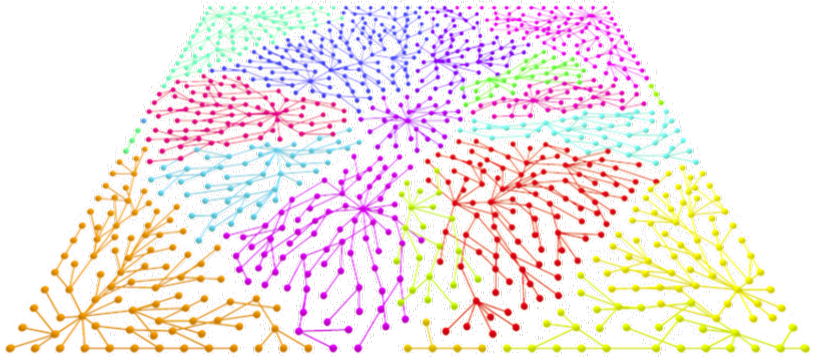
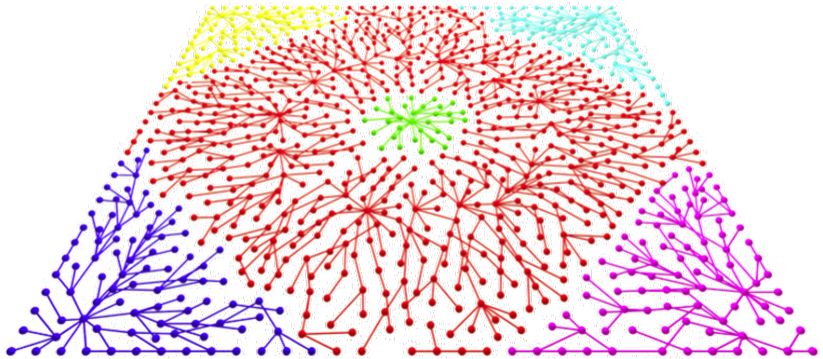
CS233, CME251: Geometric and Topological Data Analysis

Leonidas Guibas
Computer Science Department
Stanford University

Lecture 10
26 April 2021

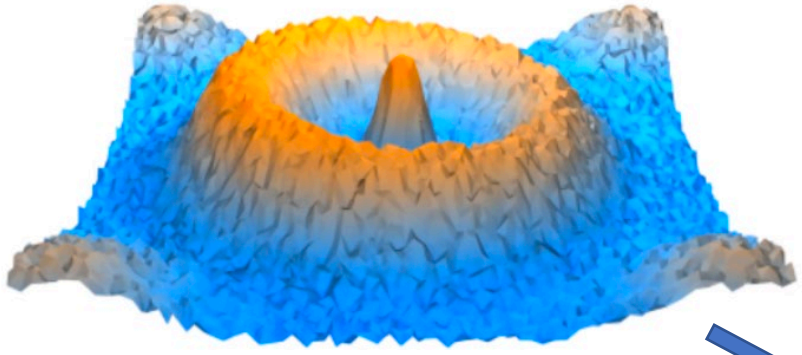


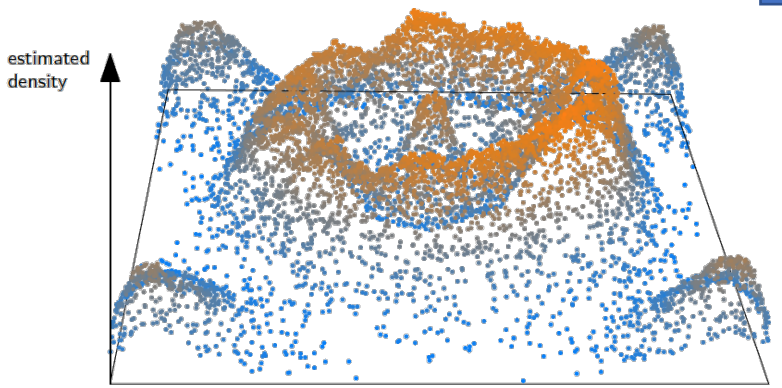


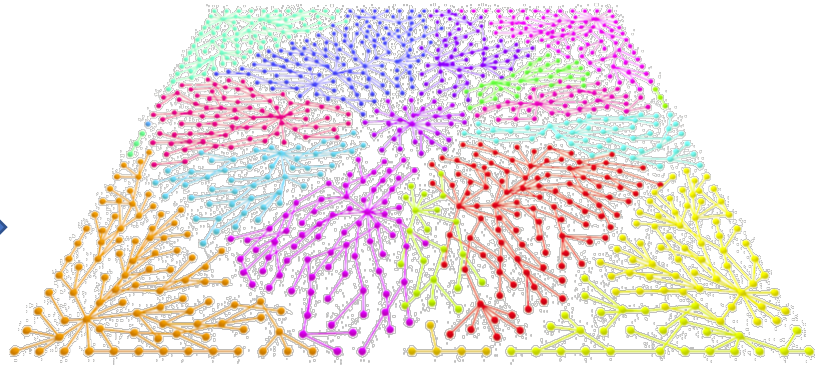
Last Time: Persistent Homology and Applications

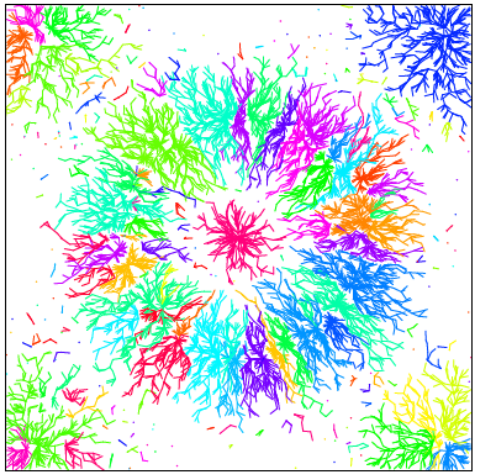
Scalar Field Analysis

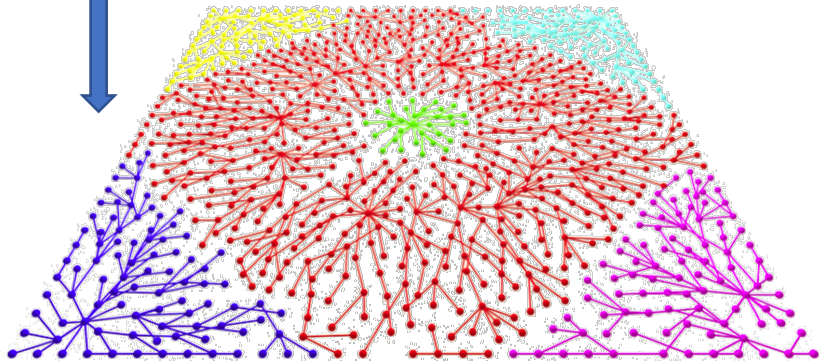
Chazal, Oudot, Skraba, Guibas *Persistence-Based Clustering in Riemannian Manifolds*





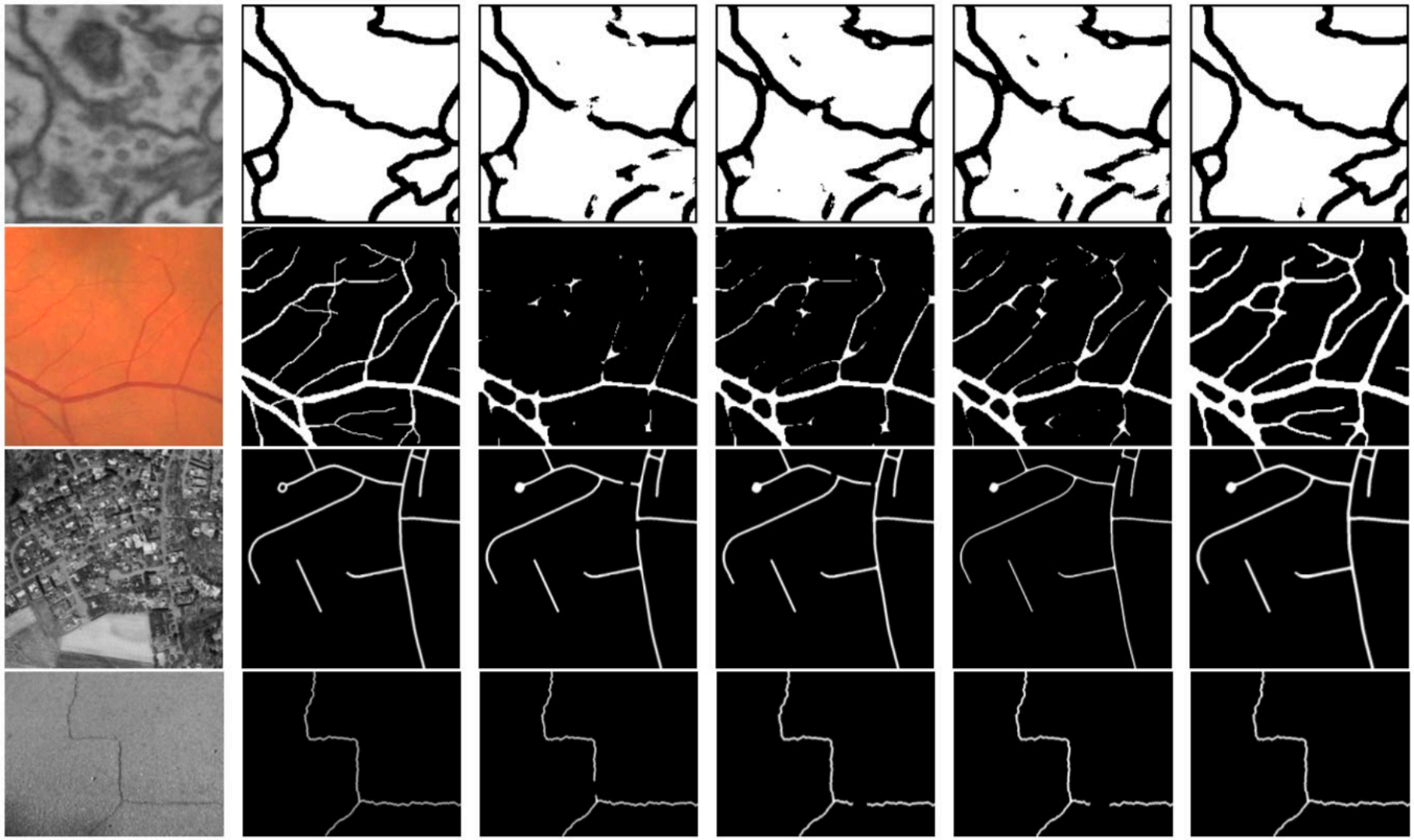






Topology for Image Segmentation

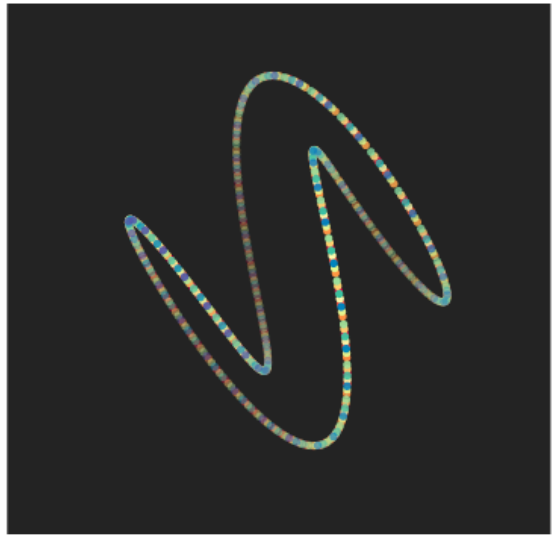
Hu, Fuxin, Samaras, Chen. *Topology-preserving deep image segmentation. NeuRIPS 2019*



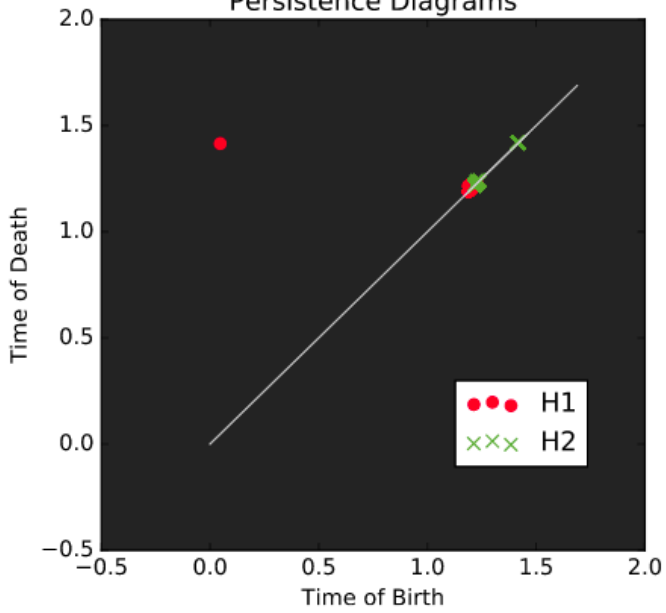
Topology for Dynamical Systems

Chris Tralie and Jose Perea, *(Quasi)Periodicity Quantification in Video Data, Using Topology*

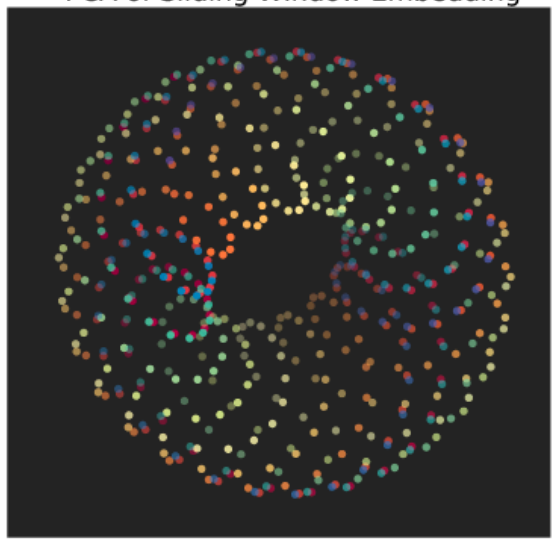
PCA of Sliding Window Embedding



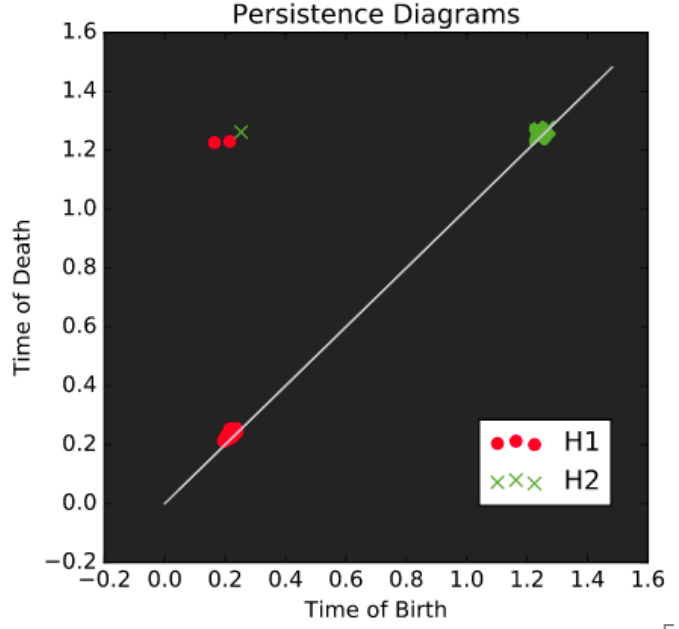
Persistence Diagrams



PCA of Sliding Window Embedding



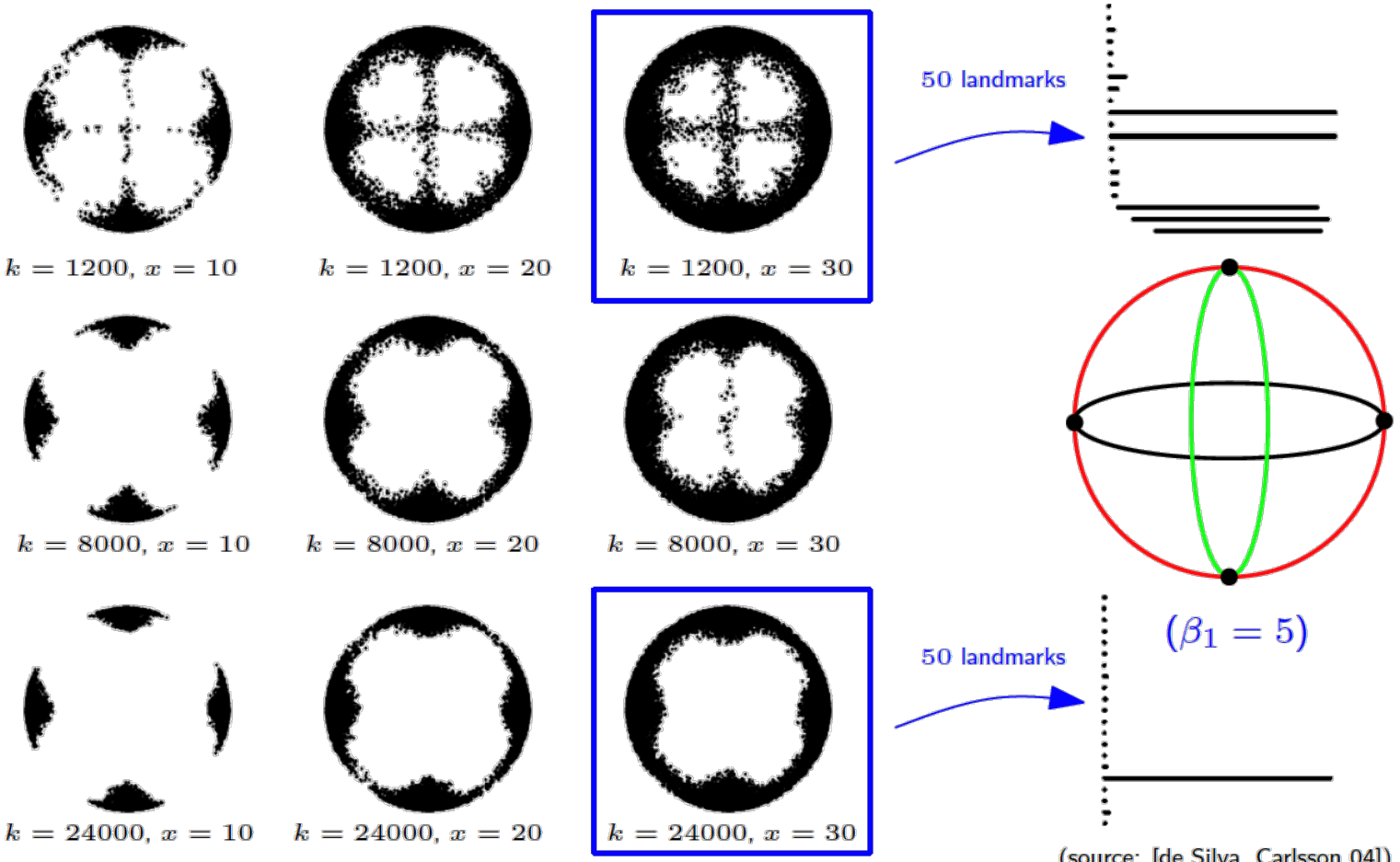
Persistence Diagrams



The Space of Natural Images

Carlsson, Ishkanov, de Silva, Zomorodian, *On the local behavior of spaces of natural images*

Preprocessing: - select bottom $x\%$ of data points according to k-NN distance
- sample 5000 points uniformly at random from filtered point set



(source: [de Silva, Carlsson 04])

Adding More Geometry: Tangent Complexes

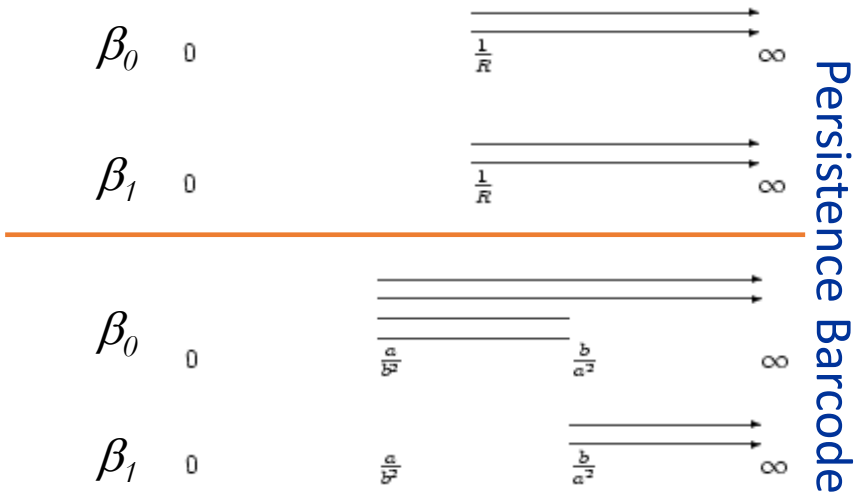
Collins, Zomorodian, Carlsson, Guibas, *A barcode shape descriptor for curve point cloud data*, CaG'04

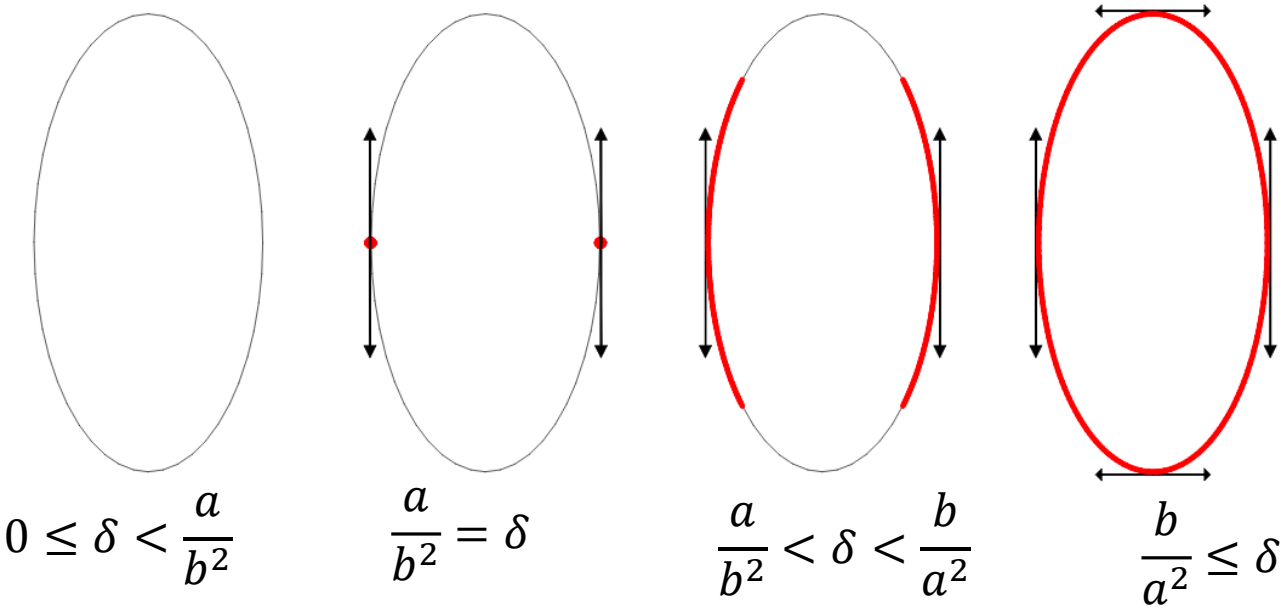
Circle of radius R: Tangent complex is $S^1 \times S^0$, as there are two tangent directions at each point.

- T^{filt} is empty until $1/R$, then full complex enters at once

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

- evolves through four stages: points at *lower* curvature appear earlier.

