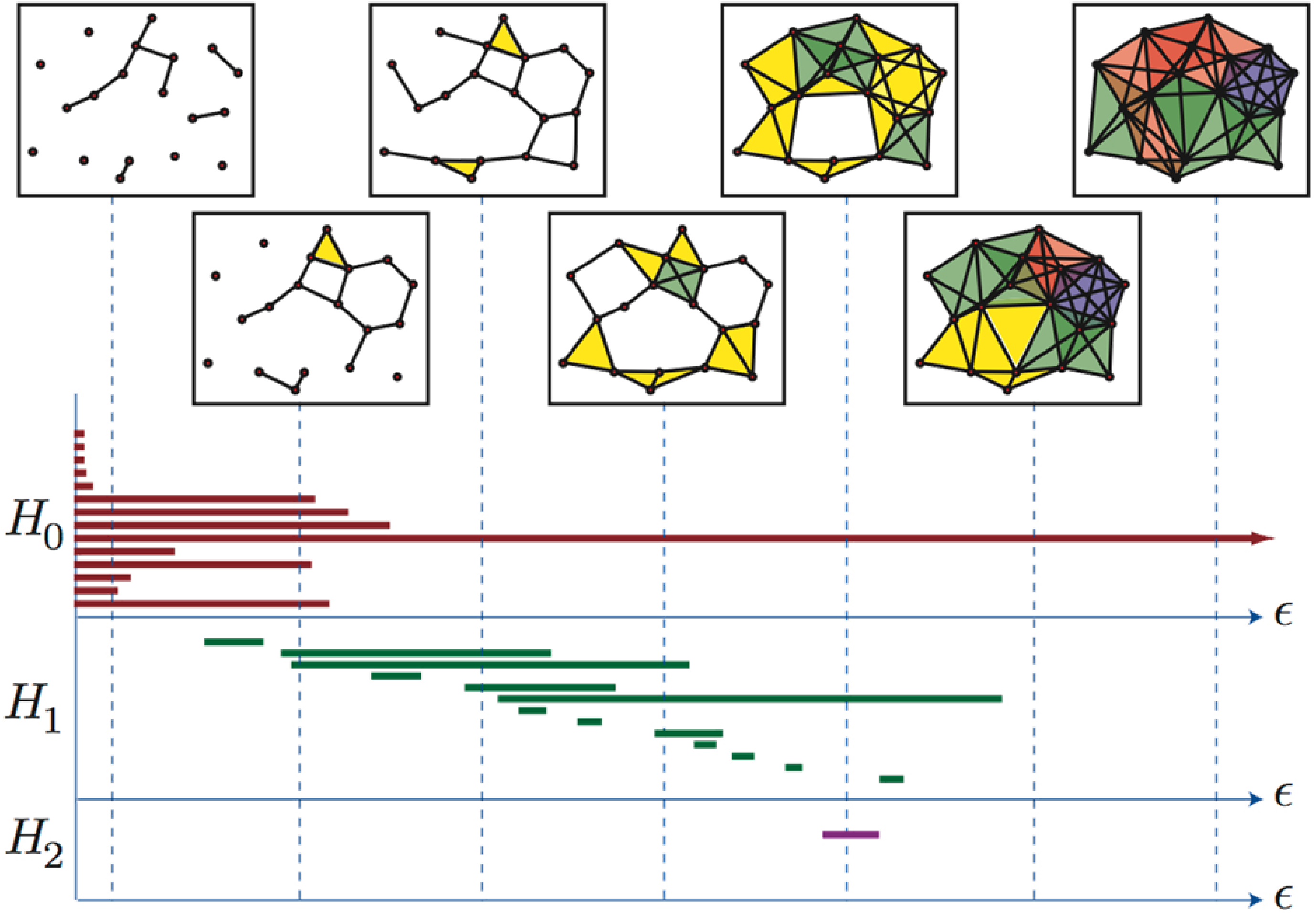
$$c_p = z_p + b_{p-1}$$

$$G/\ker(f) \equiv \mathrm{im}(f)$$



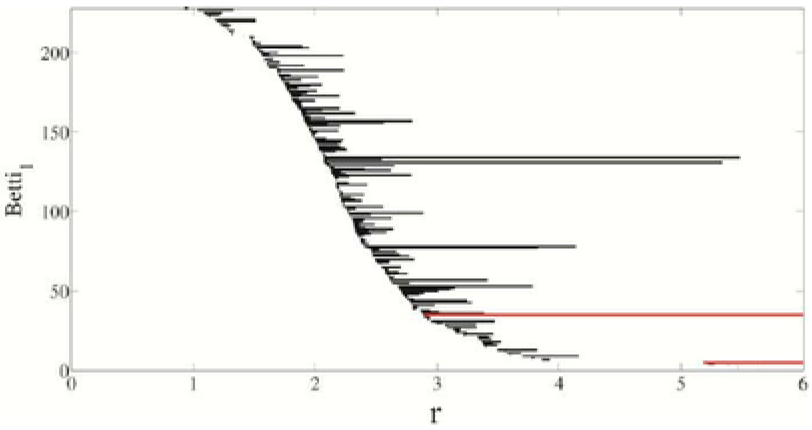
Lifetime of topological feature γ

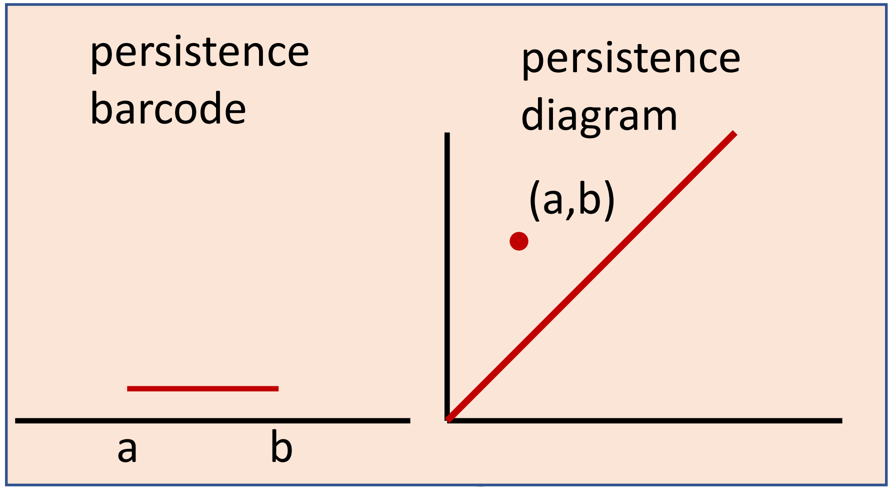
Barcodes are the Lifetimes of Topological Features



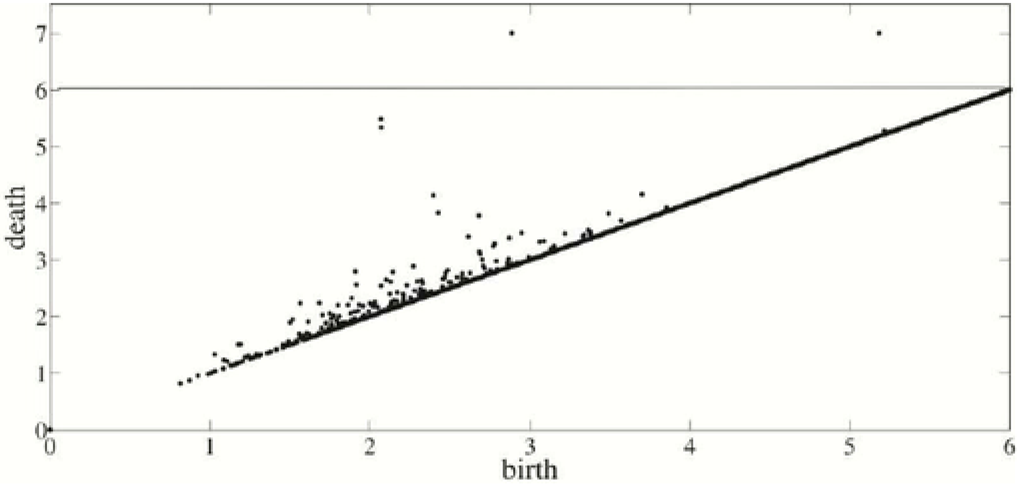
Barcodes are the output of persistent homology

Another View: Persistence Diagrams





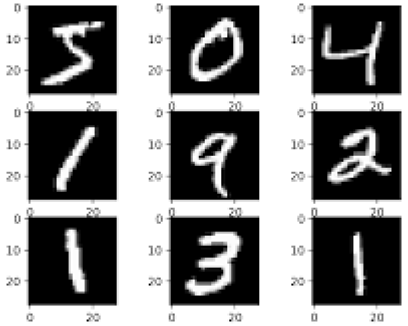
long barcodes = points away from the diagonal = robust features