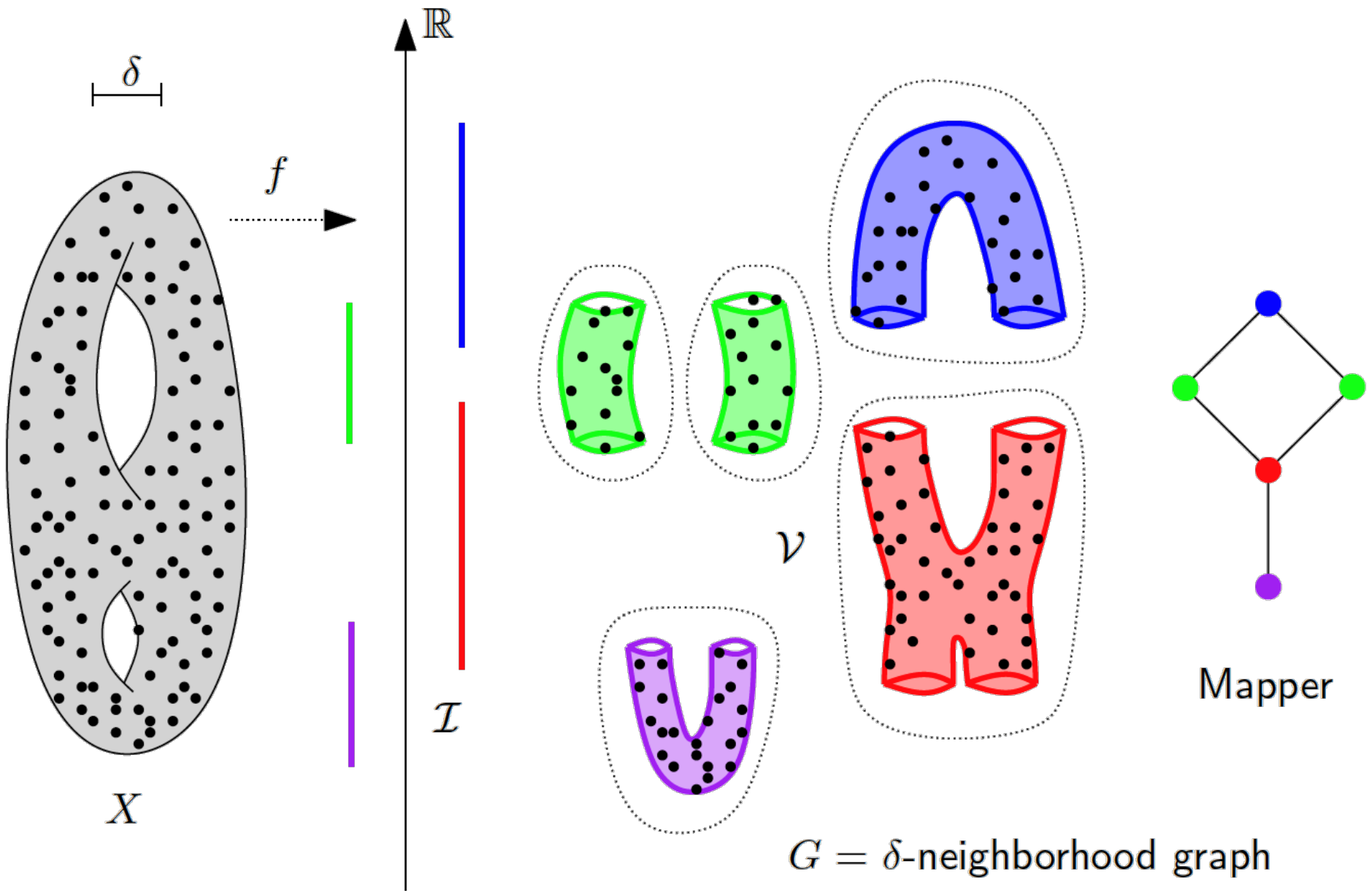




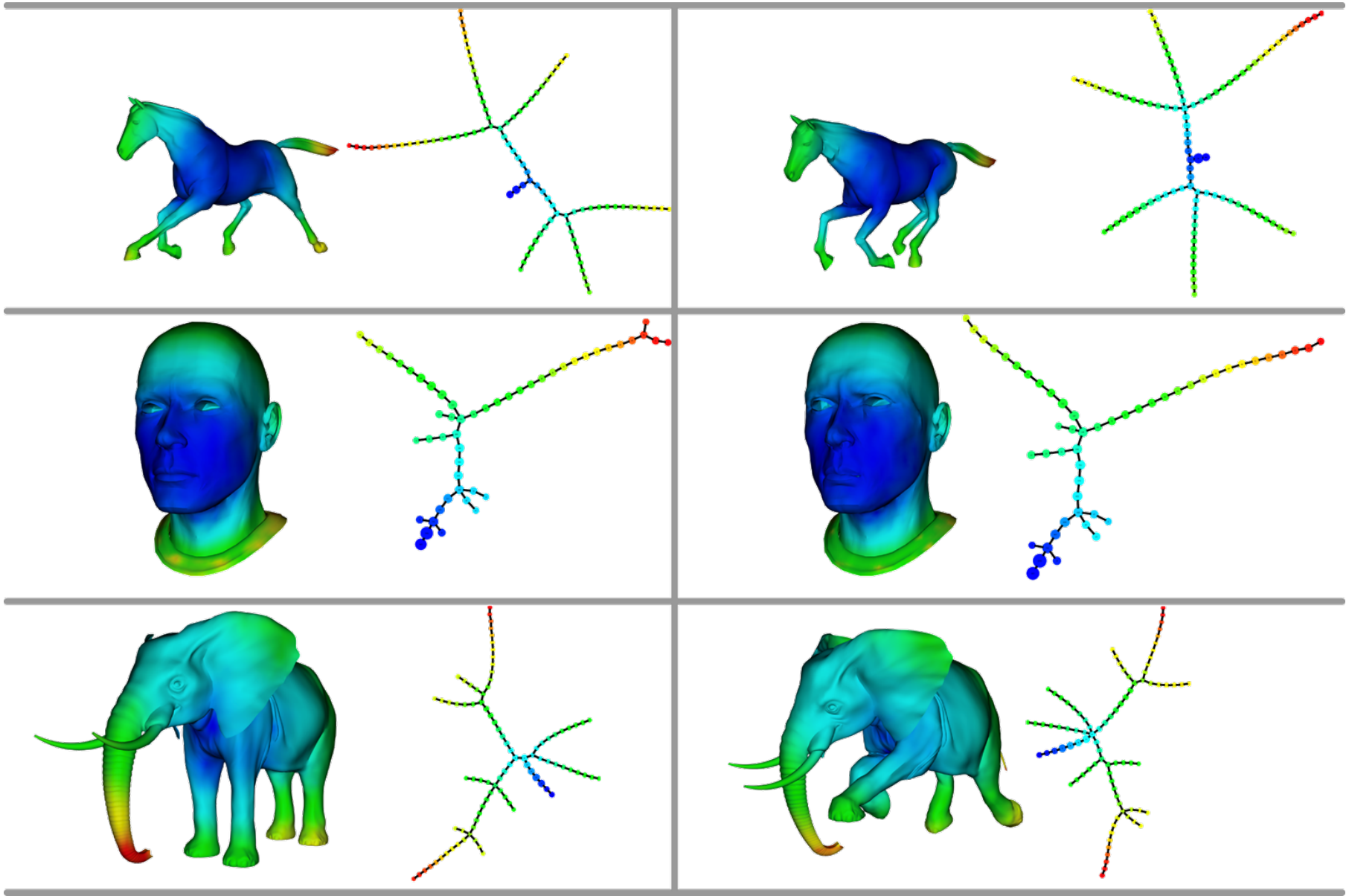
$0 \le \delta < \frac{a}{b^2}$ $\quad \frac{a}{b^2} = \delta$ $\quad \frac{a}{b^2} < \delta < \frac{b}{a^2}$ $\quad \frac{b}{a^2} \le \delta$

Mapper Algorithm

Singh, Mémoli, Carlsson - *Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition*

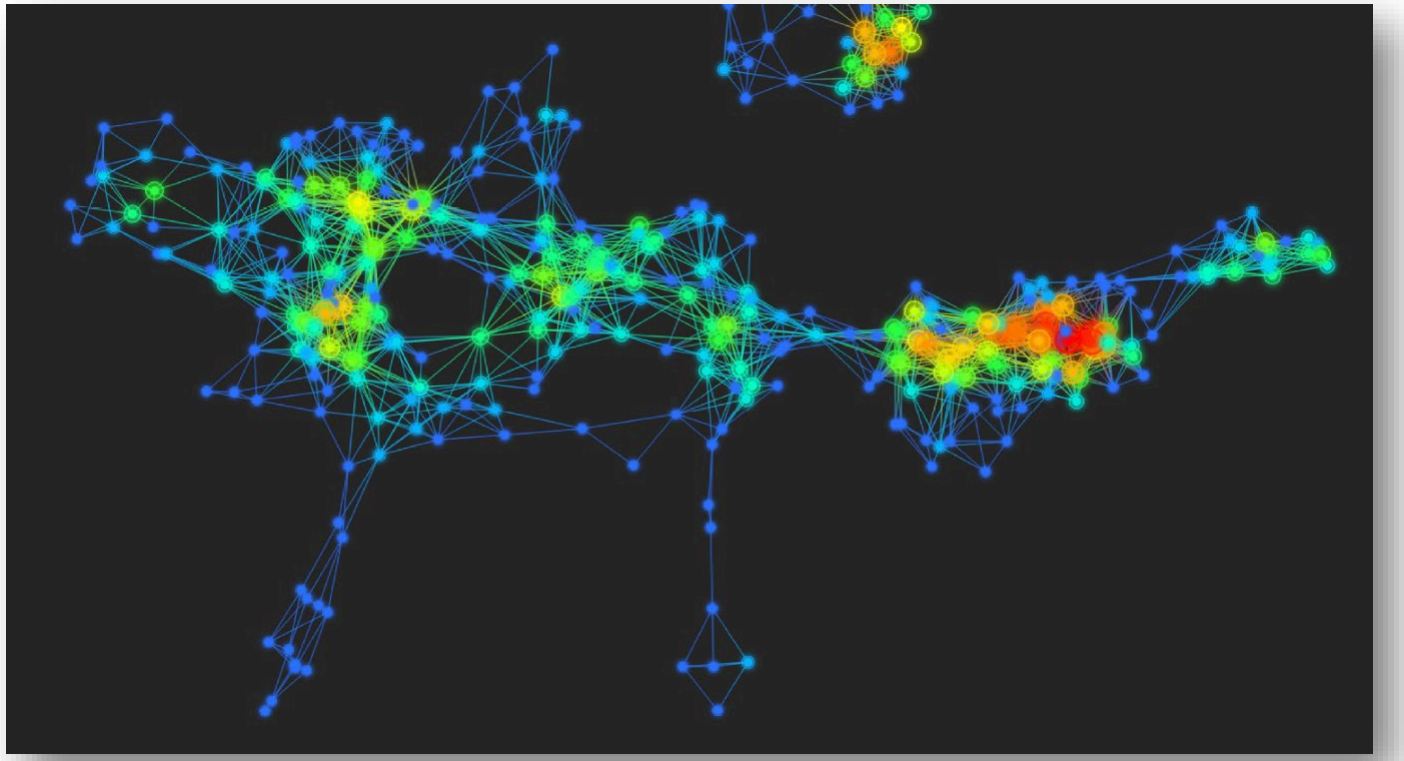


Filter: Centrality Filter Under Deformation

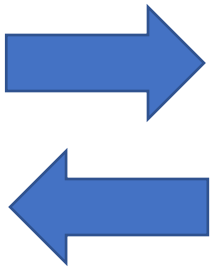


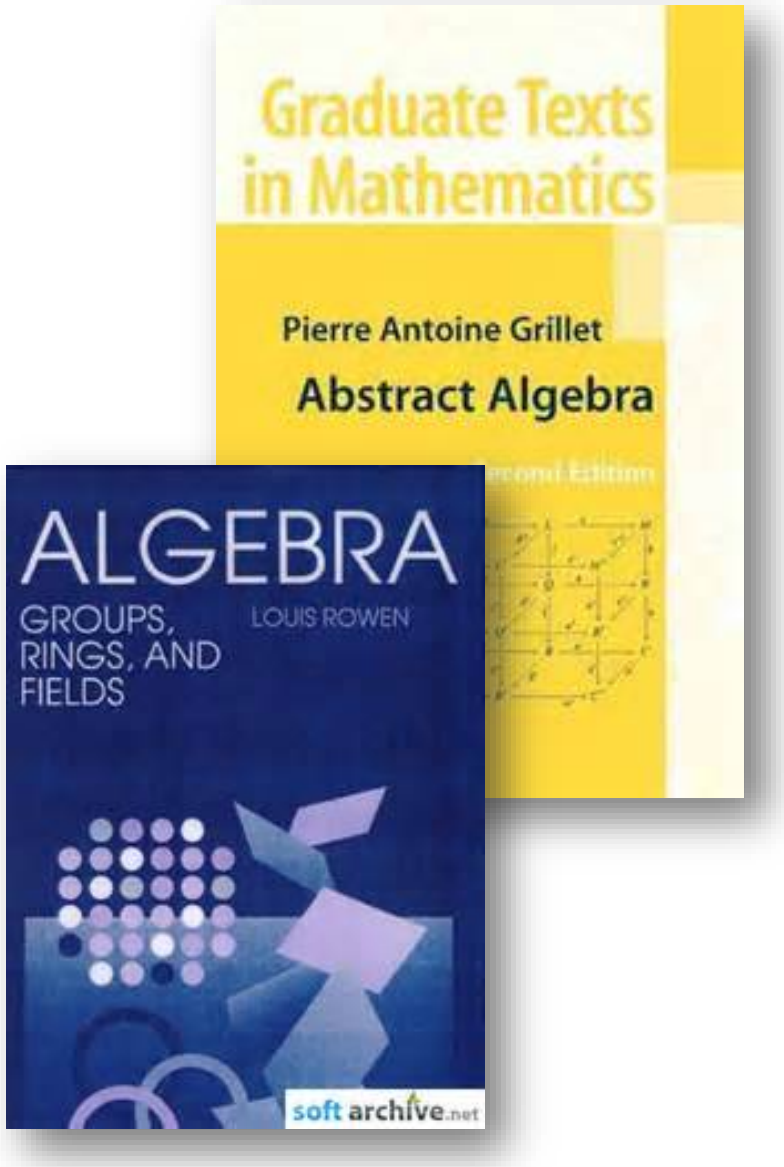
Many, Many Filter Choices

“It is useful to think of Mapper as a camera, with lens adjustments and other settings. A different filter function may generate a network with a different shape, thus allowing one to explore the data from a different mathematical perspective.”



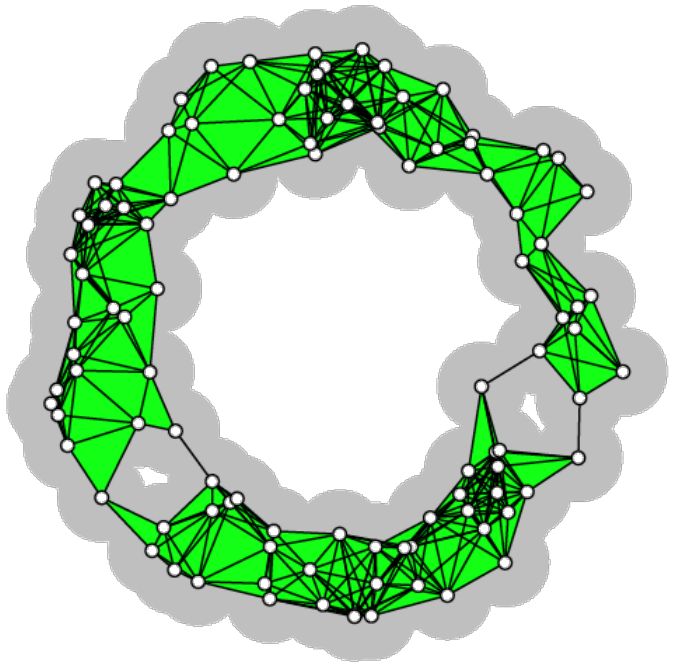
What to Remember: From Data to Algebraic Objects





Algebraic entities as data descriptors

Čech Complex



$\check{C}_{.6}$

The Čech complex $\check{C}_\epsilon$ encodes the intersection pattern of M_ϵ: Encode:

- Points as *vertices* (0-cells)
- Pairwise intersections as *edges* (1-cells)
- Threeway intersections as *triangles* (2-cells)
- k-way intersections as *(k+1)-cells*

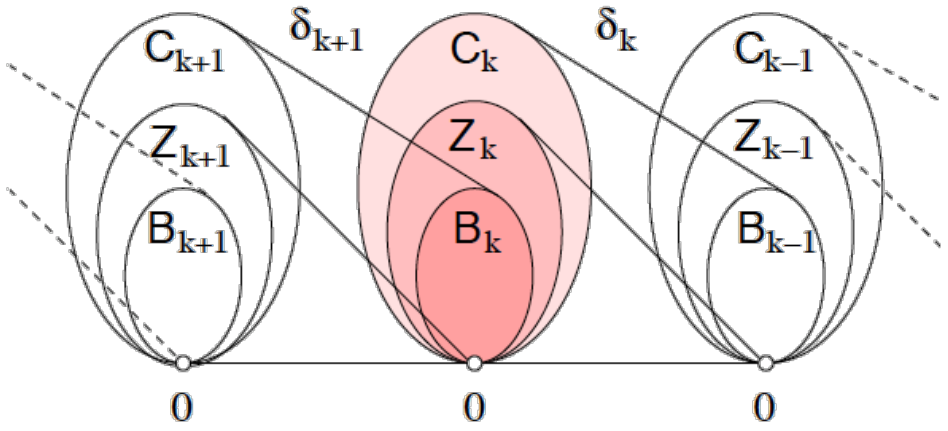
Can be hard to compute …

Simplicial Homology

- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k/\mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

- If $z_1 = z_2 + \mathsf{B}_k, z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are homologous
- $z_1 \sim z_2$.

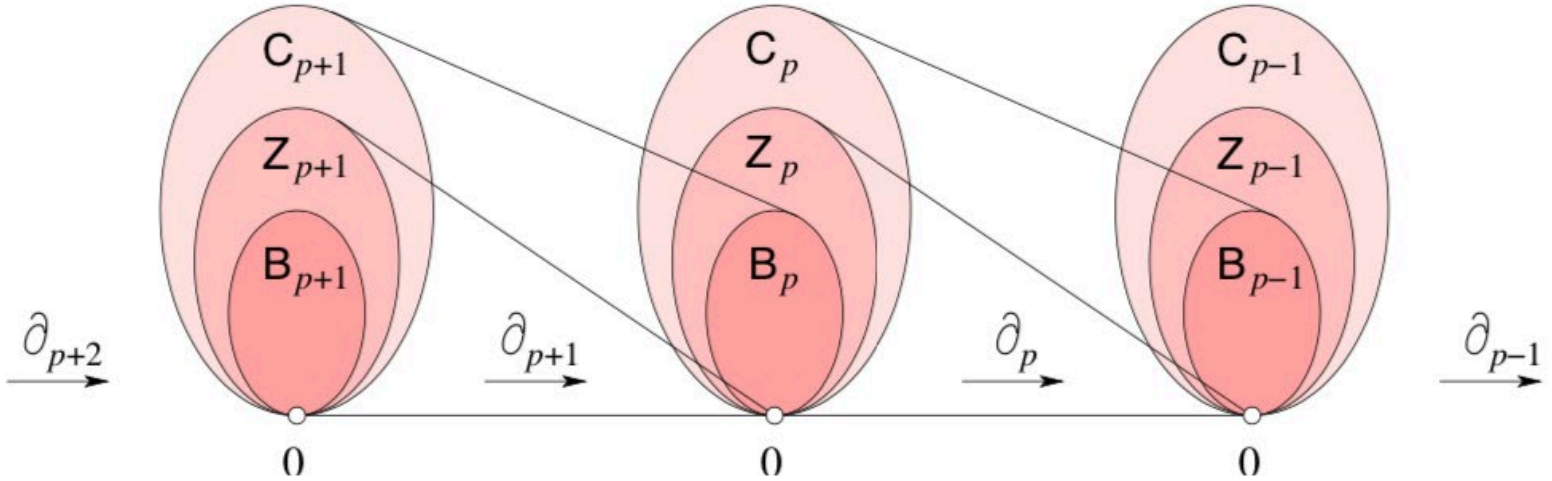


Recall: the Chain Complex

- We now have a sequence of vector spaces and linear maps:

$$\dots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \dots$$

- Notation: $Z_p := \ker \partial_p = \{v \in C_p : \partial_p(v) = 0\}$ the p-cycles, $B_p := im\ \partial_{p+1} \subseteq C_p$ the p-boundaries.



pth homology vector space:

$$H_p := \frac{\ker \partial_p}{im\ \partial_{p+1}}$$

Intuition: p-dimensional connectivity

Reduction Algorithm for Homology*

*Cohen-Steiner, Edelsbrunner, Morozov - **Vines and Vineyards by Updating Persistence in Linear Time**

To get H_p, we need: (1) a basis for $\ker \partial_p$, and (2) a sub-basis for $im\ \partial_{p+1}$.

	ab	ca	bc	cd	de	eb	abc
a	1	1	0	0	0	0	0
b	1	0	1	0	0	1	0
c	0	1	1	1	0	0	0
d	0	0	0	1	1	0	0
e	0	0	0	0	1	1	0
ab	0	0	0	0	0	0	1
ca	0	0	0	0	0	0	1
bc	0	0	0	0	0	0	1
cd	0	0	0	0	0	0	0
de	0	0	0	0	0	0	0
eb	0	0	0	0	0	0	0

R

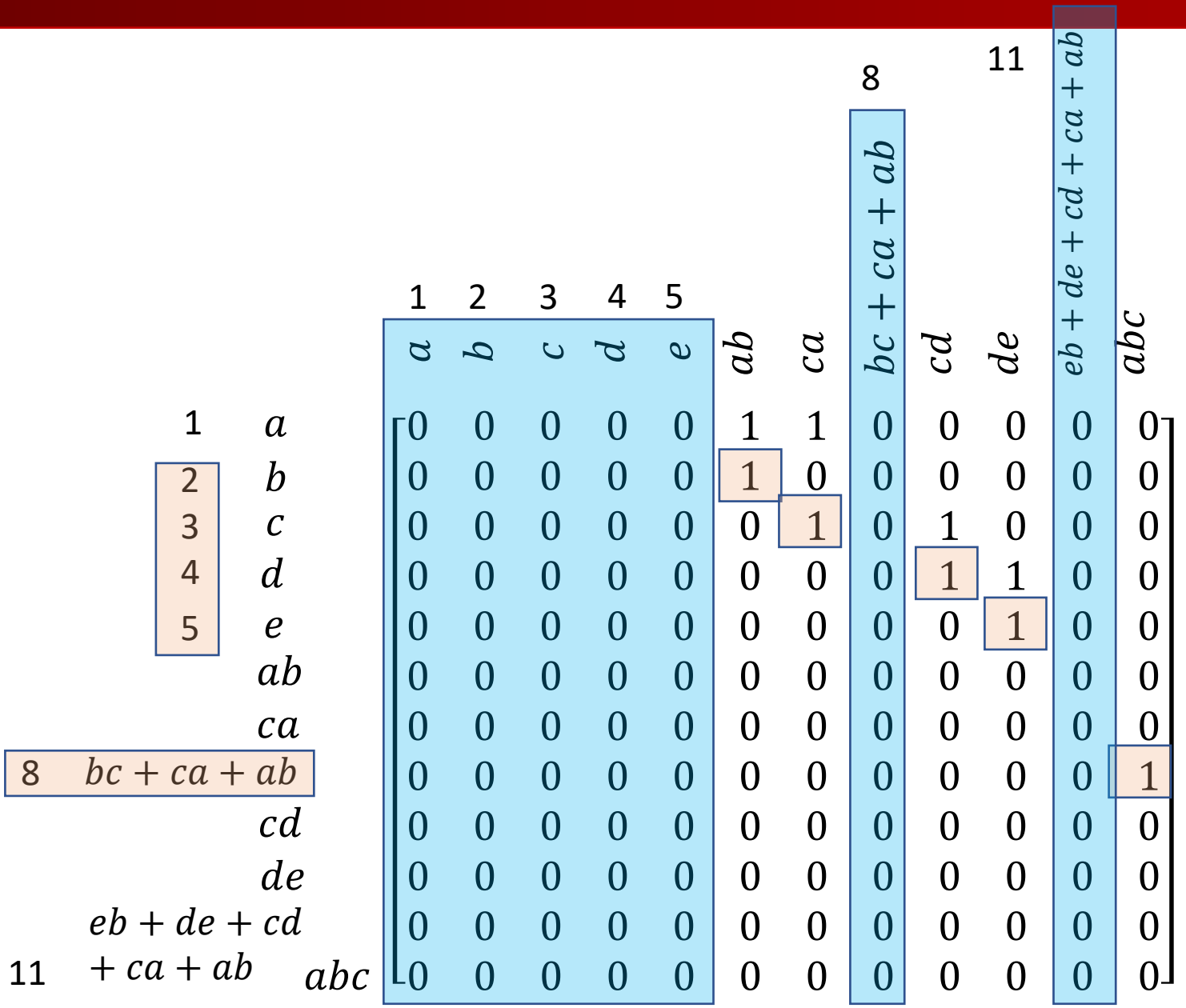
Notation: $low_R(j)$= row index of bottom 1 in column j of matrix R

Algorithm:

Initialize boundary information in incidence matrix D
Initialize $R \coloneqq D$

For column $j = 1, 2, \ldots, n$
  while $\exists j' < j, low_R(j') = low_R(j)$ do
    add column j' to column j

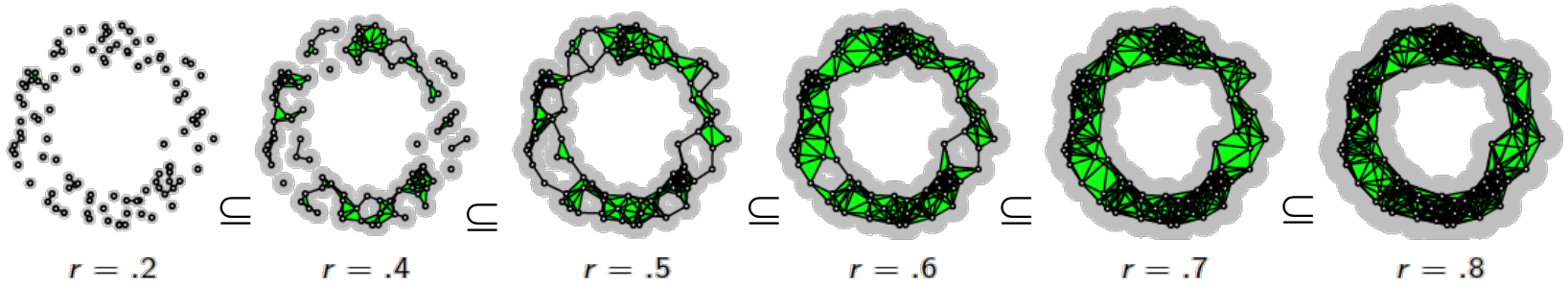
Reduction Algorithm for Homology



Full matrix at termination. Observe:

- (Proposition) No two columns with the same low_R value
- Zero columns can be read off immediately to get $\ker \partial_p$
- (Proposition) Indices i such that column i is zero but $i \neq low_R(j)$ for any j correspond exactly to the cycles that are not boundaries!
- Thus we get a basis for homology in each dimension.