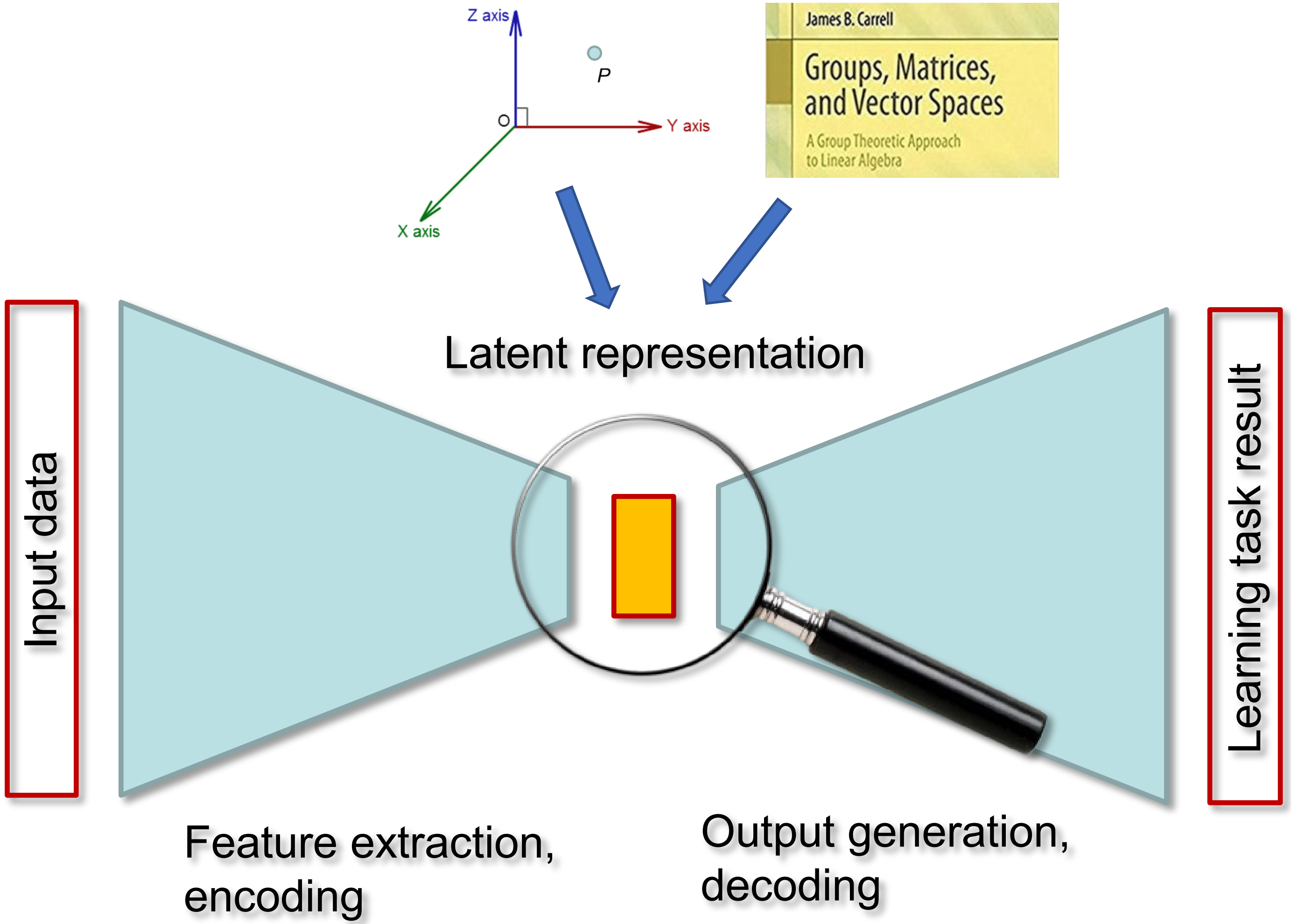


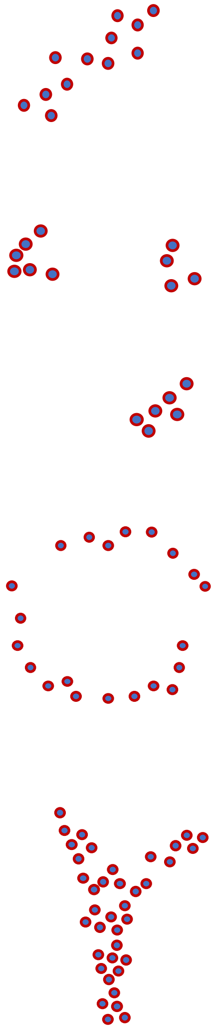
Short barcodes = points near the diagonal = noise

Map 1-D intervals to points in 2-D

Summary: Algebraic Descriptors for Data







That’s All