Persistent Homology is Functorial Homology



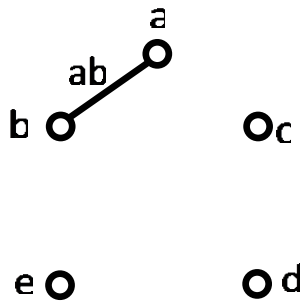
$$H_d(\check{C}_*) = \bigoplus_{\epsilon} H_d(\check{C}_\epsilon)$$

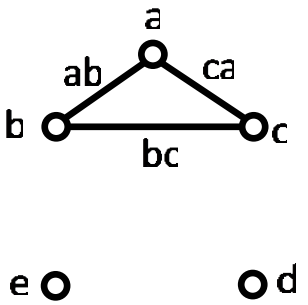
Homology of the entire filtration

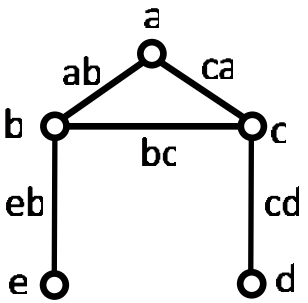
Homomorphisms at the homology level allow us
to track homology classes – i.e., topological features

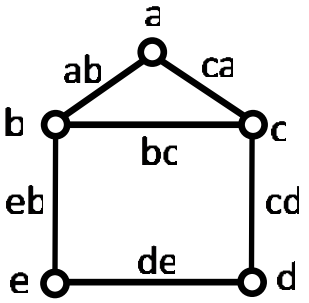
Back to the House Example

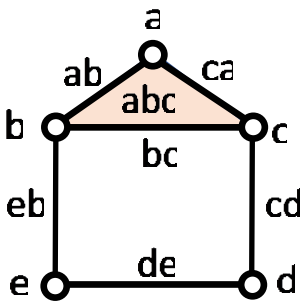
		1	2	3	4	5			8		11	
		a	b	c	d	e	ab	ca	$bc+ca+ab$	cd	eb	$de+eb+cd+ca+ab$	abc
1	a	0	0	0	0	0	1	1	0	0	0	0	0
2	b	0	0	0	0	0	1	0	0	0	1	0	0
3	c	0	0	0	0	0	0	1	0	1	0	0	0
4	d	0	0	0	0	0	0	0	0	1	0	0	0
5	e	0	0	0	0	0	0	0	0	0	1	0	0
	ab	0	0	0	0	0	0	0	0	0	0	0	0
	ca	0	0	0	0	0	0	0	0	0	0	0	0
8	$bc+ca+ab$	0	0	0	0	0	0	0	0	0	0	0	1
	cd	0	0	0	0	0	0	0	0	0	0	0	0
	eb	0	0	0	0	0	0	0	0	0	0	0	0
11	$de+eb+cd+ca+ab$	0	0	0	0	0	0	0	0	0	0	0	0
	abc	0	0	0	0	0	0	0	0	0	0	0	0











Simplices are “Creators” or “Destroyers”

- Every simplex either creates a homology class or destroys one
- Specifically, a k-simplex either creates a k-dimensional feature or destroys a $(k-1)$-dimensional feature

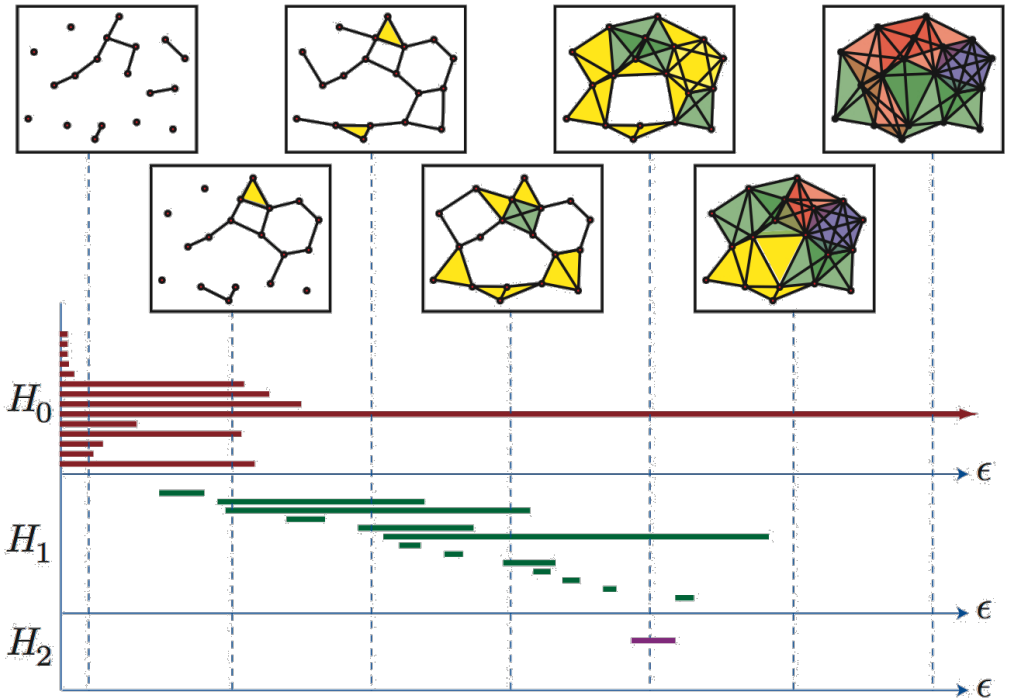
What to Remember: The Persistence Pipeline

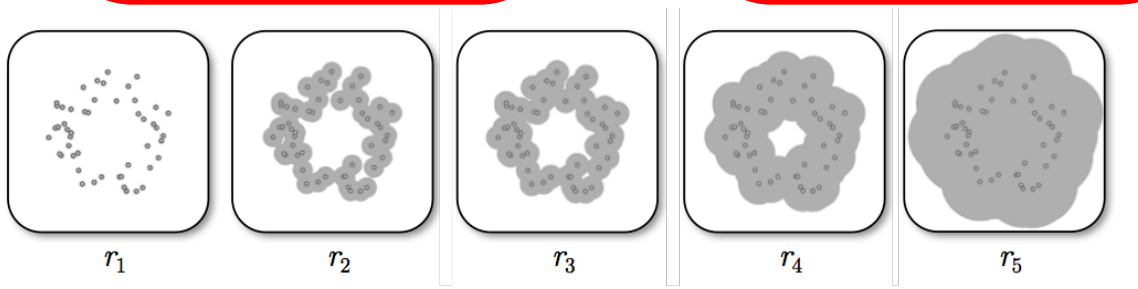
Dataset:
- **Euclidean point cloud data**
- **Metric spaces**
- **Triangulated shape**
- Graphs, matrices…

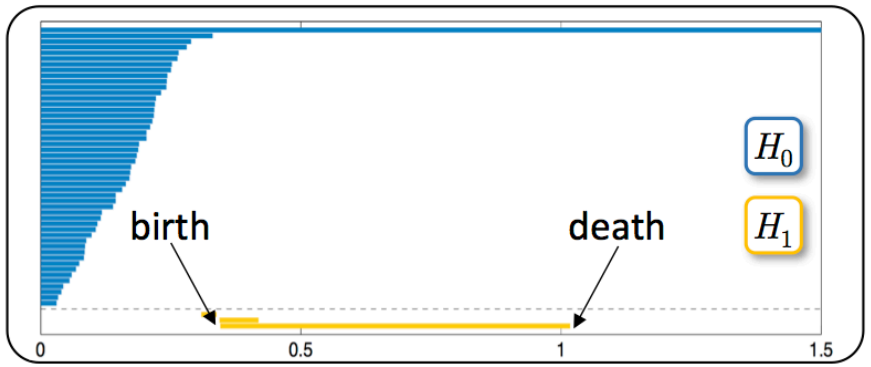
Filtered simplicial complex
- Possible to have both inclusions and deletions (dynamic data)

Vector spaces with linear maps: a **persistent vector space**

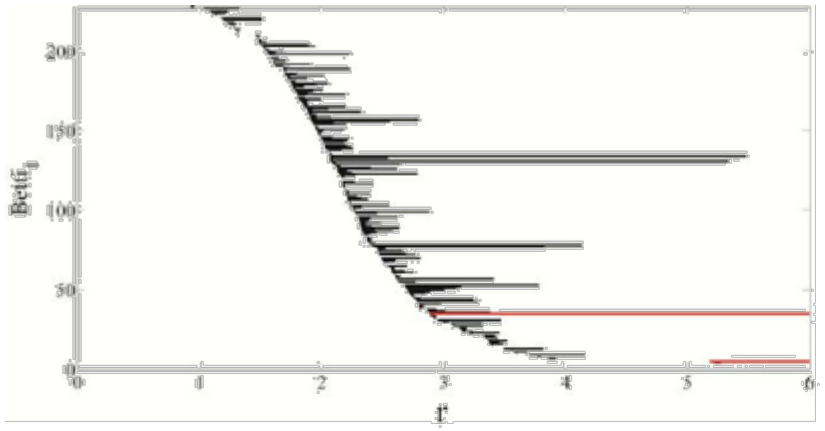
Visual summary: a **persistence diagram** or **persistence barcode**
- Persistence landscapes for ML-friendly output

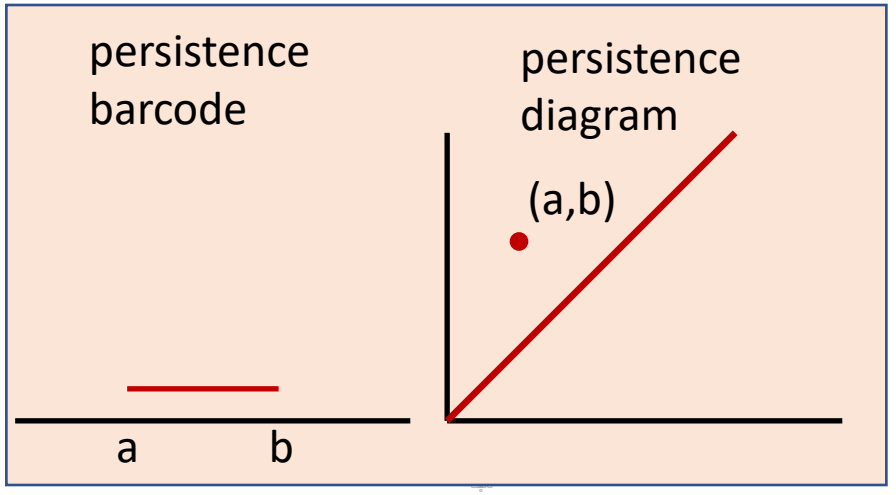




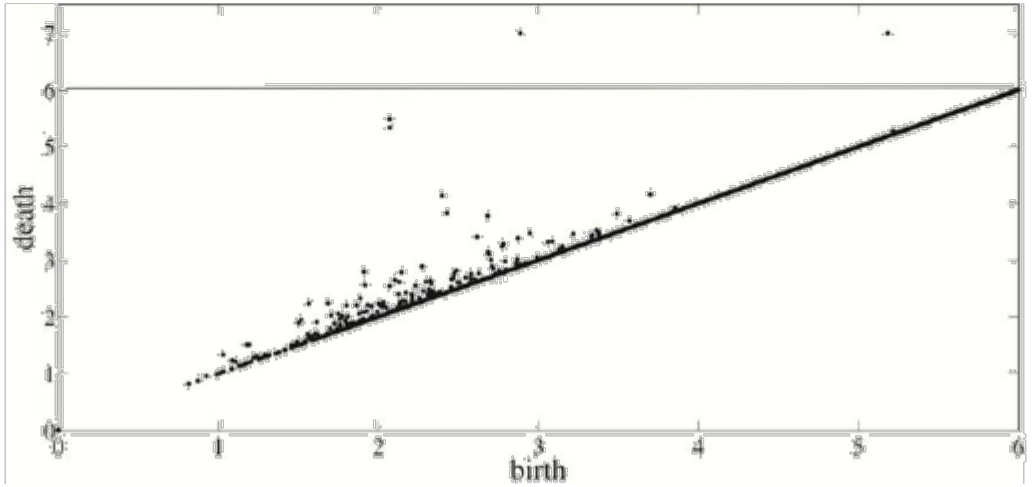


Another View: Persistence Diagrams





long barcodes = points away from the diagonal = robust features



Short barcodes = points near the diagonal = noise

Map 1-D intervals to points in 2-D

Midterm
Monday, May 10

Class Midterm

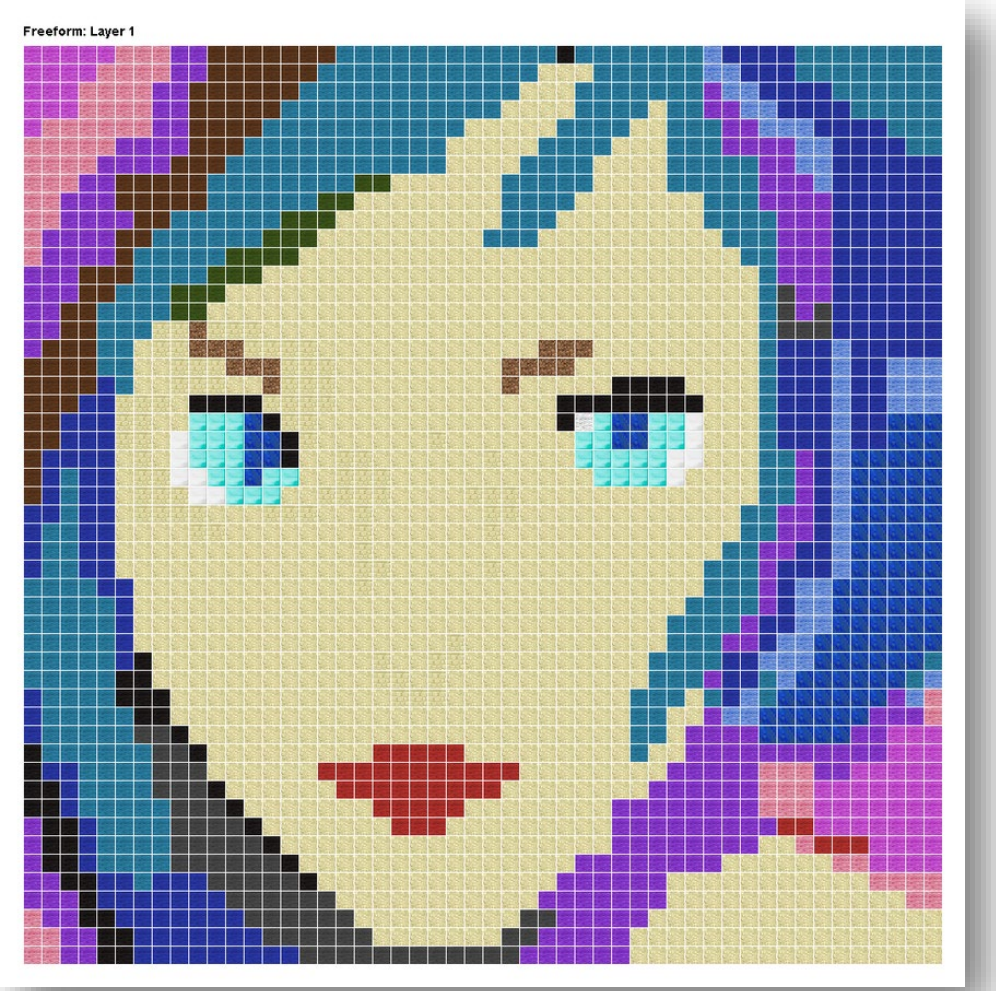
- The class midterm will be on **Monday, May 10**, during the regular class time. It may cover material relevant to any previous lecture.
- Unlike the homeworks, the exam will consist of a number of very short problems (that also have short answers) testing basic understanding of the concepts covered in class. All problems will have the same weight and there will be some choice (do X out of Y).
- In doing the exam, you may use any of the on-line class materials, as well as your own notes. Internet searches, however, are not allowed -- nor is any kind of human help or collaboration on the solutions.
- You may use the period of 3:50 -- 4:00 pm to get your solutions in digital form (e.g., by scanning paper, taking photos, etc...). Your must submit your solutions as a single PDF file via Gradescope, where the problem solutions appear in the same order as the problems in the exam.

Class Grading

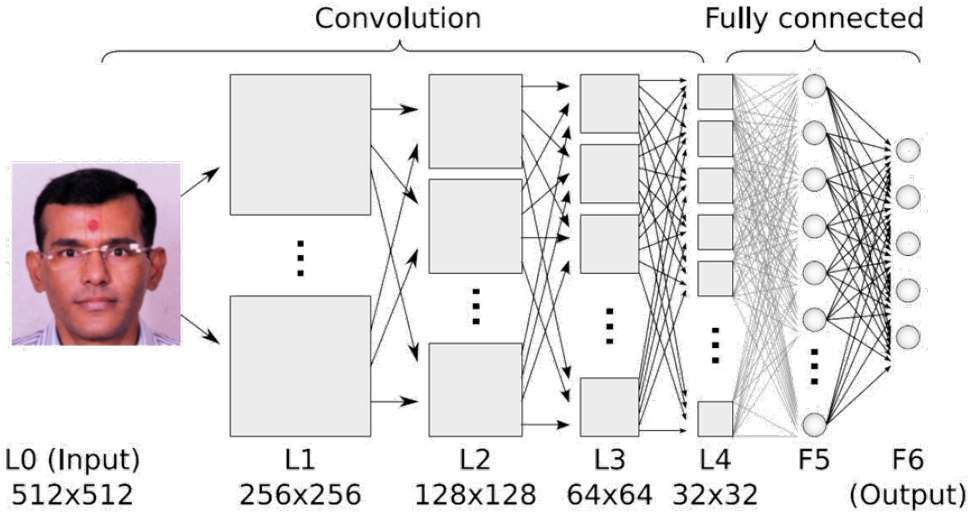
- The four homeworks and the midterm will count for 20% of the grade each.

Today:
Data with Internal Structure & Geometry Representations in 3D

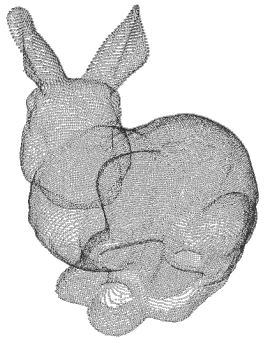
Images Have Canonical Representations



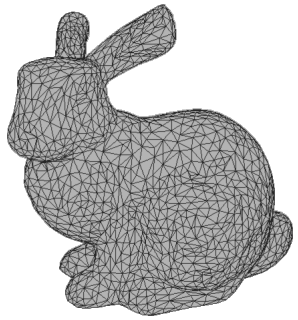
Regular representations
aid ML algorithms, e.g.
convolutional deep networks



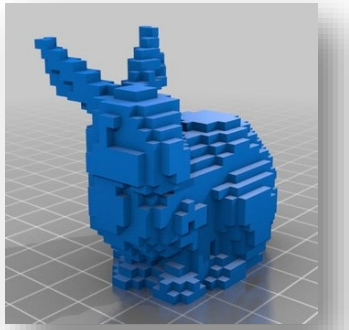
In 3D There is Representation Diversity



Point Cloud



Mesh

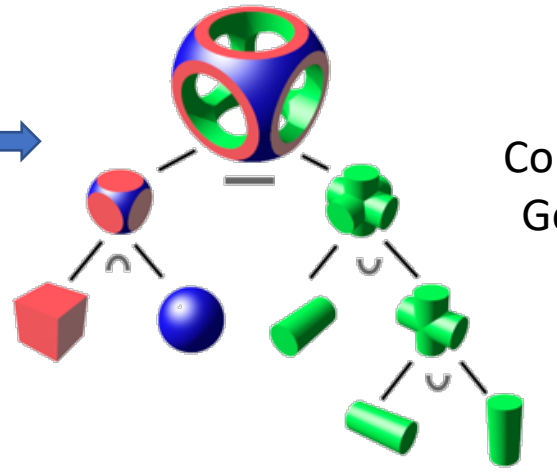


Voxels

Multiple Projected Views
RGB(D)

...

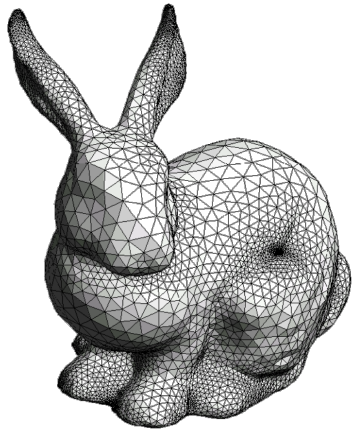
These are irregular representations

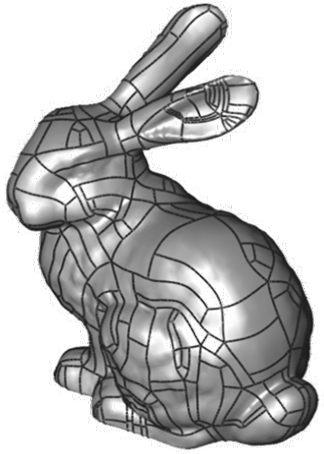


Constructive Solid Geometry (CSG)

3D: Boundary or Volumetric?

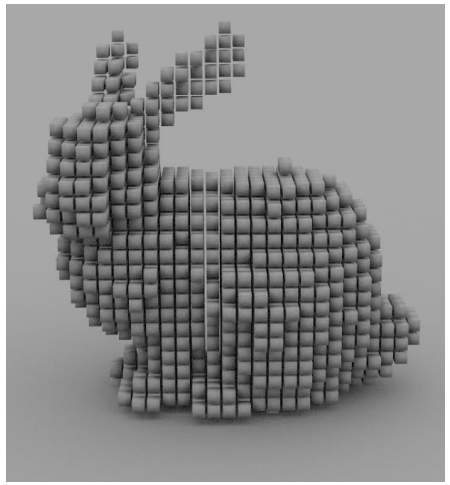
- B(oundary)-Reps

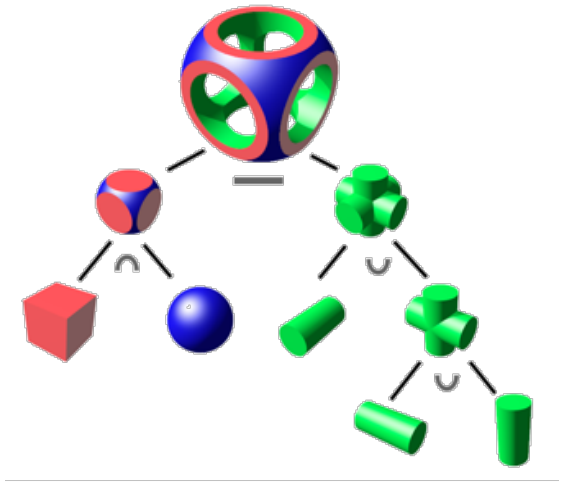




  - more efficient
  - closer to semantics, support local editing

- V(olume)-Reps





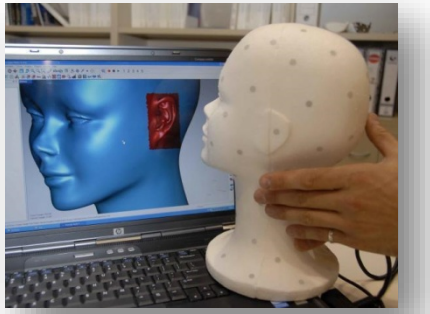
  - more regular
  - support unions and intersections

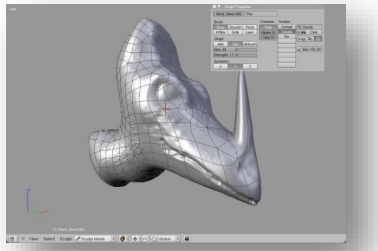
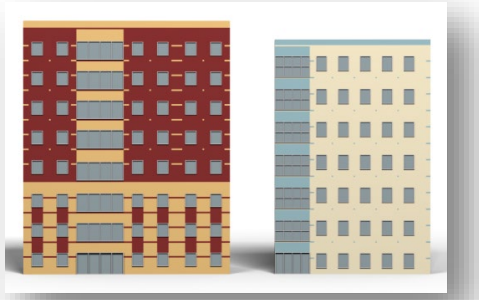
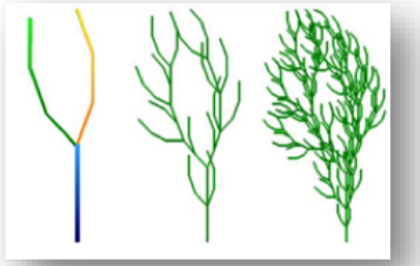
Decoding/Generation from Latent Vectors

Generator/Decoder: generating shapes from latent vectors via deep networks

Because 3D has Many Sources

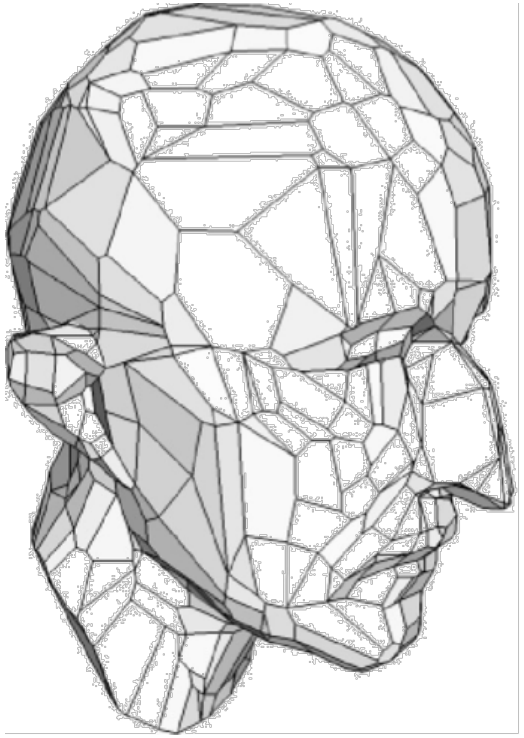
- Acquired real-world objects:
  - RGB or RGBD data, scanning
  - Points, meshes
- Modeling “by hand”:
  - Higher-level representations, amendable to modification, control
  - Parametric surfaces, subdivision surfaces, implicits
- Procedural modeling
  - Algorithms, grammars
  - Primitives, Polygons, application-dependent elements
- Neural generators

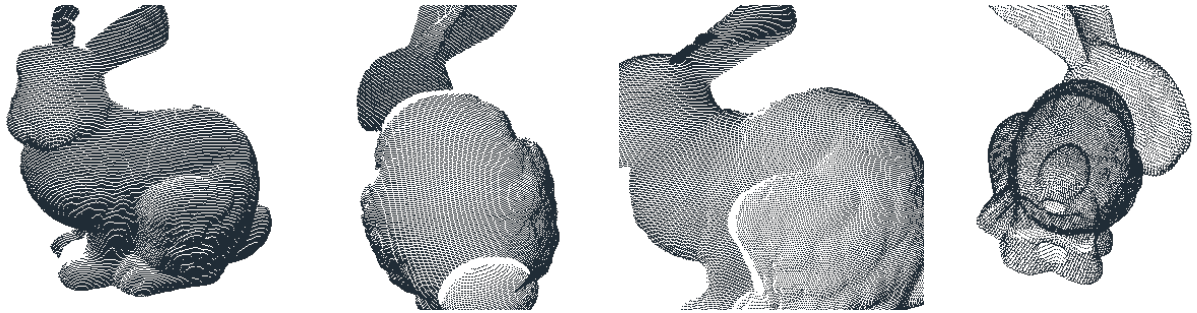




Representation Considerations

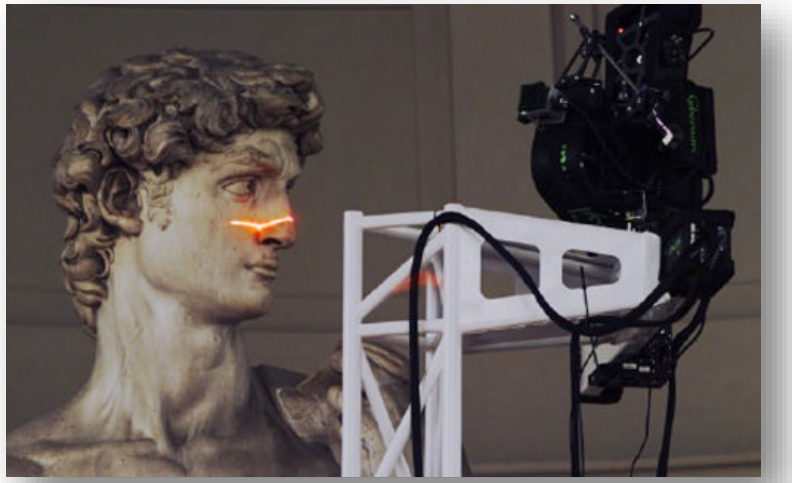
- How should we represent geometry?
  - Be derivable from sensor data
  - Support storage efficiency
  - Support editing:
    - Modification, simplification, smoothing, filtering, repair…
  - Support creativity:
    - Input metaphors, UIs…
  - Support rendering:
    - Rasterization, raytracing…
  - Support ML:
    - Share info across related shapes

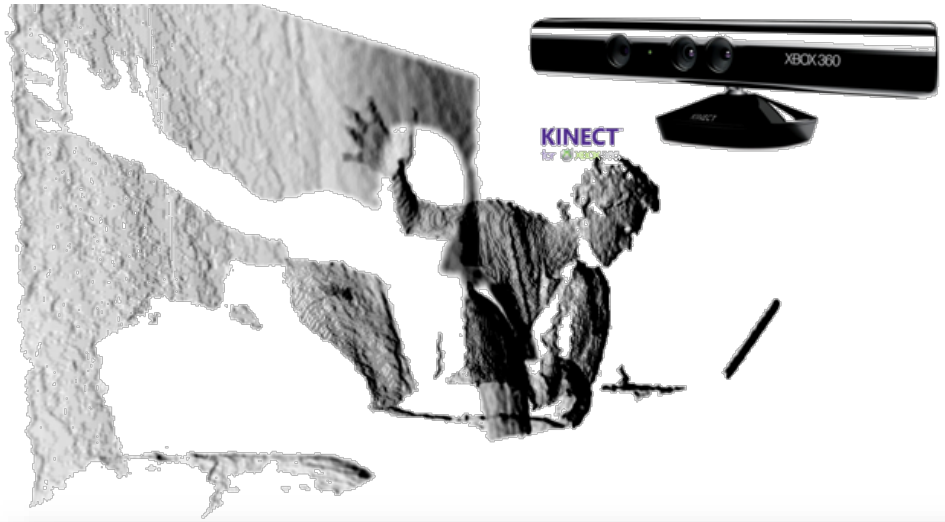


Point Clouds



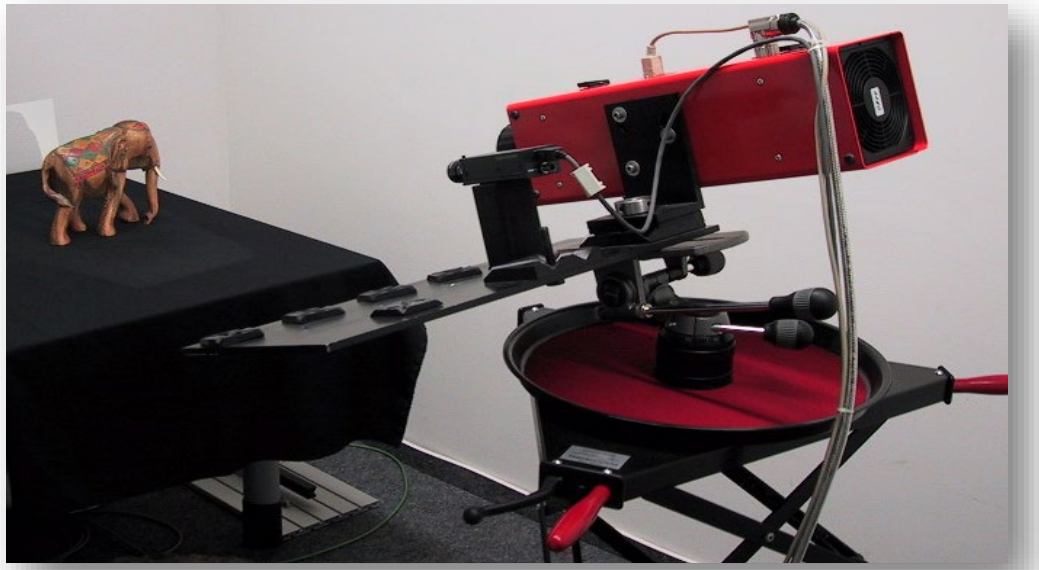
Output of Acquisition Process





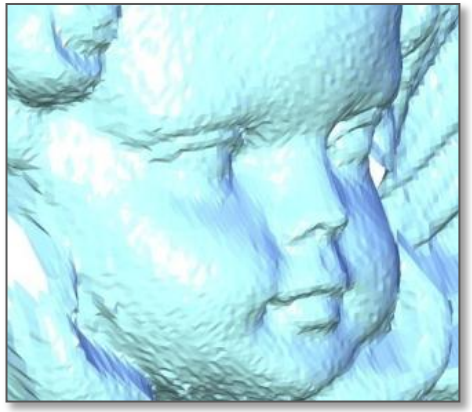
Triangulation, time-of-flight, structured light scanners

but also from classic computer algorithms like stereo



Point Cloud Issues

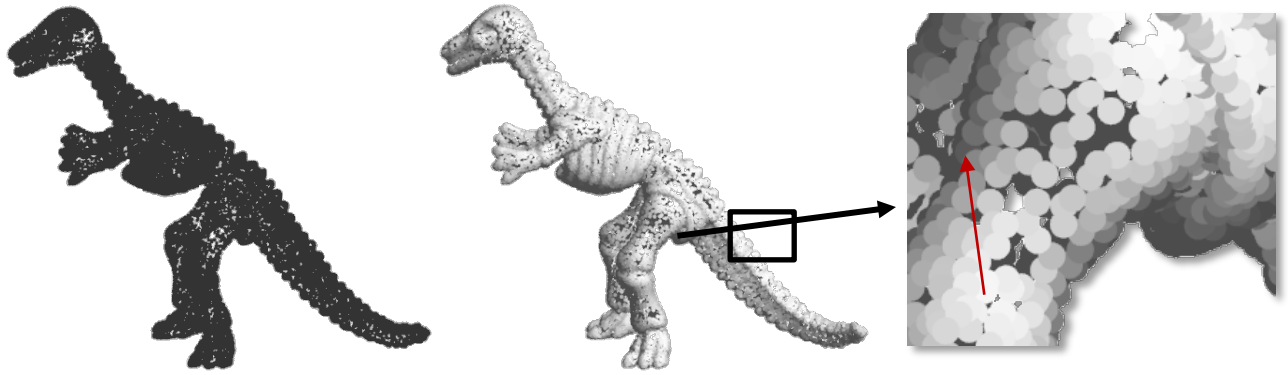
- Standard 3D data from a variety of sources/scanners -- but
  - Potentially noisy



  - Can have holes
  - Registration of multiple images is required

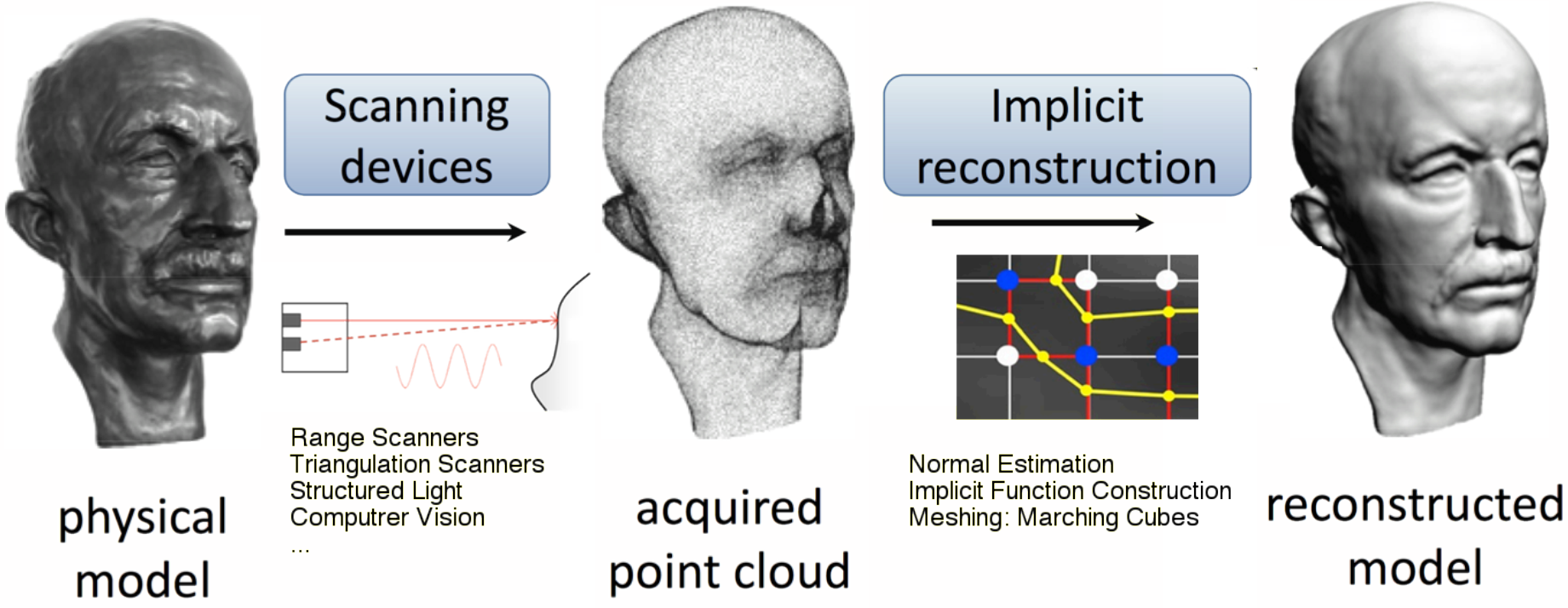
Point Clouds, Often an Intermediate Representation

- Points = unordered set of 3-tuples – no structure
- Frequently converted to other reps
  - Meshes, implicits, parametric surfaces
  - Easier to process, edit and/or render
- Efficient point processing / modeling requires spatial partitioning data structure
  - E.g., to figure out neighborhoods



shading needs normals!

3D Point Cloud Processing

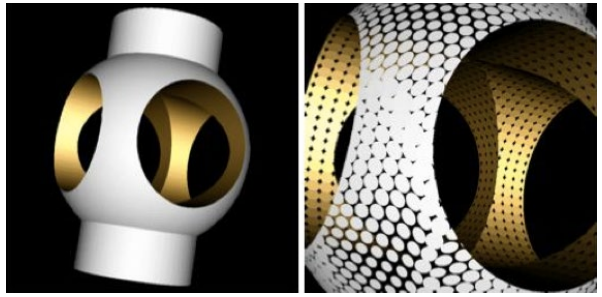


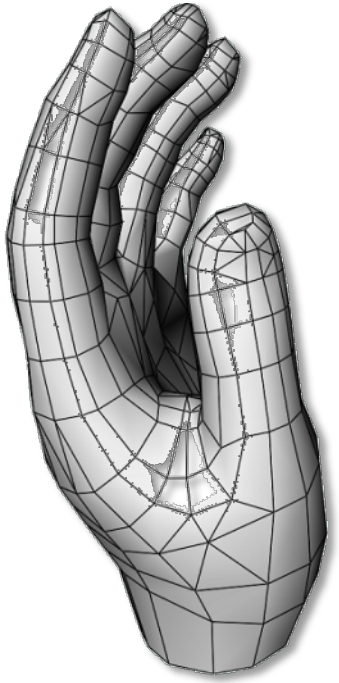
Traditional 3D Acquisition Pipeline

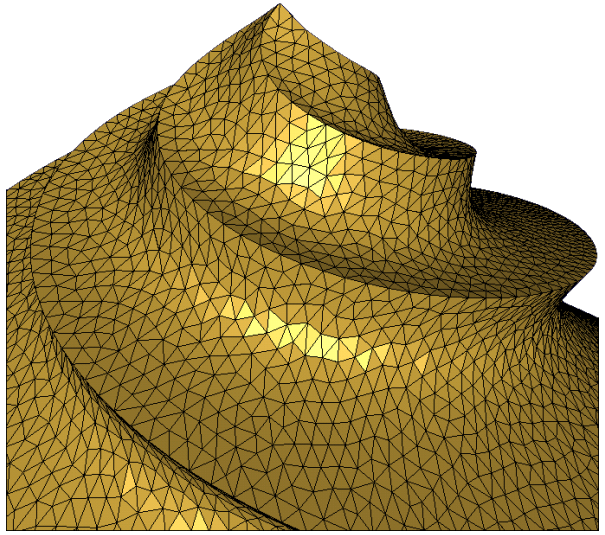
Now ML directly on Point Cloud Data

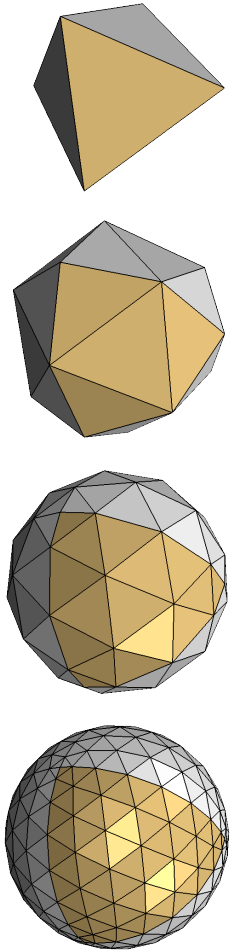
B-Reps: Low-Level

- Triangle meshes
- Quad meshes





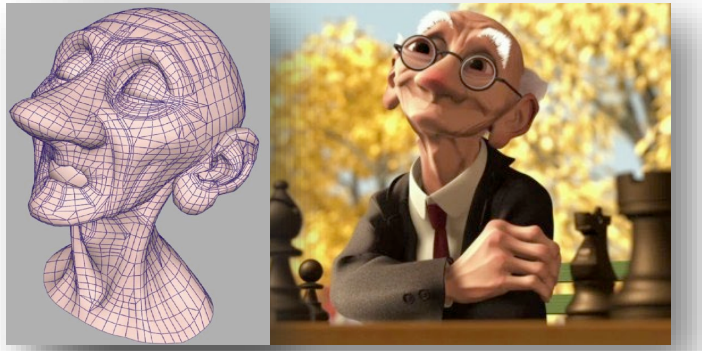


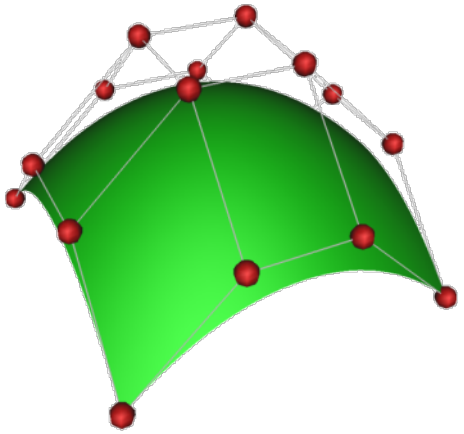


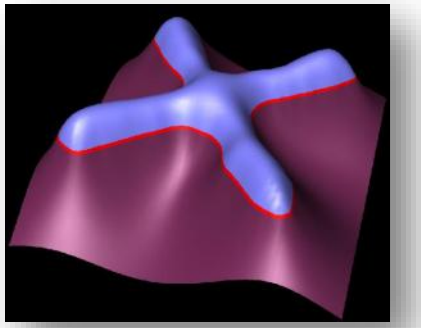
trade-offs

B-Reps: High-Level

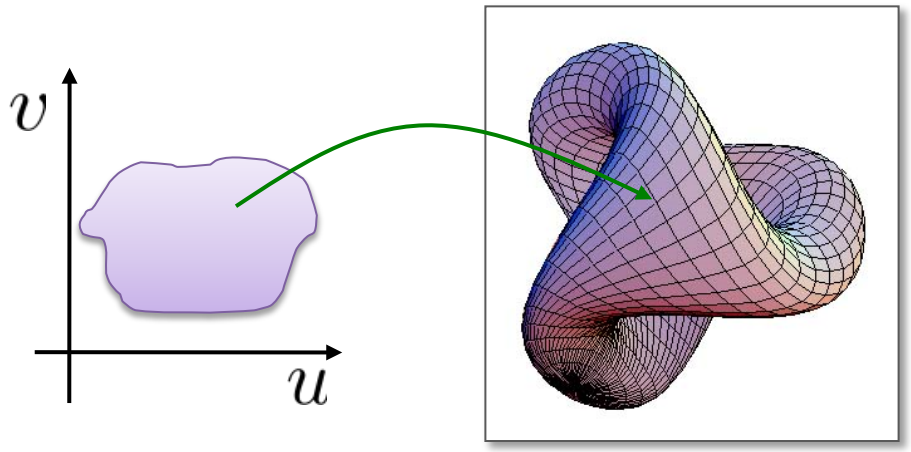
- Parametric surfaces
- Implicit functions
- Subdivision surfaces







Parametric Curves and Surfaces

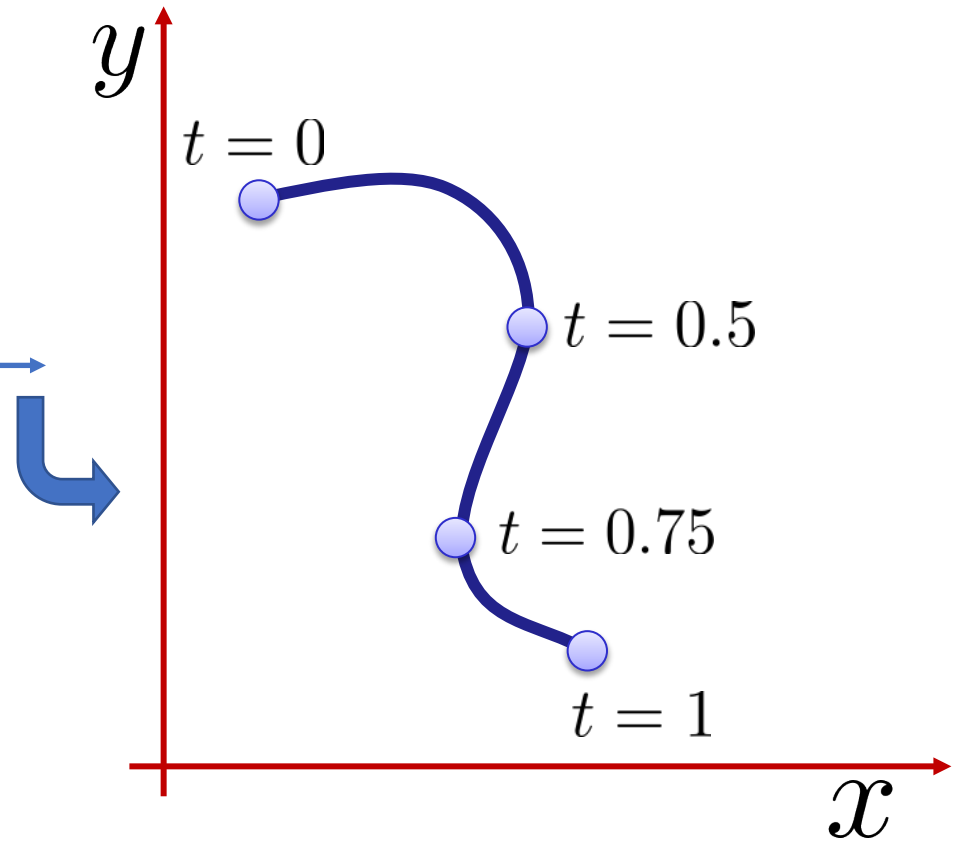


Parametric Representation: 1D Curves

- Range of a function $f : X \to Y, X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$
  - Planar curve: $m = 1, n = 2$

$$s(t) = (x(t), y(t))$$

t



  - Space curve: $m = 1, n = 3$

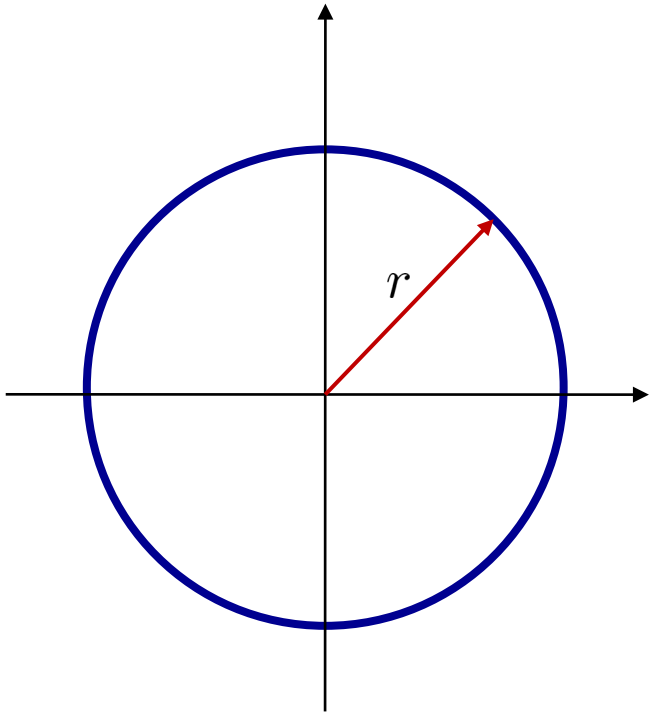
$$s(t) = (x(t), y(t), z(t))$$

1-D Parametric Curves

- Example: Explicit curve/circle in 2D

$$\mathbf{p} : \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto \mathbf{p}(t) = (x(t), y(t))$$

$$\mathbf{p}(t) = r\,(\cos(t), \sin(t))$$
$$t \in [0, 2\pi)$$

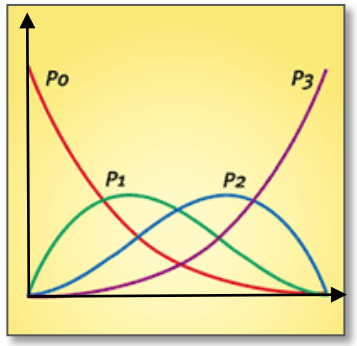


→ CS348a

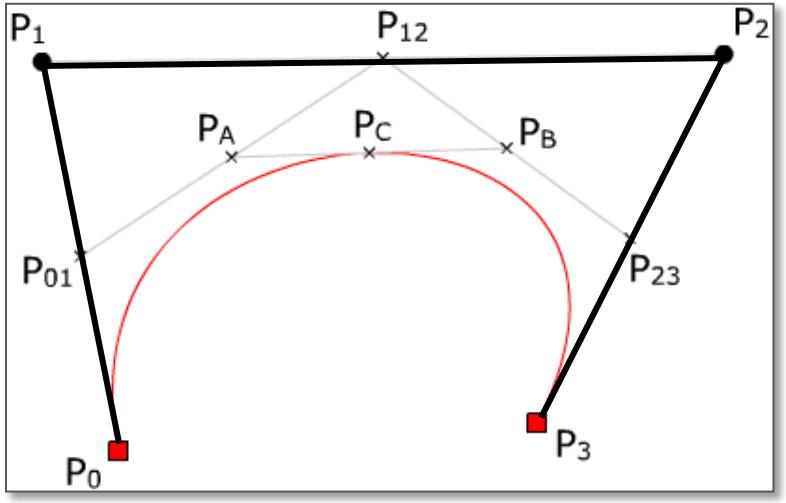
Parametric Curves: Splined Representations

- Bézier curves, splines (use multiple parametric functions)

$$s(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t) \qquad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



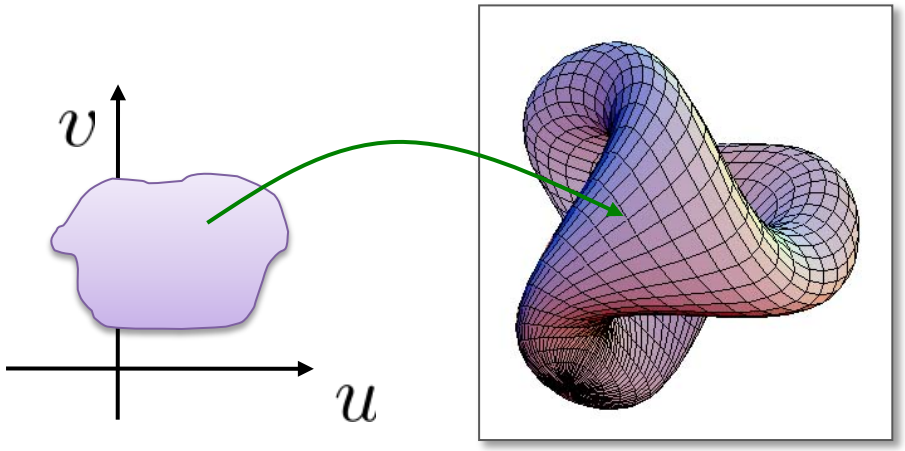
Basis functions



Curve and control polygon

Parametric Representation: 2D Surfaces

- Range of a function $f : X \to Y, X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$
  - Surface in 3D: $m = 2, n = 3$

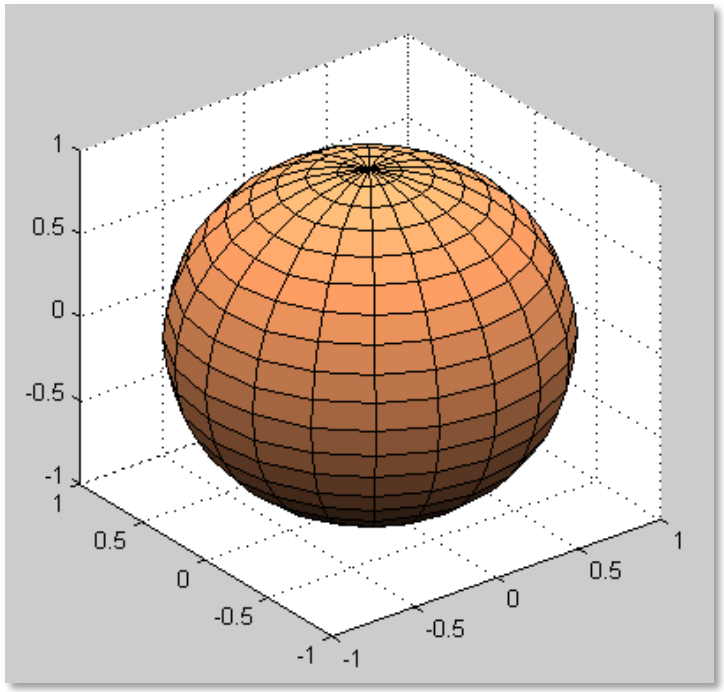


$$s(u, v) = (x(u, v), y(u, v), z(u, v))$$

Parametric Surfaces

- Sphere in 3D

$$s : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



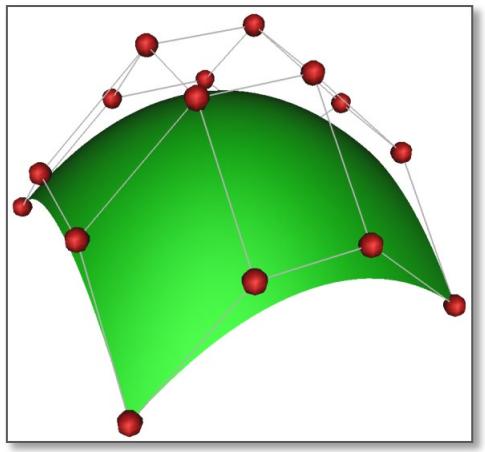
$$s(u, v) = r\left(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)\right)$$

$$(u, v) \in [0, 2\pi) \times [-\pi/2, \pi/2]$$

Tensor Product Parametric Surfaces

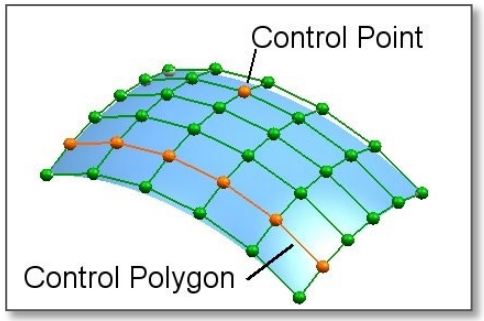
- Curve swept by another curve

$$s(u,v) = \sum_{i,j} \mathbf{p}_{i,j} B_i(u) B_j(v)$$



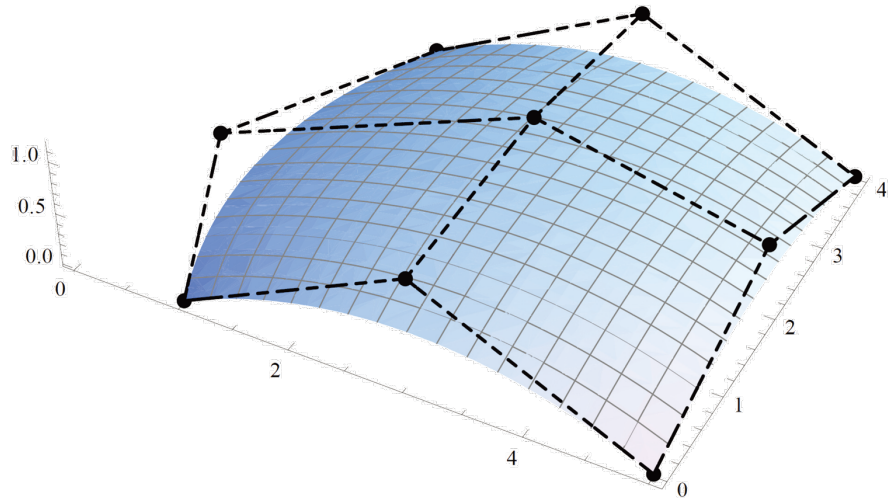
- Bézier surface:

$$s(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}_{i,j} B_i^m(u) B_j^n(v)$$

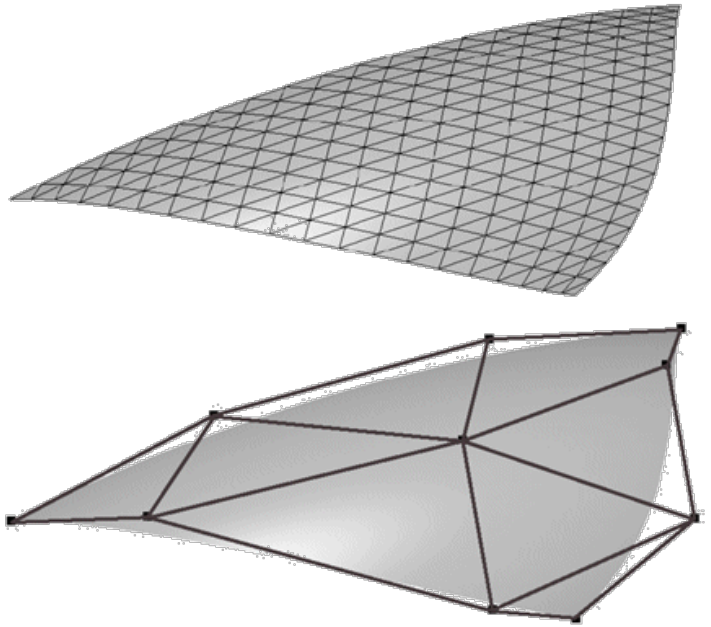


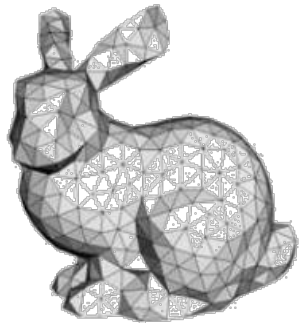
- Also : triangular patch surfaces, subdivision surfaces

Tensor Product vs. Triangular Patch Surfaces



tensor product,
regular quad mesh





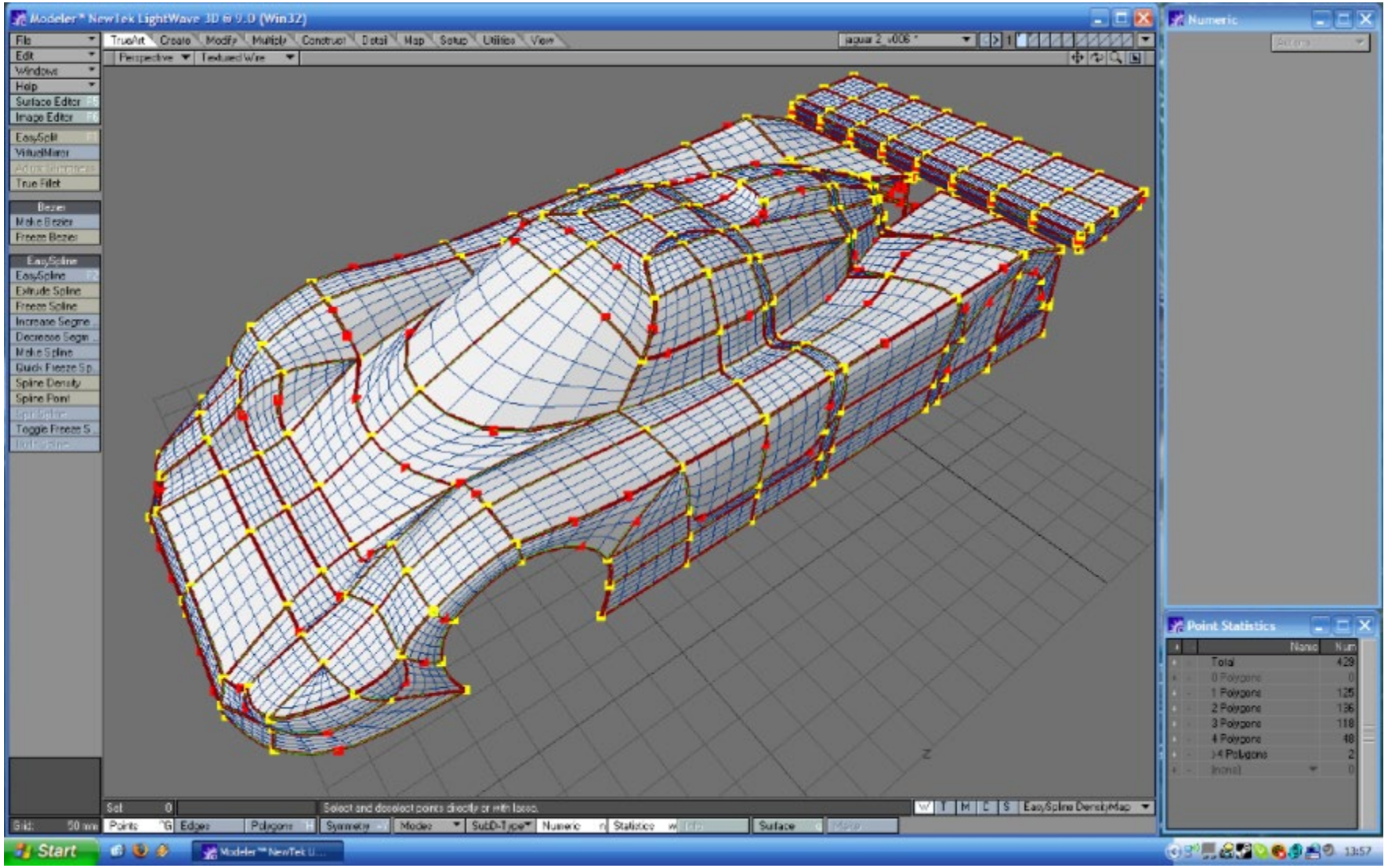
triangular patch,
triangular mesh

Parametric Curves and Surfaces

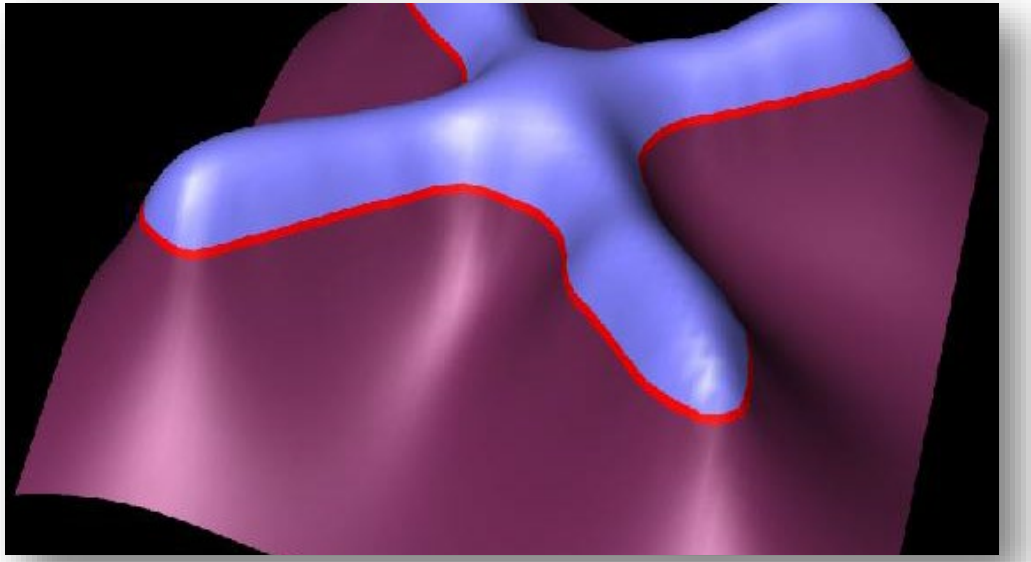
- Advantages
  - Easy to generate points on the curve/surface
  - Separates x/y/z components
  - Names each point

- Disadvantages
  - Hard to determine inside/outside
  - Hard to determine if a point is **on** the curve/surface
  - Hard to express more complex curves/surfaces!
    →cue: piecewise parametric surfaces (e.g., mesh)

Splined Surfaces for CAD



Implicit Curves and Surfaces



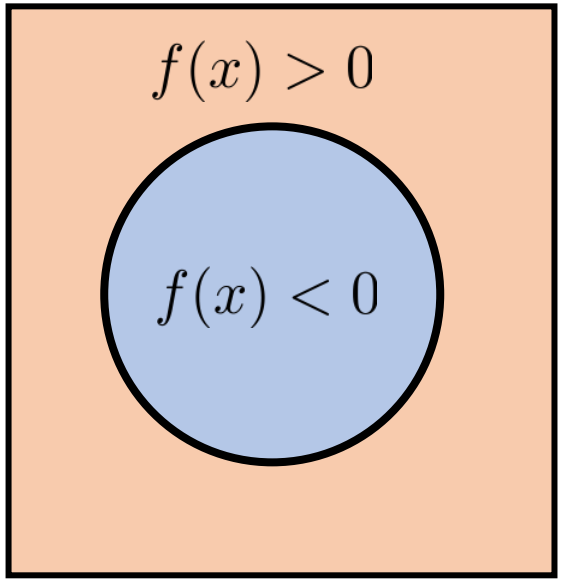
Implicit Curves and Surfaces

- Kernel of a scalar function $f : \mathbb{R}^m \rightarrow \mathbb{R}$
  - Curve in 2D: $S = \{x \in \mathbb{R}^2 | f(x) = 0\}$
  - Surface in 3D: $S = \{x \in \mathbb{R}^3 | f(x) = 0\}$
- Space partitioning

$\{x \in \mathbb{R}^m | f(x) > 0\}$ Outside

$\{x \in \mathbb{R}^m | f(x) = 0\}$ Curve/Surface

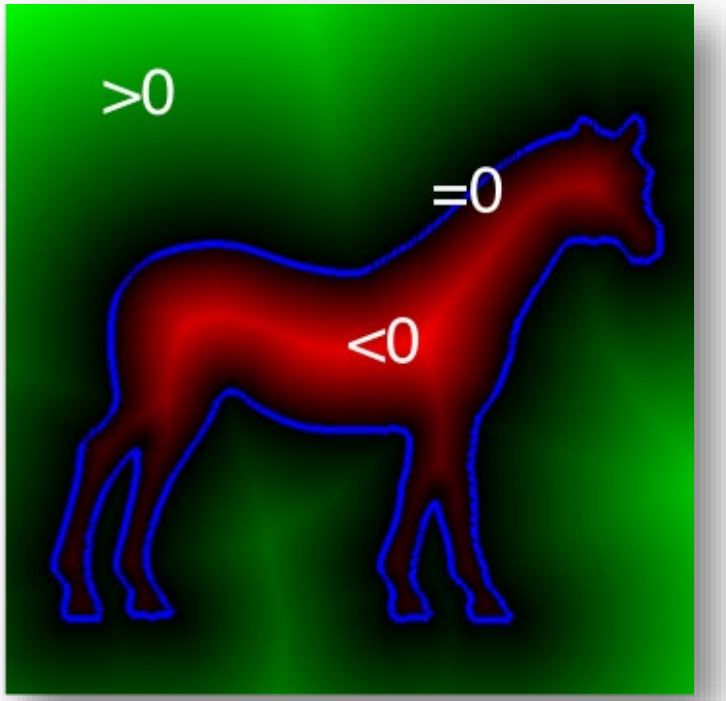
$\{x \in \mathbb{R}^m | f(x) < 0\}$ Inside



Implicit Curves and Surfaces

- Kernel of a scalar function $f : \mathbb{R}^m \rightarrow \mathbb{R}$
  - Curve in 2D: $S = \{x \in \mathbb{R}^2 | f(x) = 0\}$
  - Surface in 3D: $S = \{x \in \mathbb{R}^3 | f(x) = 0\}$

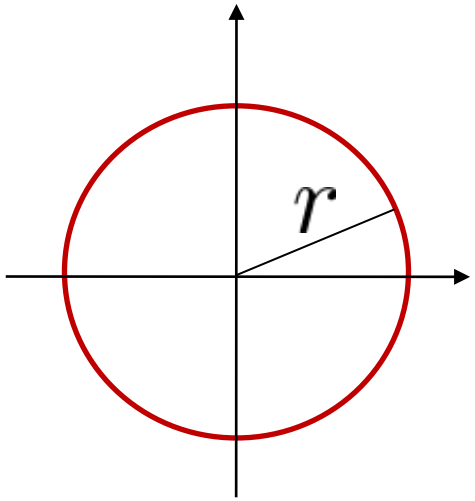
- Zero level set of
  signed distance function



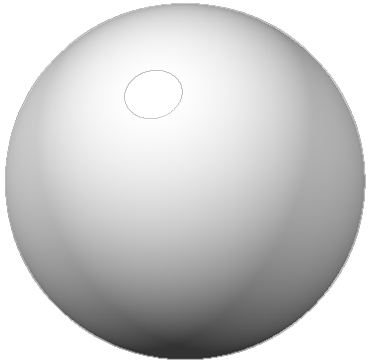
Implicit Curves and Surfaces

- Implicit circle and sphere

$$f(x, y) = x^2 + y^2 - r^2$$



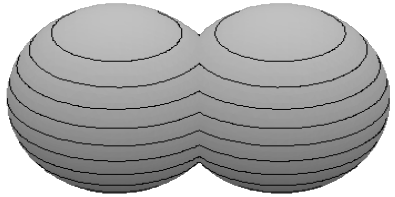
$$f(x, y, z) = x^2 + y^2 + z^2 - r^2$$

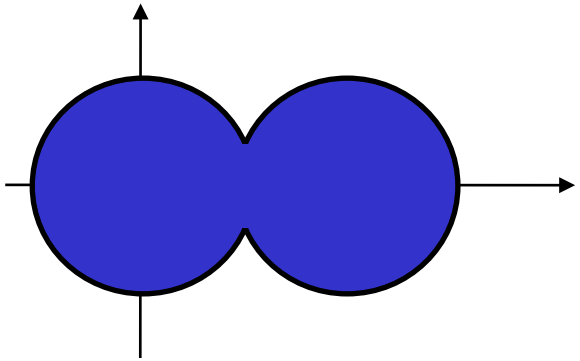


Boolean Set Operations

- Union:

$$\bigcup_i f_i(x) = \min f_i(x)$$





- Intersection:

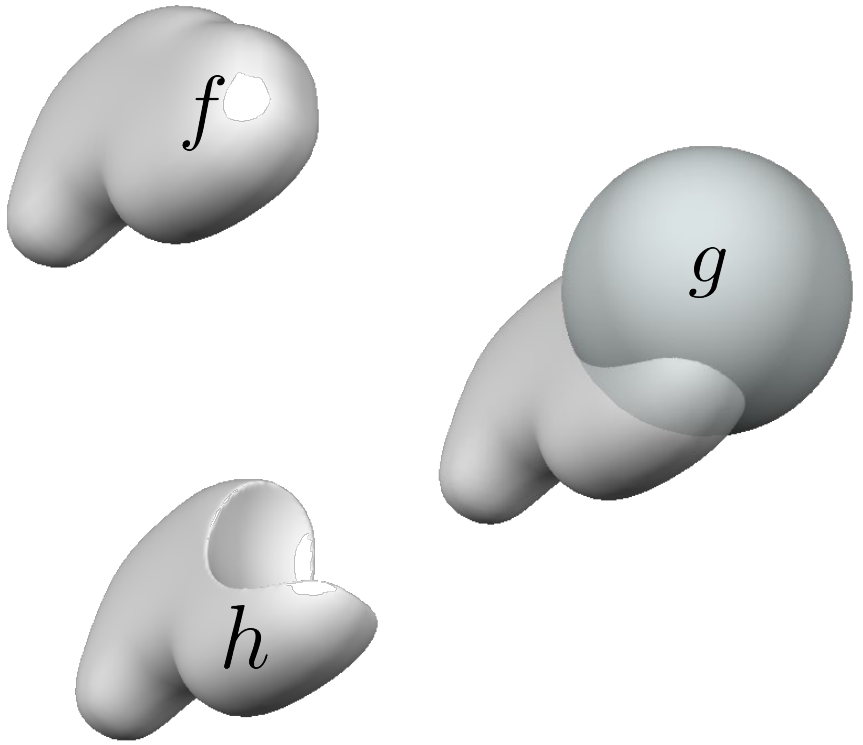
$$\bigcap_i f_i(x) = \max f_i(x)$$

More Boolean Set Operations

- Positive = outside, negative = inside
- Boolean subtraction:

	$f > 0$	$f < 0$
$g > 0$	$h > 0$	$h < 0$
$g < 0$	$h > 0$	$h > 0$

- Much easier than for parametric surfaces!

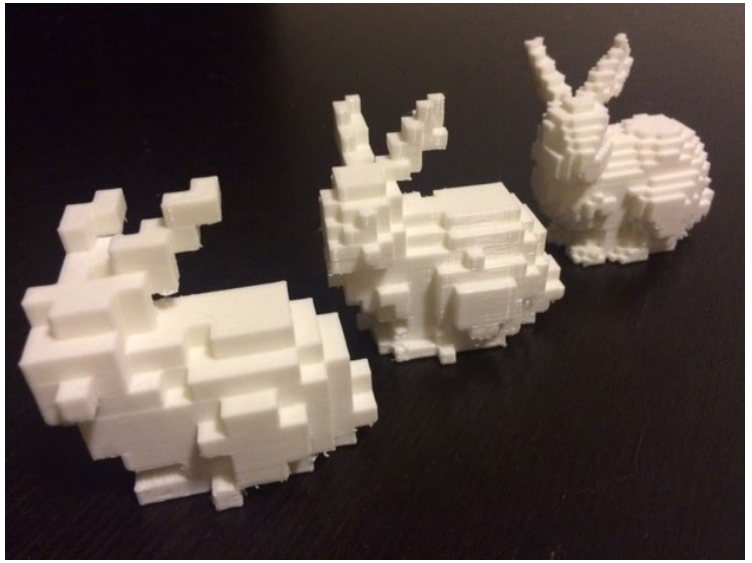


$$h = max(f, -g)$$

Implicit Curves and Surfaces

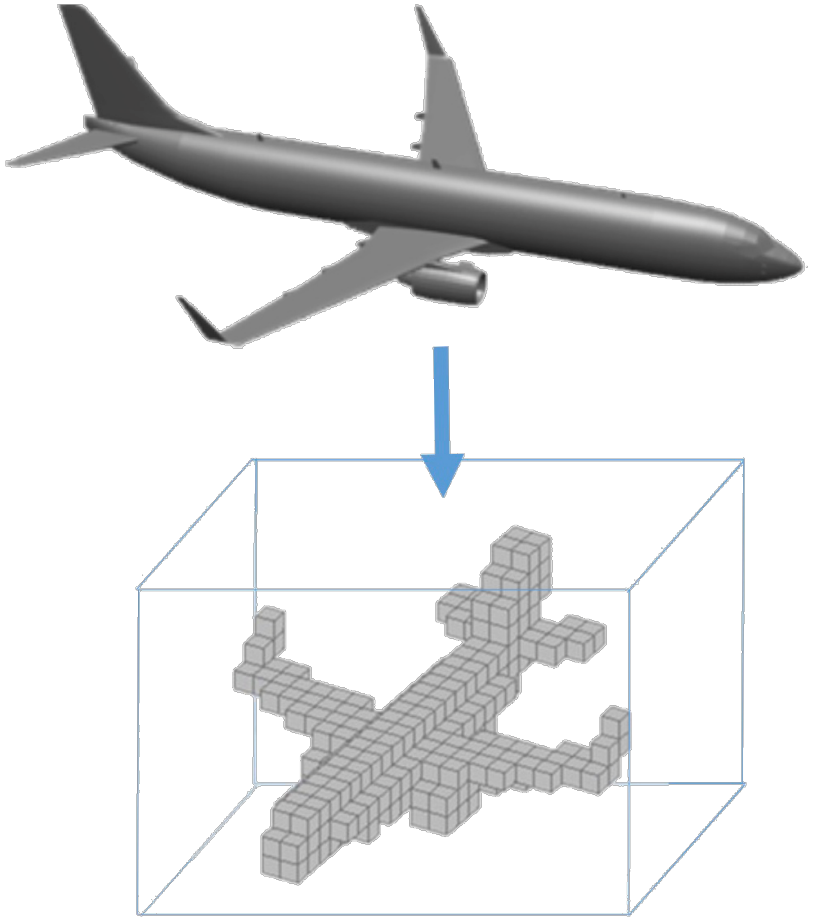
- Advantages
  - Easy to determine inside/outside
  - Easy to determine if a point is **on** the curve/surface, on what side of the surface
- Disadvantages
  - Hard to generate points on the curve/surface
  - Do not lend to (real-time) rendering

Volumetric Representations



V-Rep: Volumetric Grids

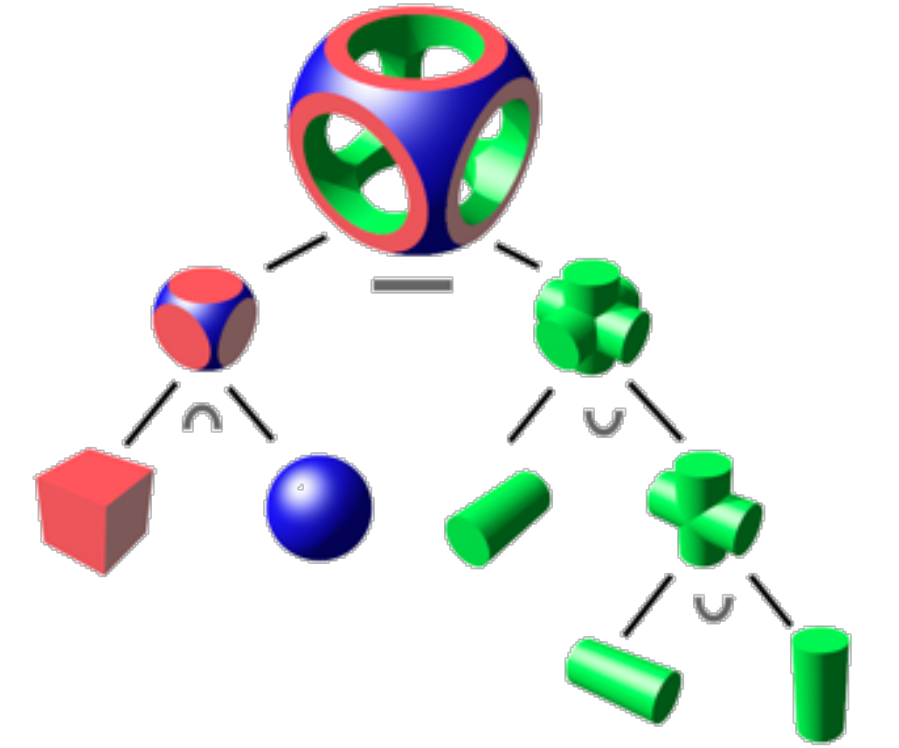
- Binary volumetric grids
- Can be produced by thresholding the distance function, or from the scanned points directly
- N^3 gets expensive fast



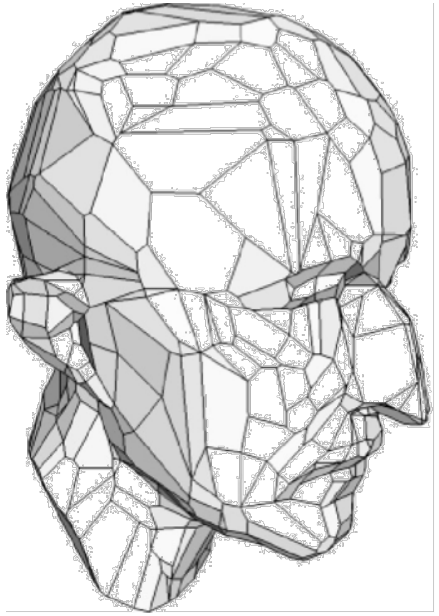
Also represents space of little informational value

V-Rep: CSG

- Constructive Solid Geometry
- Boolean ops over geometric primitives (spheres, boxes, cylinders, cones, …)
- Often non-unique

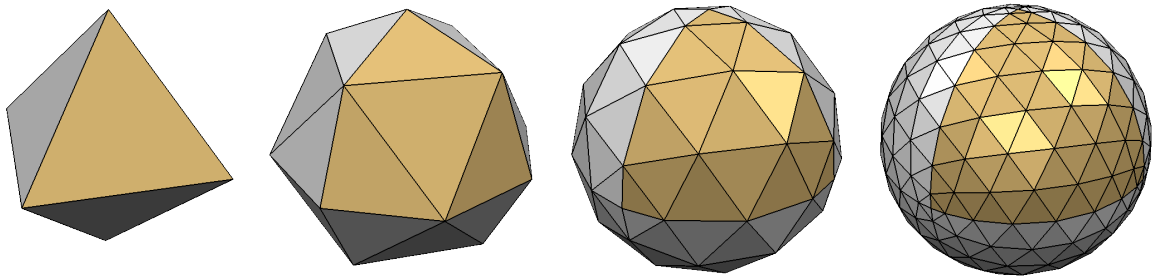


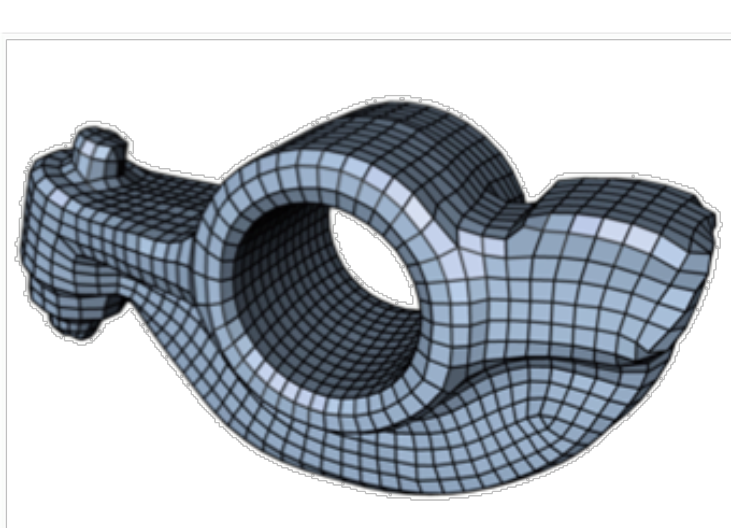
Polygonal Meshes



Polygonal Meshes

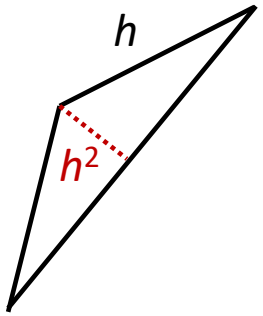
- Boundary representations of objects using polygonal primitives

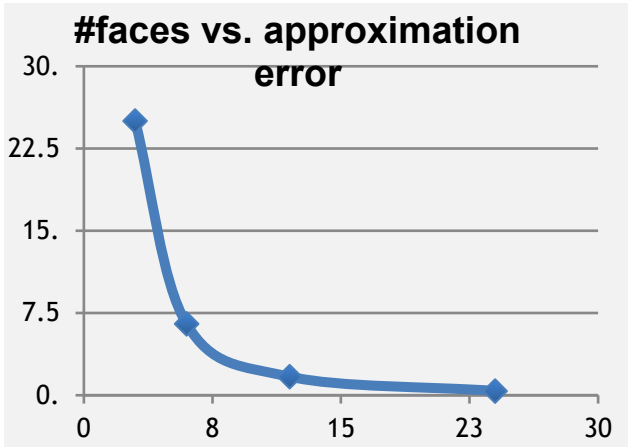


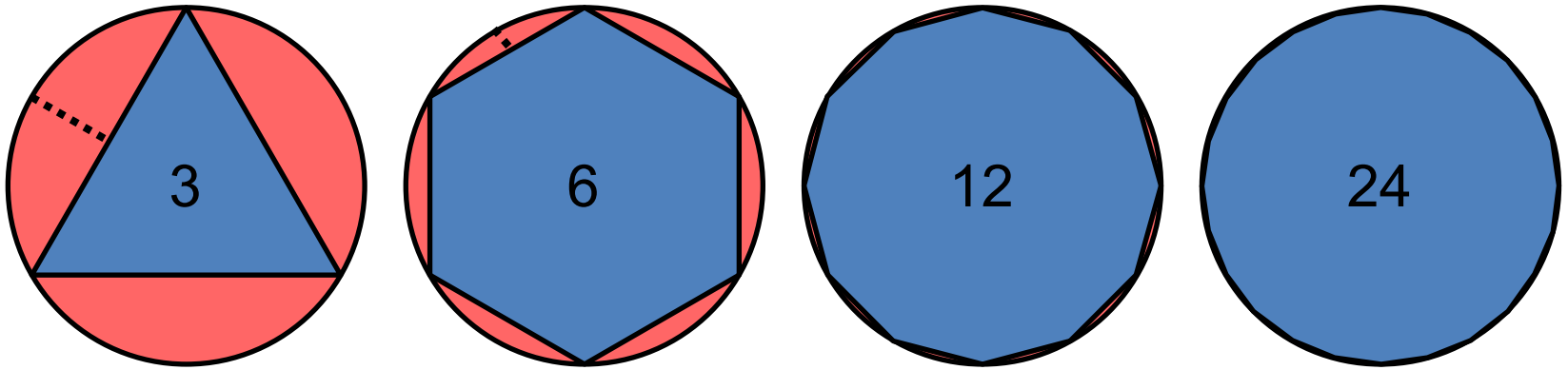


Meshes as Approximations of Smooth Surfaces

- Piecewise linear approximation
  - Error is $O(h^2)$ [$O(h)$ for points]



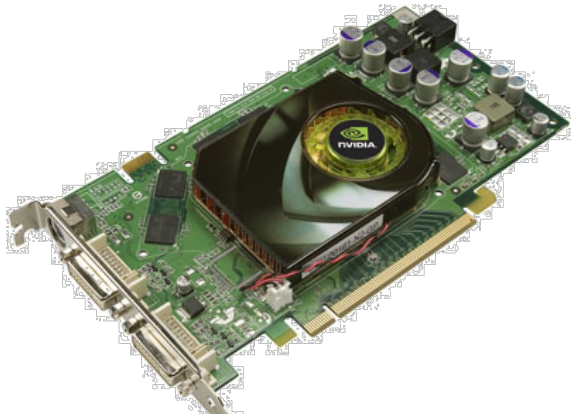


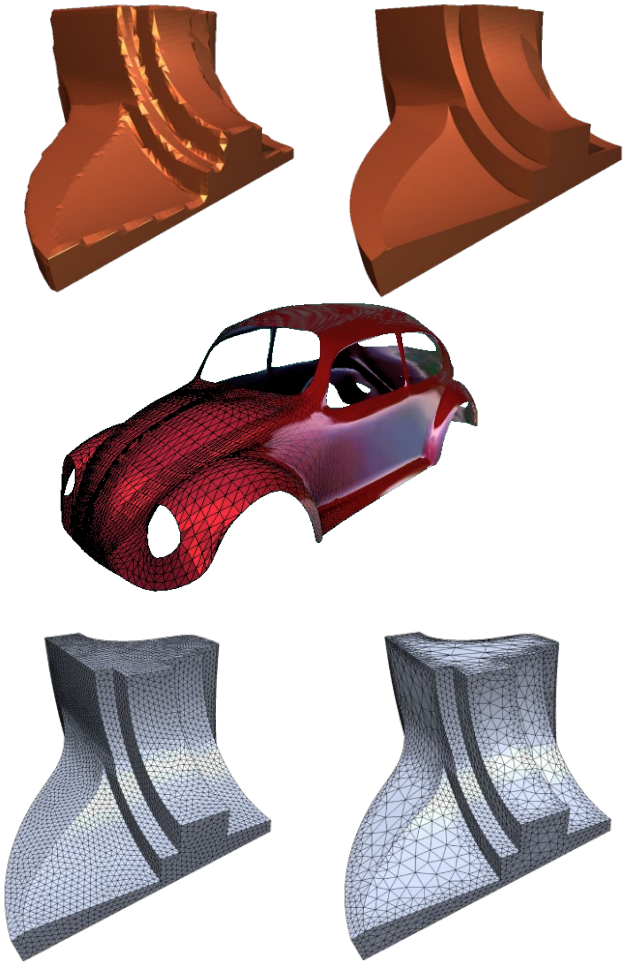


25% 6.5% 1.7% 0.4%

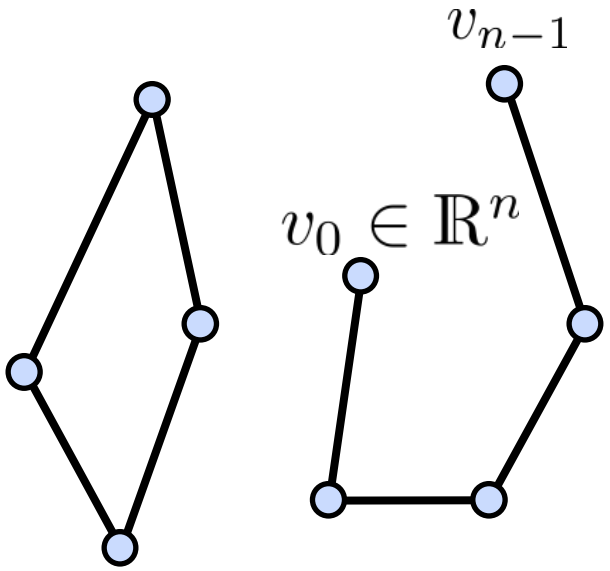
Polygonal Meshes

- Polygonal meshes are a good representation
  - approximation $O(h^2)$
  - arbitrary topology
  - adaptive refinement
  - efficient rendering



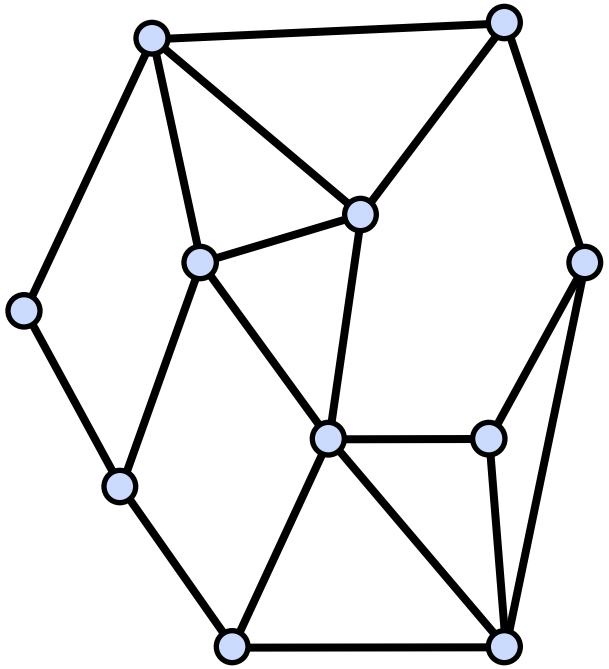


Planar Polygons



- Vertices: $v_0, v_1, \ldots, v_{n-1}$
- Edges: $\{(v_0, v_1), \ldots, (v_{n-2}, v_{n-1})\}$
- Closed: $v_0 = v_{n-1}$
- Planar: all vertices on a plane
- Simple: not self-intersecting

Polygonal Meshes

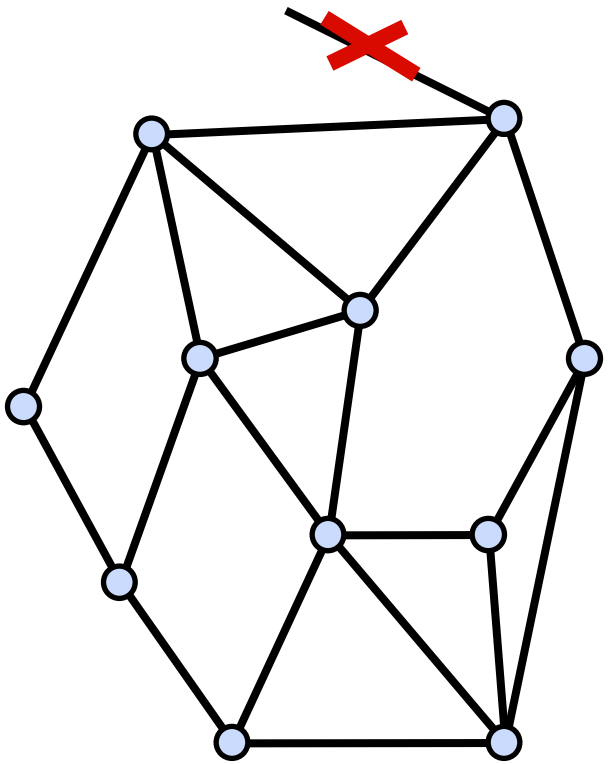


- A finite set M of closed, simple polygons Q_i is a polygonal mesh
- The intersection of two polygons in M is either empty, a vertex, or an edge

$$M = \langle V, E, F \rangle$$

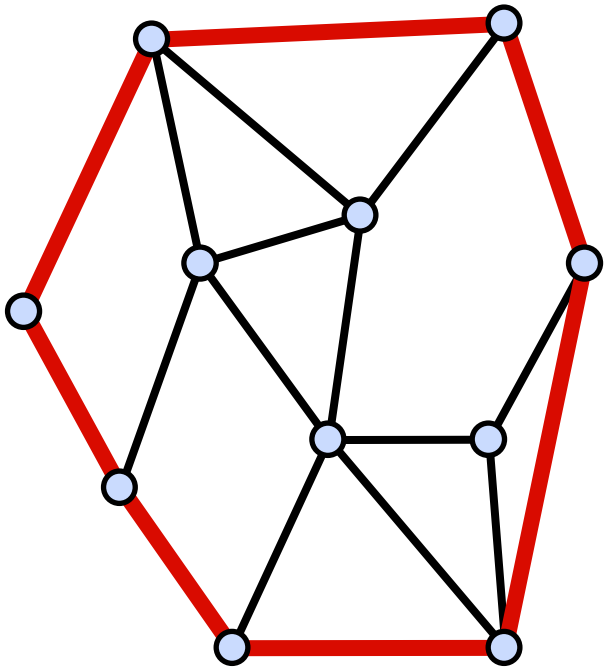
vertices edges faces

Polygonal Mesh

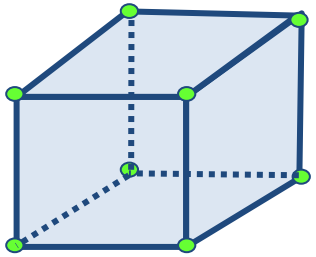


- A finite set M of closed, simple polygons Q_i is a polygonal mesh
- The intersection of two polygons in M is either empty, a vertex, or an edge
- Every edge belongs to at least one polygon

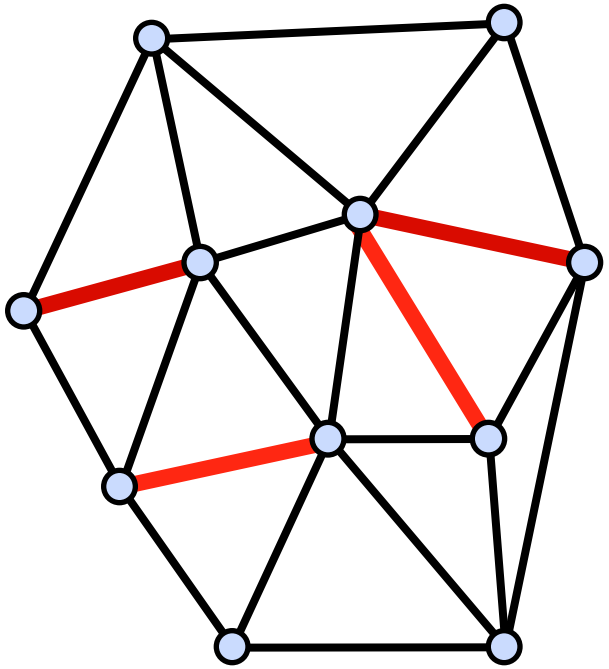
Polygonal Mesh



- **Boundary**: the set of all edges that belong to only one polygon
  - Either empty or forms closed loops
  - If empty, then the polygonal mesh is closed



Triangulation

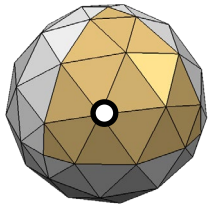


- Polygonal mesh where every face is a triangle

- Simplifies data structures
- Simplifies rendering
- Simplifies algorithms
- Each face planar and convex
- Any polygon can be triangulated

Triangle Meshes

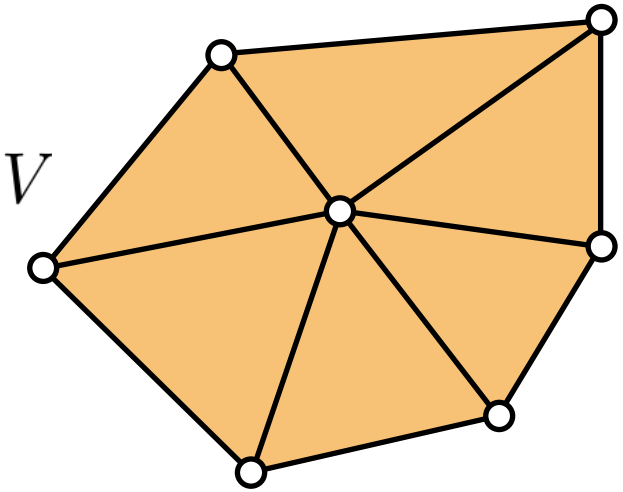
- Connectivity: vertices, edges, triangles
- Geometry: vertex positions



$$V = \{v_1, \ldots, v_n\}$$

$$E = \{e_1, \ldots, e_k\}, \quad e_i \in V \times V$$

$$F = \{f_1, \ldots, f_m\}, \quad f_i \in V \times V \times V$$



$$P = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}, \quad \mathbf{p}_i \in \mathbb{R}^3$$

Data Structures

- What should be stored?
  - Geometry: 3D coordinates
  - Connectivity
    - Adjacency relationships
  - Attributes
    - Normal, color, texture coordinates
    - Per vertex, face, edge

Simple Data Structures: Triangle List

- STL ("Standard Triangle Language") format (used in CAD)
- Storage
  - Face: 3 positions
  - 4 bytes per coordinate
  - 36 bytes per face
    - on average: $f = 2v$ *(Euler)*
    - $72*v$ bytes for a mesh with v vertices
- No explicit connectivity information

Triangles			
0	x0	y0	z0
1	x1	x1	z1
2	x2	y2	z2
3	x3	y3	z3
4	x4	y4	z4
5	x5	y5	z5
6	x6	y6	z6
…	…	…	…

Simple Data Structures: Triangle List

- Used in formats

  OBJ, OFF, WRL

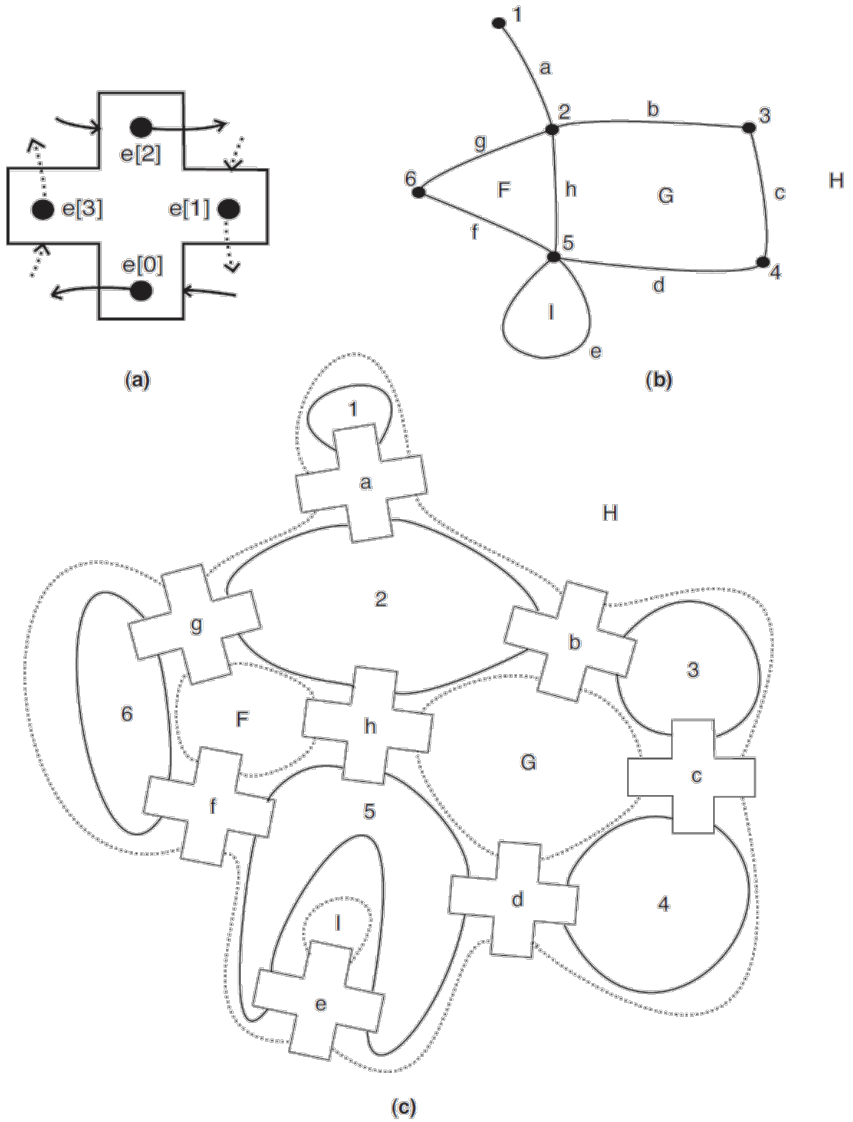
- Storage
  - Vertex: position
  - Face: vertex indices
  - 12 bytes per vertex
  - 12 bytes per face
  - 36**v* bytes for the mesh

- No explicit neighborhood info

Vertices			
v0	x0	y0	z0
v1	x1	x1	z1
v2	x2	y2	z2
v3	x3	y3	z3
v4	x4	y4	z4
v5	x5	y5	z5
v6	x6	y6	z6
…	..	..	..
	.	.	.

Triangles			
t0	v0	v1	v2
t1	v0	v1	v3
t2	v2	v4	v3
t3	v5	v2	v6
…	..	..	..
	.	.	.

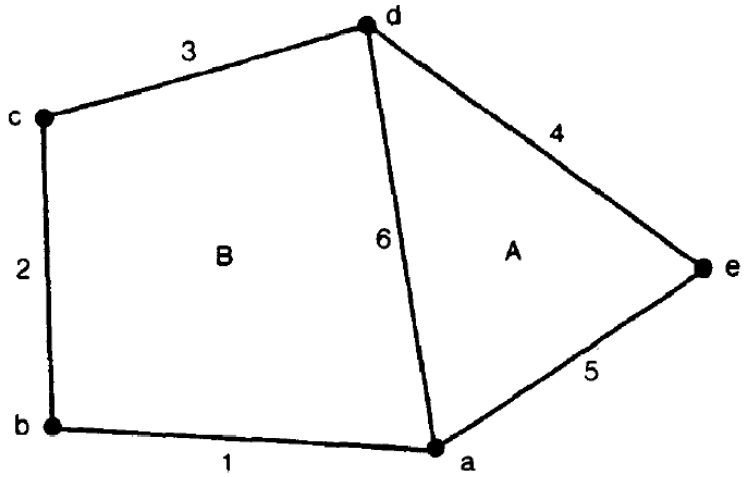
Quad-Edge: Encoding Mesh Topology

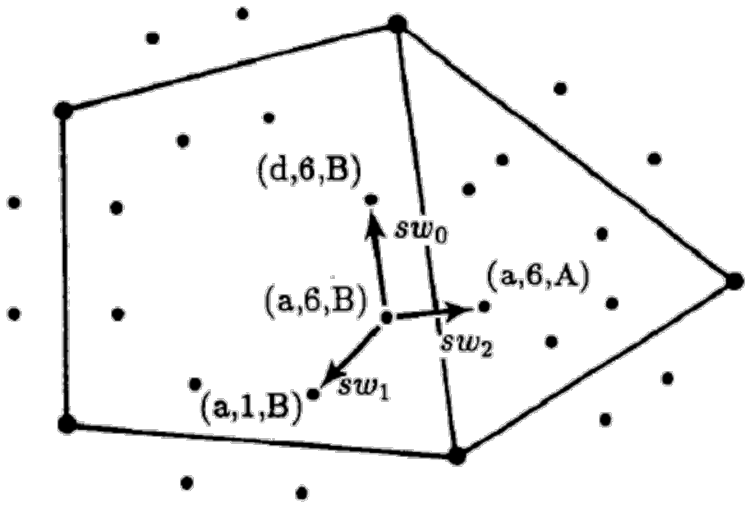


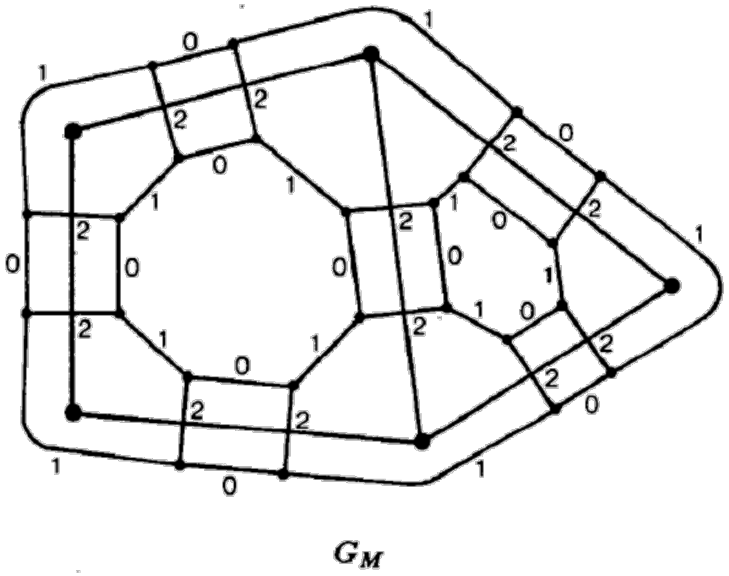
Edge-based
(many variants,
half-edge, etc)

Brisson: Cell-Tuple

(*v*,*e*,*f*)

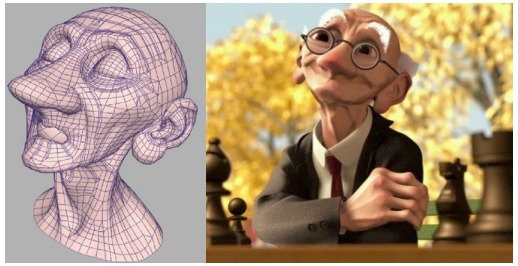
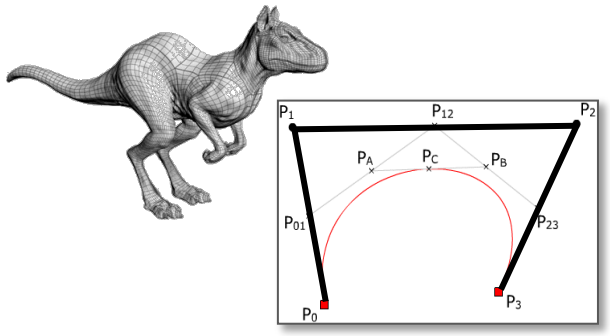
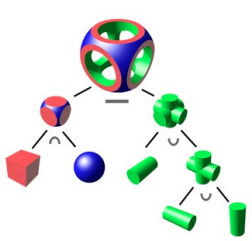
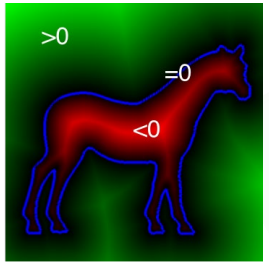
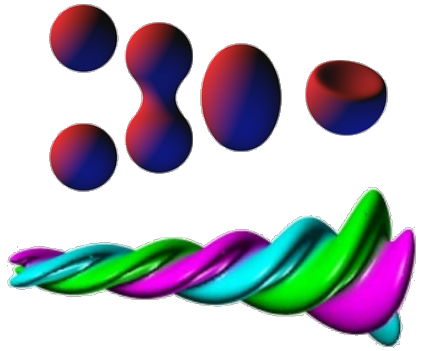
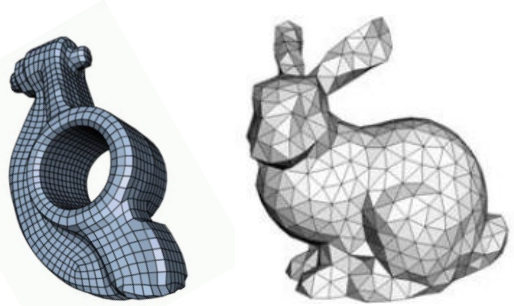
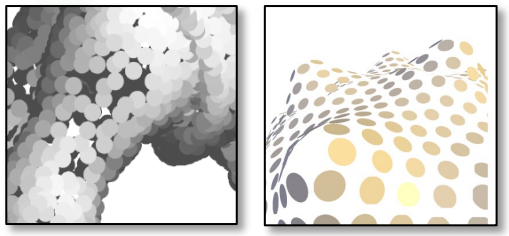






G_M

Summary

Parametric	Implicit	Discrete/Sampled
 	  	 
• Splines, tensor-product surfaces • Subdivision surfaces	• Distance fields • Metaballs/blobs	• Meshes • Point set surfaces

Representation Conversions

Points → Implicit
Implicit → Mesh
Mesh → Points

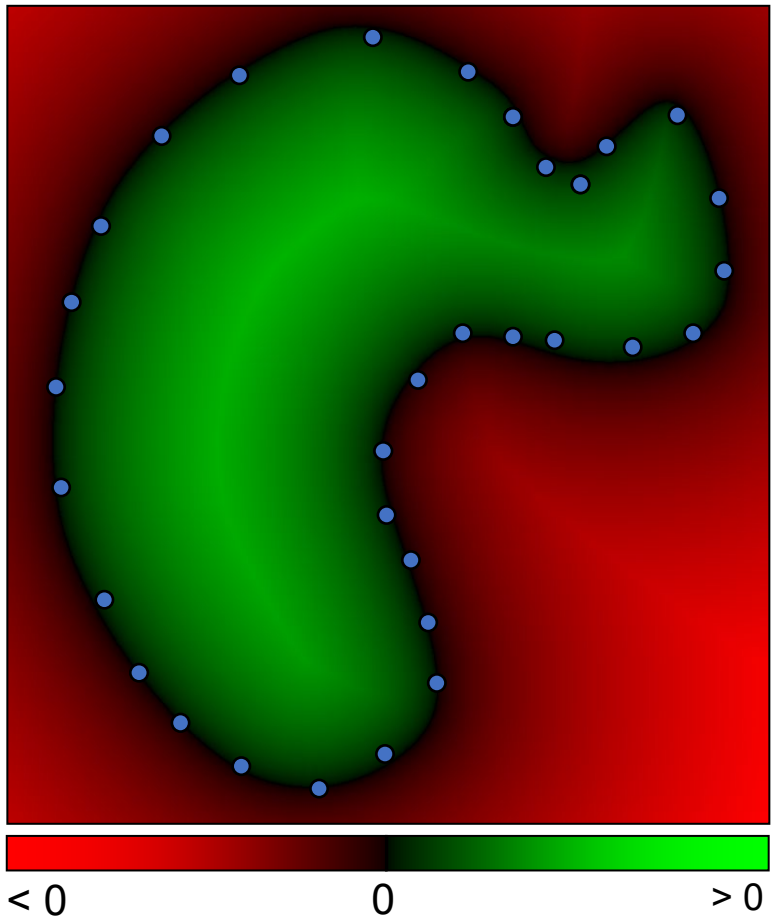
POINTS → IMPLICIT

Implicit Surface Reconstruction

Implicit Function Approach

- Given a point cloud
- Define a function $f : R^3 \rightarrow R$

  with value > 0 outside the shape and < 0 inside



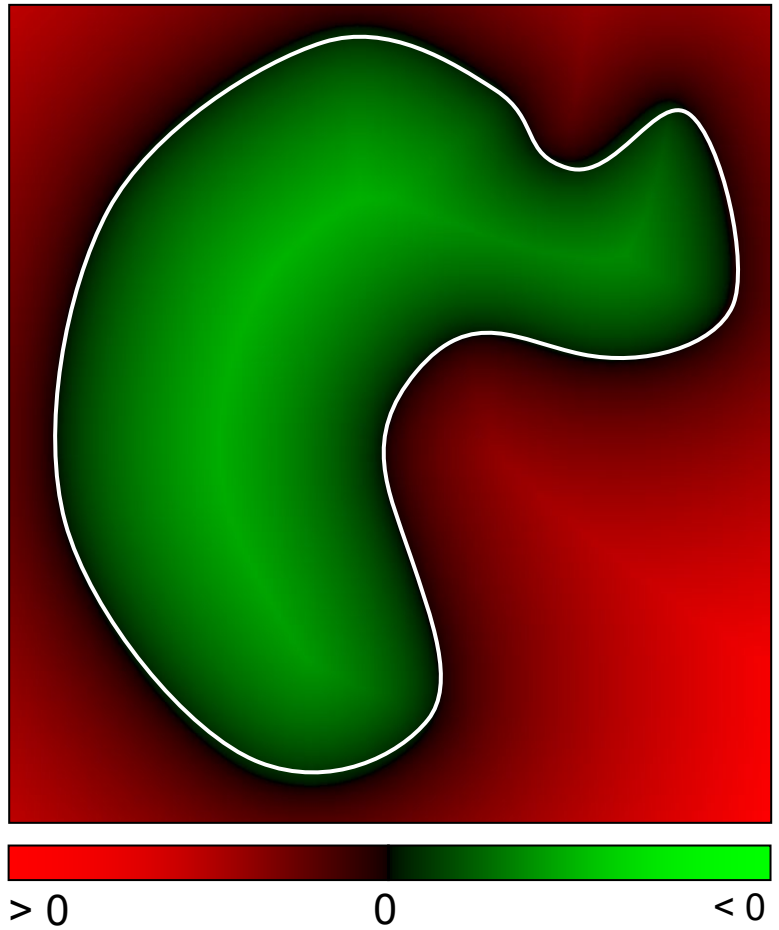
Implicit Function Approach

- Define a function $f : R^3 \rightarrow R$

with value > 0 outside the shape
and < 0 inside

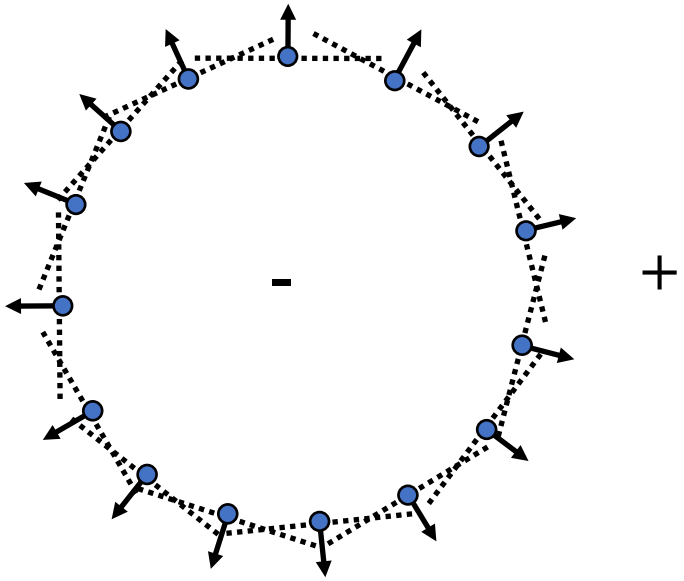
- Extract the zero-set

$$\{x : f(x) = 0\}$$



SDF from Points and Normals

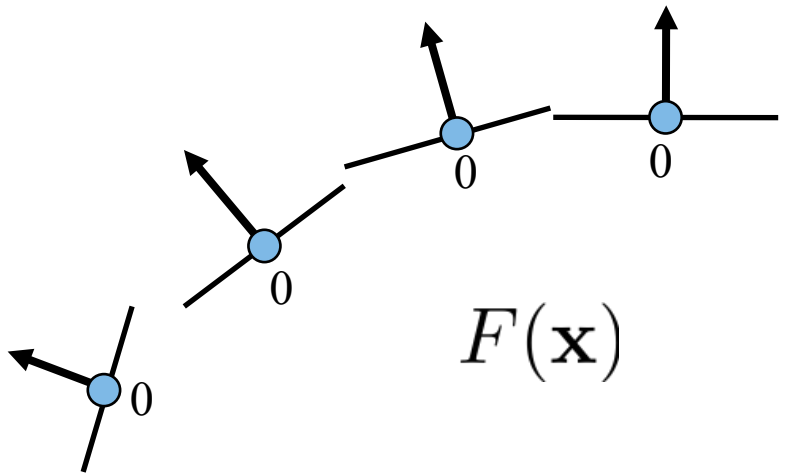
- Input: Points + Normals
- Normals help to distinguish between inside and outside
- Computed via locally fitting planes at the points (but orientation can be tricky)



"Surface reconstruction from unorganized points", Hoppe et al., ACM SIGGRAPH 1992
http://research.microsoft.com/en-us/um/people/hoppe/proj/recon/

Smooth SDF

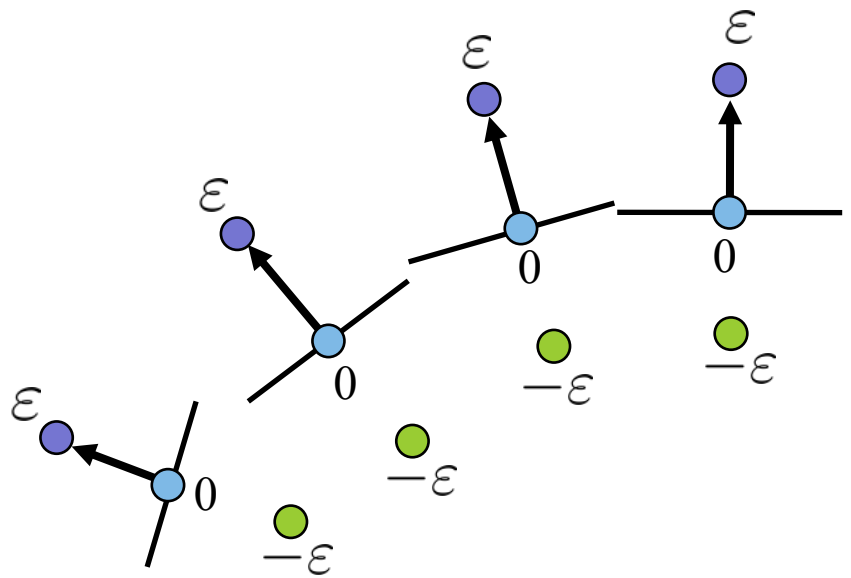
- Find smooth implicit F
- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - F is smooth
  - Avoid trivial $F \equiv 0$



"Reconstruction and representation of 3D objects with radial basis functions", Carr et al., ACM SIGGRAPH 2001

Smooth SDF

- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - F is smooth
  - Avoid trivial $F \equiv 0$
- Add off-surface constraints



$$F(\mathbf{p}_i + \varepsilon \mathbf{n}_i) = \varepsilon$$

$$F(\mathbf{p}_i - \varepsilon \mathbf{n}_i) = -\varepsilon$$

"Reconstruction and representation of 3D objects with radial basis functions", Carr et al., ACM SIGGRAPH 2001

Radial Basis Function Interpolation

- **RBF**: Weighted sum of shifted, smooth kernels

$$F(\mathbf{x}) = \sum_{i=0}^{N-1} w_i \, \varphi(\|\mathbf{x} - \mathbf{c}_i\|) \quad N = 3n$$

Scalar weights
Unknowns

Smooth kernels
(basis functions)
centered at constrained
points.
For example:

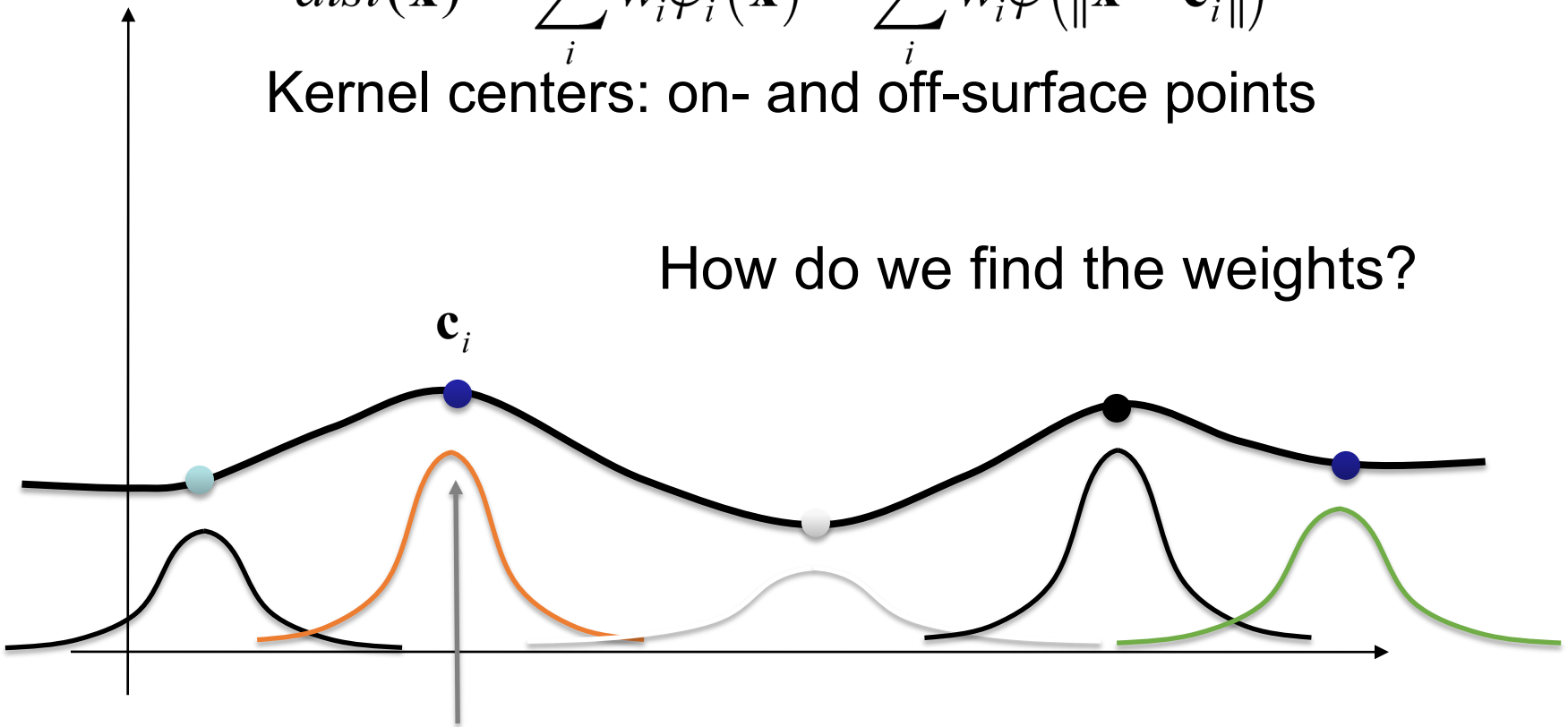
$$\varphi(r) = r^3$$

Radial Basis Function Interpolation

$$dist(\mathbf{x}) = \sum_i w_i \varphi_i(\mathbf{x}) = \sum_i w_i \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

Kernel centers: on- and off-surface points

How do we find the weights?

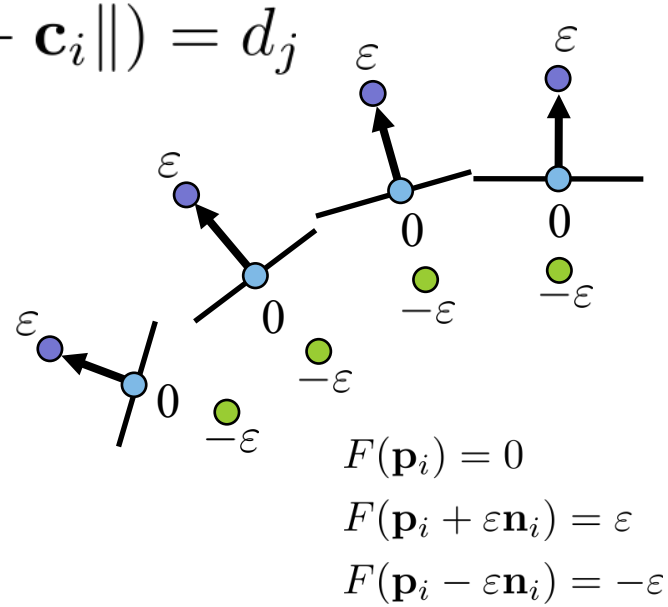


Radial Basis Function Interpolation

- Interpolate the constraints:

$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} = \{\mathbf{p}_i,\ \mathbf{p}_i + \varepsilon\mathbf{n}_i,\ \mathbf{p}_i - \varepsilon\mathbf{n}_i\}$$

$$\forall j = 0, \ldots, N-1, \quad \sum_{i=0}^{N-1} w_i\, \varphi(\|\mathbf{c}_j - \mathbf{c}_i\|) = d_j$$



Radial Basis Function Interpolation

- Interpolate the constraints:

$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} = \{\mathbf{p}_i,\ \mathbf{p}_i + \varepsilon\mathbf{n}_i,\ \mathbf{p}_i - \varepsilon\mathbf{n}_i\}$$

  - Symmetric linear system to get the weights:

$$\begin{pmatrix} \varphi(\|\mathbf{c}_0 - \mathbf{c}_0\|) & \cdots & \varphi(\|\mathbf{c}_0 - \mathbf{c}_{N-1}\| \\ \vdots & \ddots & \vdots \\ \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_0\|) & \cdots & \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_{N-1}\|) \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \end{pmatrix}$$

$3n$ equations

$3n$ variables

RBF Kernels

- Triharmonic: $\varphi(r) = r^3$
  - Globally supported
  - Leads to dense symmetric linear system
  - C^2 smoothness
  - Works well for highly irregular sampling

RBF Kernels

- Polyharmonic spline
  - $\varphi(r) = r^k \log(r),\ k = 2, 4, 6 \ldots$
  - $\varphi(r) = r^k,\ k = 1, 3, 5 \ldots$
- Multiquadratic

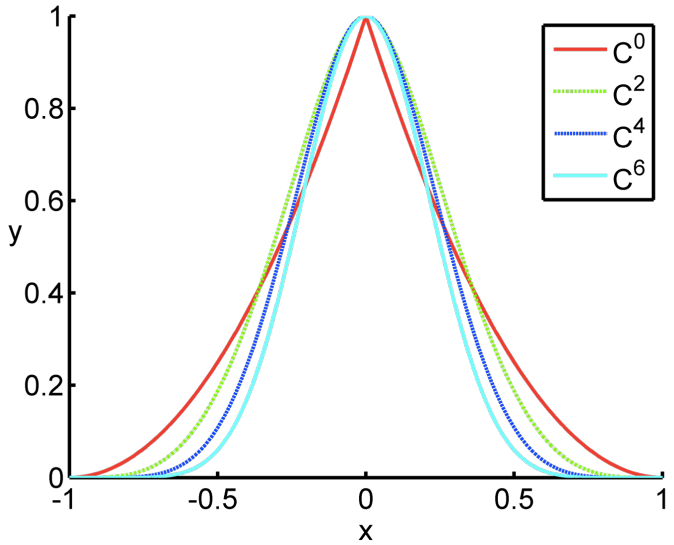
$$\varphi(r) = \sqrt{r^2 + \beta^2}$$

- Gaussian

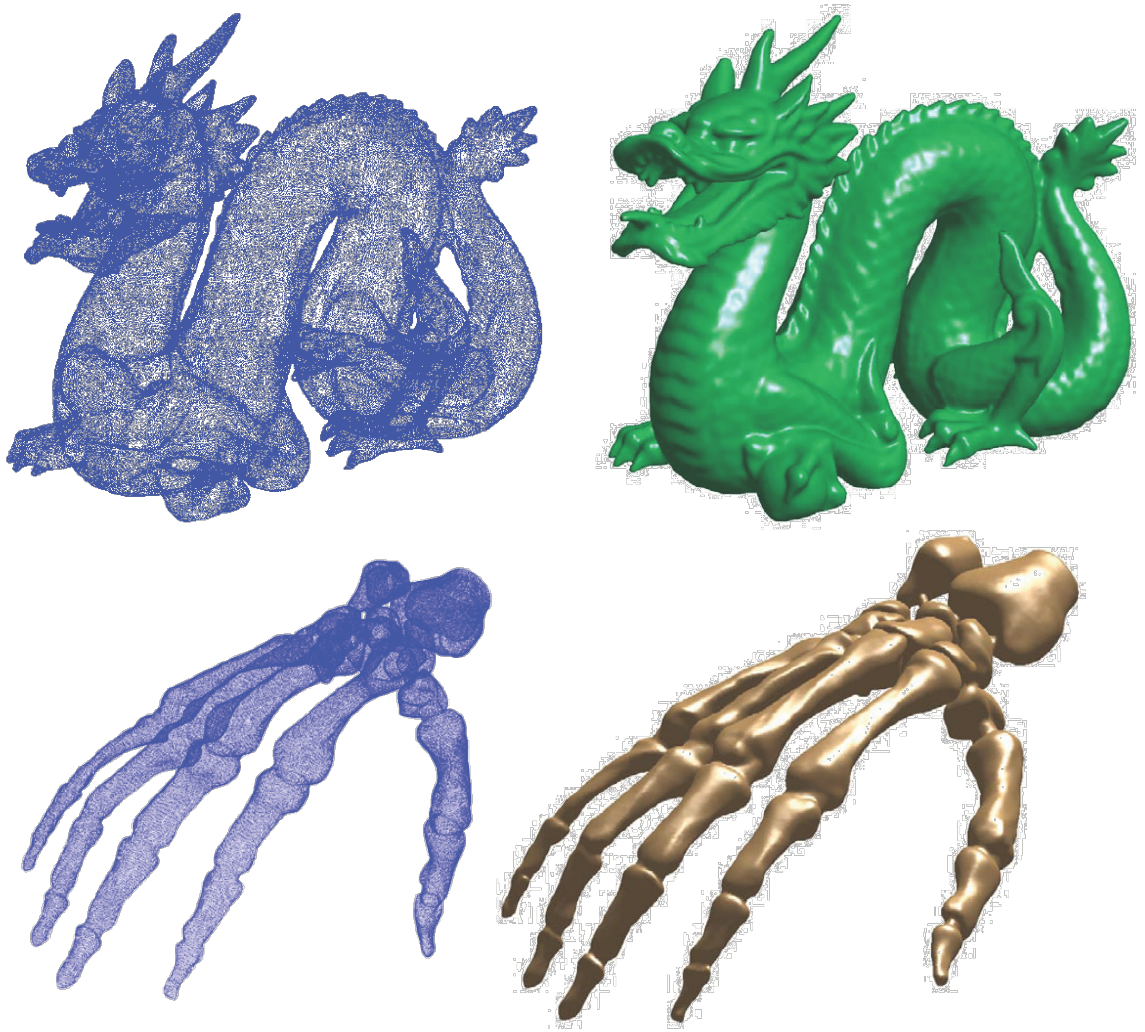
$$\varphi(r) = e^{-\beta r^2}$$

- B-Spline (compact support)

$$\varphi(r) = \text{piecewise-polynomial}(r)$$

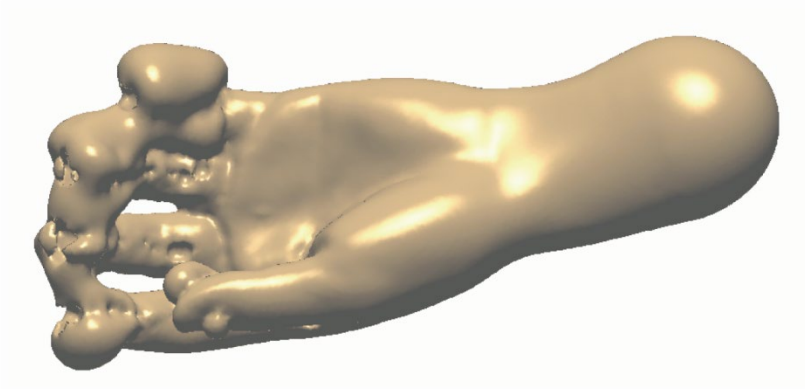


RBF Reconstruction Examples

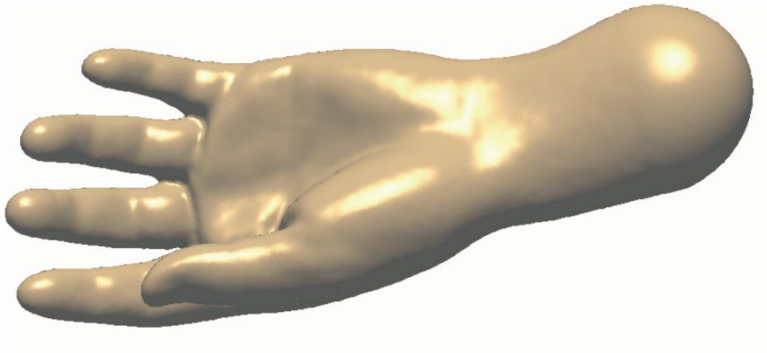


"Reconstruction and representation of 3D objects with radial basis functions", Carr et al., ACM SIGGRAPH 2001

Off-Surface Points



Insufficient number/
badly placed off-surface points



Properly chosen off-surface points

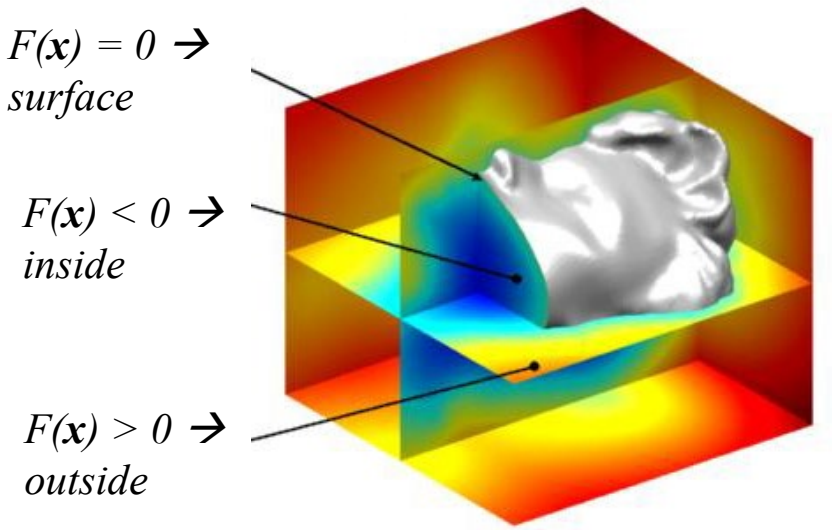
"Reconstruction and representation of 3D objects with radial basis functions", Carr et al., ACM SIGGRAPH 2001

IMPLICT → MESH

Marching Cubes

Extracting the Surface

- Wish to compute a manifold mesh of the level set

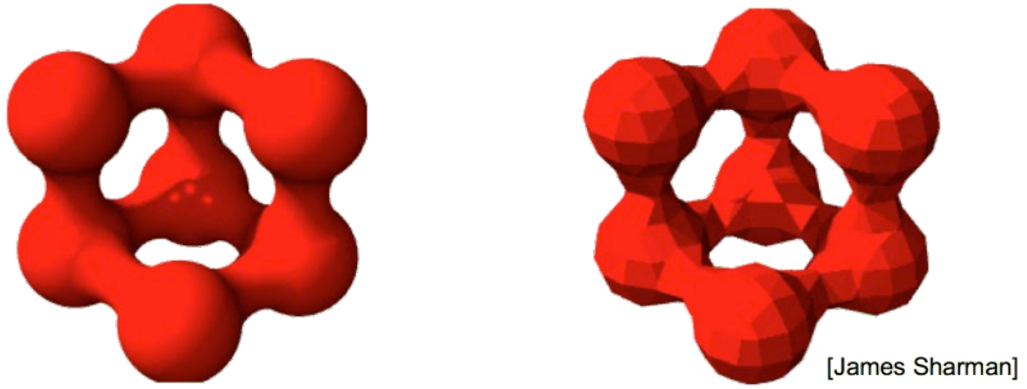


Marching Cubes

Converting from implicit to explicit representations.

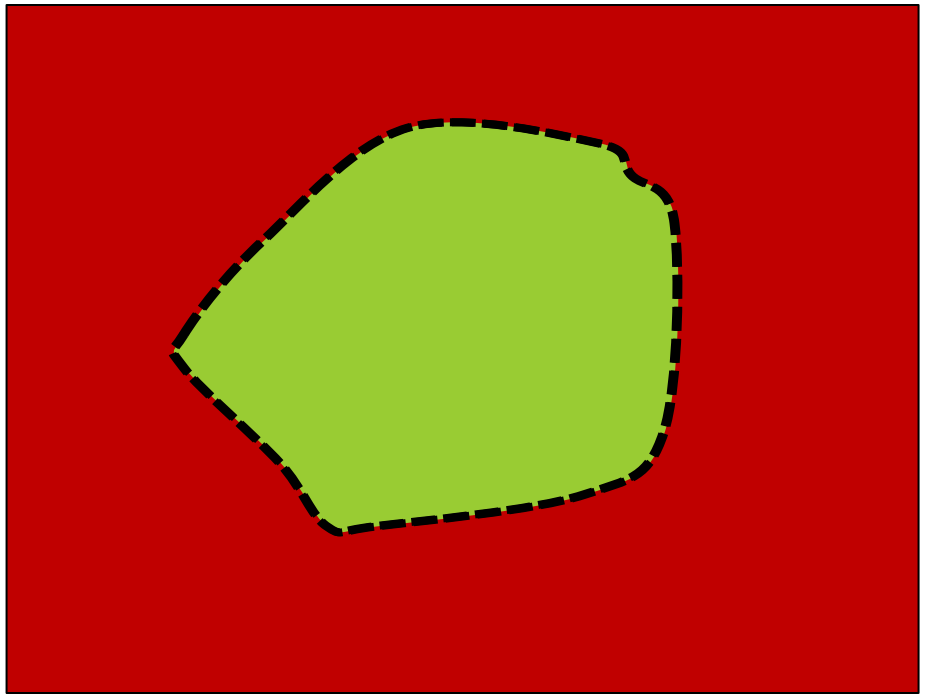
Goal: Given an implicit representation: $\{\mathbf{x}, \text{s.t.} f(\mathbf{x}) = 0\}$

Create a triangle mesh that approximates the surface.

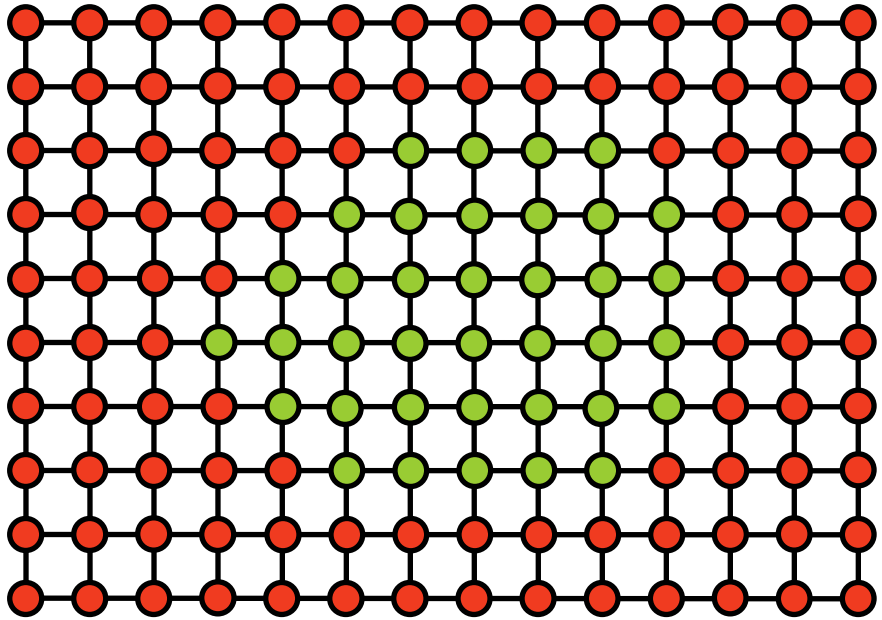


Lorensen and Cline, SIGGRAPH ‘87

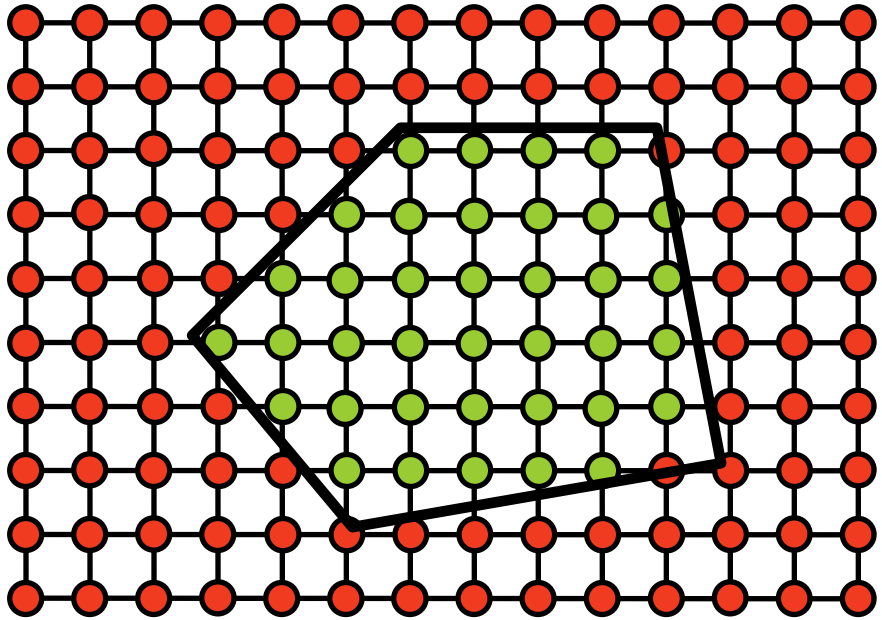
Sample the SDF



Sample the SDF



Sample the SDF

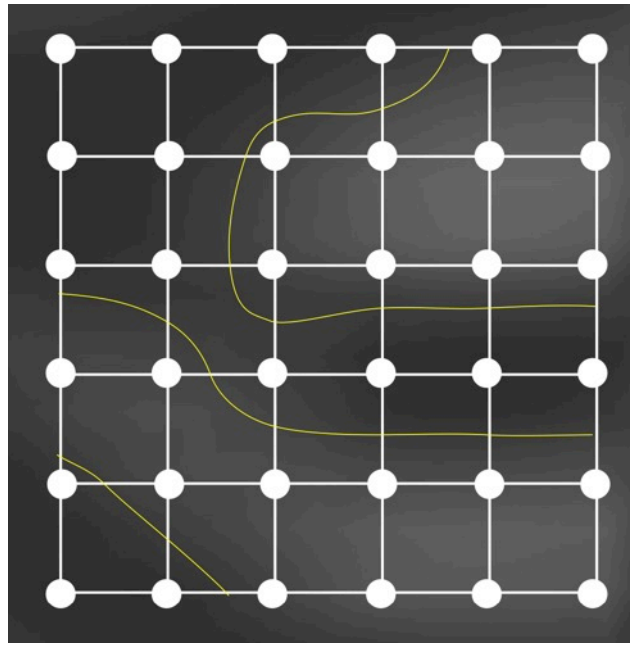


Marching Squares (2D)

Given a function: $f(x)$

- $f(\mathbf{x}) < 0$ inside
- $f(\mathbf{x}) > 0$ outside

1. Discretize space.
2. Evaluate $f(x)$ on a grid.

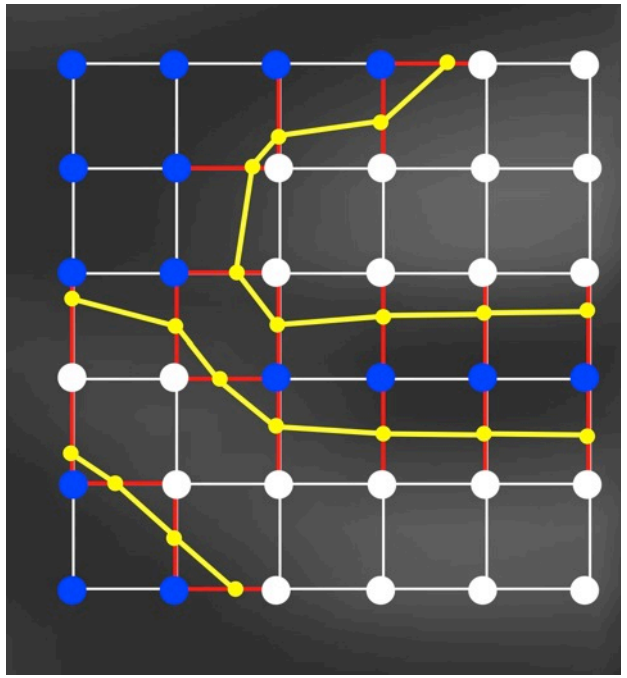


Marching Squares (2D)

Given a function: $f(x)$

- $f(\mathbf{x}) < 0$ inside
- $f(\mathbf{x}) > 0$ outside

1. Discretize space.
2. Evaluate $f(x)$ on a grid.
3. Classify grid points (+/-)
4. Classify grid edges
5. Compute intersections
6. Connect intersections



Marching Squares (2D)

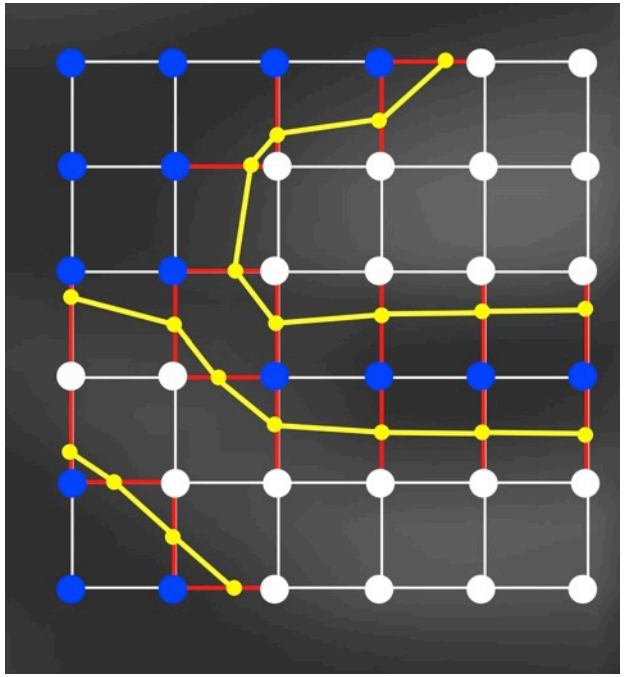
Computing the intersections:

- Edges with a sign switch contain intersections.

$$f(x_1) < 0, f(x_2) > 0 \Rightarrow$$
$$f(x_1 + t(x_2 - x_1)) = 0$$
$$\text{for some } 0 \leq t \leq 1$$

- Simplest way to compute t: assume f is linear between x1 and x2:

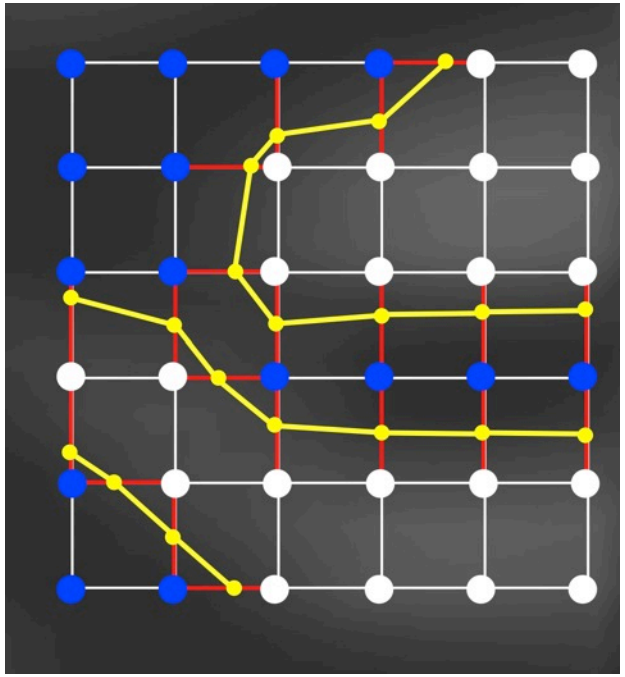
$$t = \frac{f(x_1)}{f(x_2) - f(x_1)}$$



Marching Squares (2D)

Connecting the intersections:

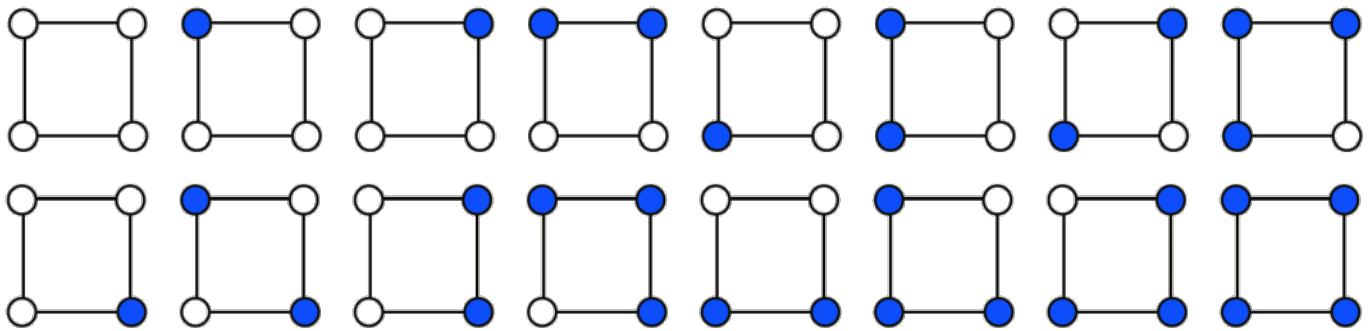
- Grand principle: treat each cell separately!
- Enumerate all possible inside/outside combinations.



Marching Squares (2D)

Connecting the intersections:

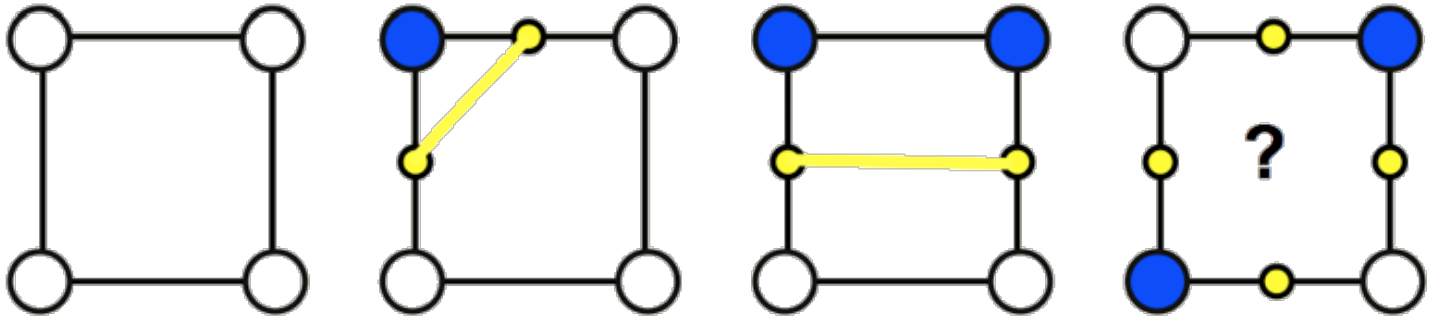
- Grand principle: treat each cell separately!
- Enumerate all possible inside/outside combinations.
- Group those leading to the same intersections



Marching Squares (2D)

Connecting the intersections:

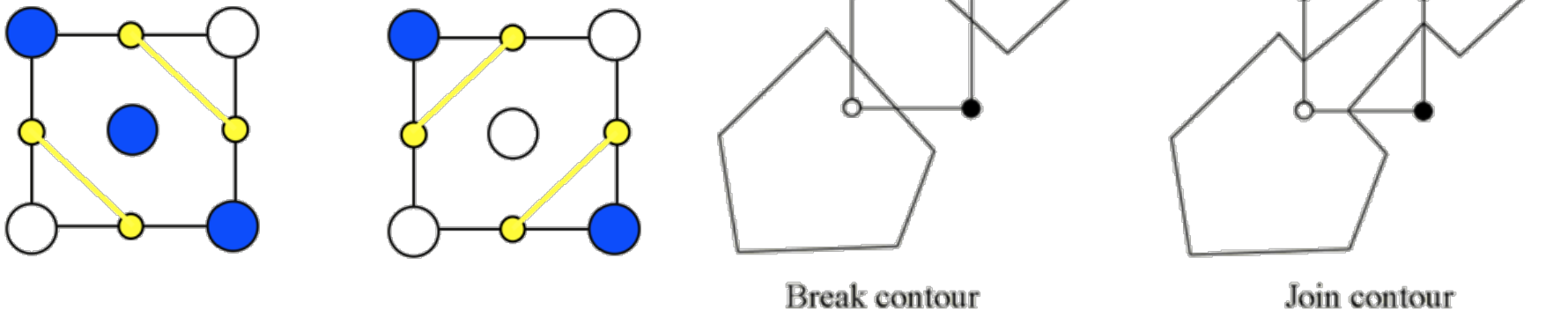
- Grand principle: treat each cell separately!
- Enumerate all possible inside/outside combinations.
- Group those leading to the same intersections.
- Group equivalent after rotation.
- Connect intersections



Marching Squares (2D)

Connecting the intersections:

Ambiguous cases:



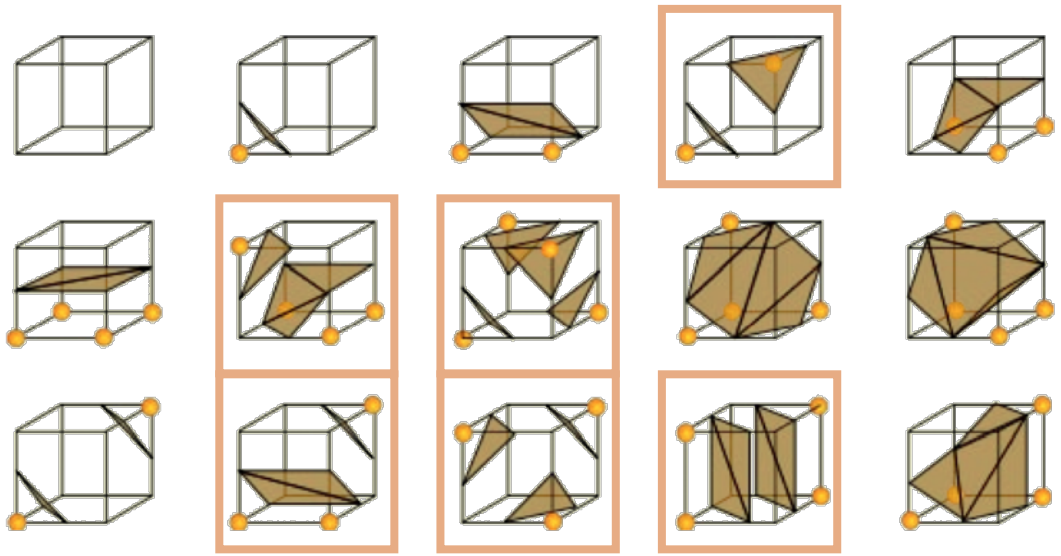
Two options:

1) Can resolve ambiguity by subsampling inside the cell.
2) If subsampling is impossible, pick one of the two possibilities.

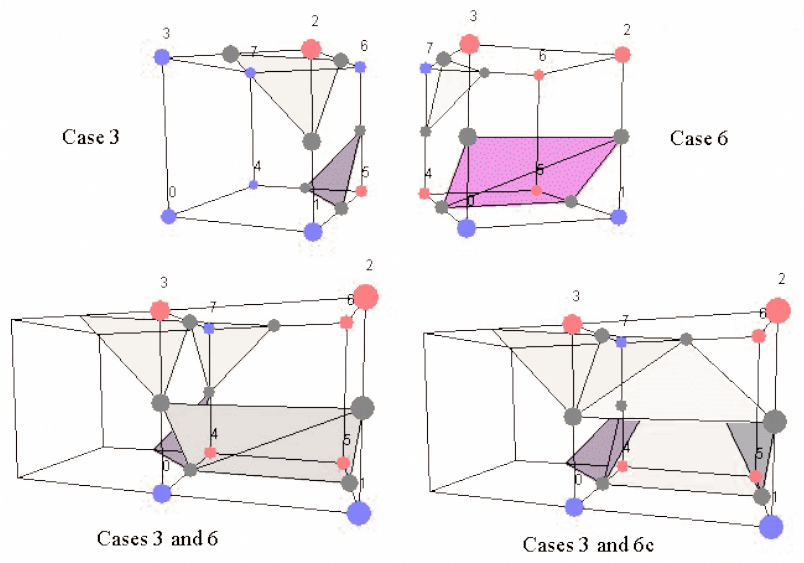
Marching Cubes (3D)

Same machinery: cells → **cubes** (voxels), lines → triangles

- 256 different cases - 15 after symmetries, 6 ambiguous cases
- More subsampling rules → 33 unique cases



the 15 cases



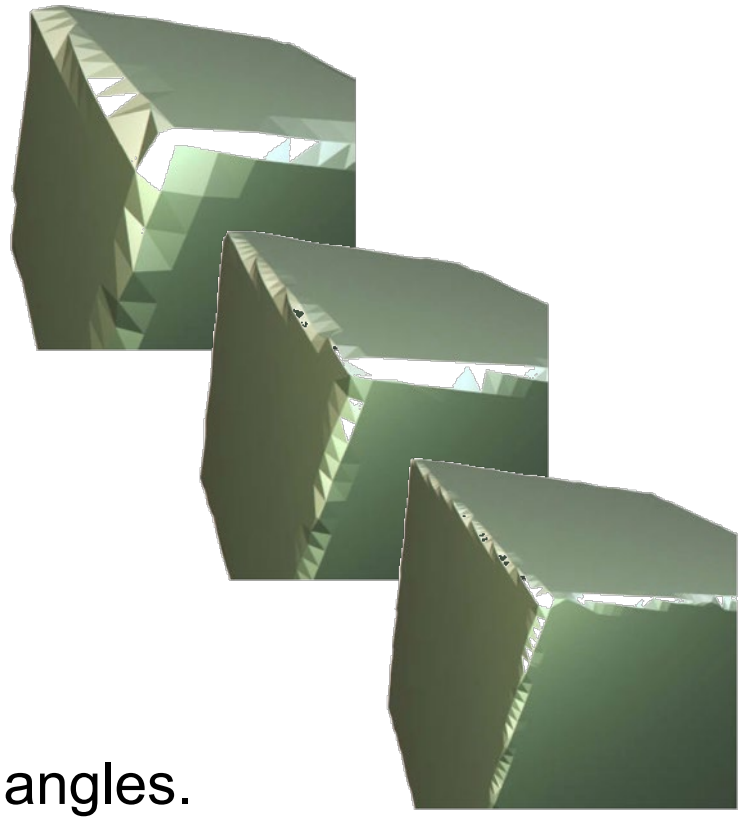
explore ambiguity to avoid holes!

Chernyaev, Marching Cubes 33,'95

Marching Cubes (3D)

Main Strengths:

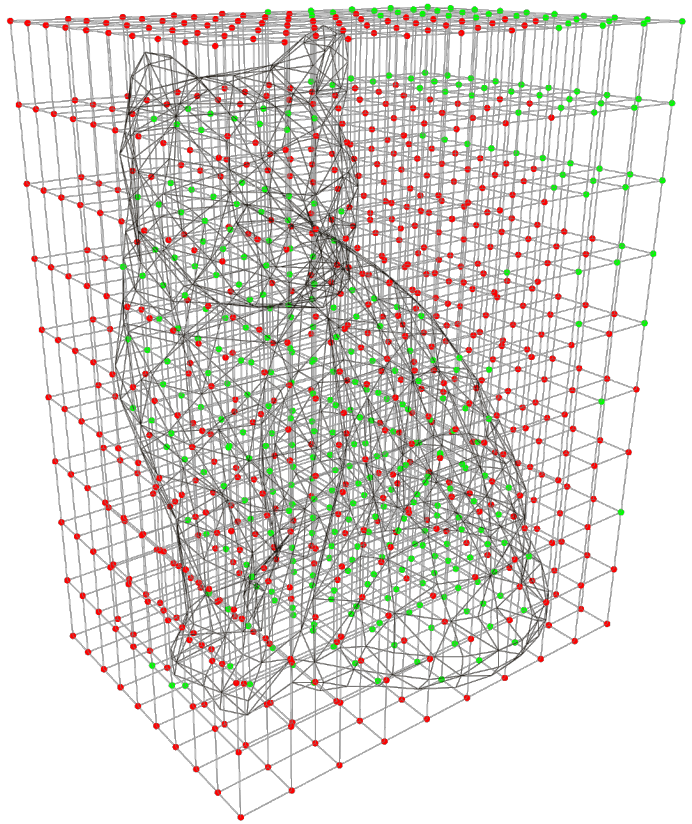
- Very multi-purpose.
- Extremely fast and parallelizable.
- Relatively simple to implement.
- Virtually parameter-free

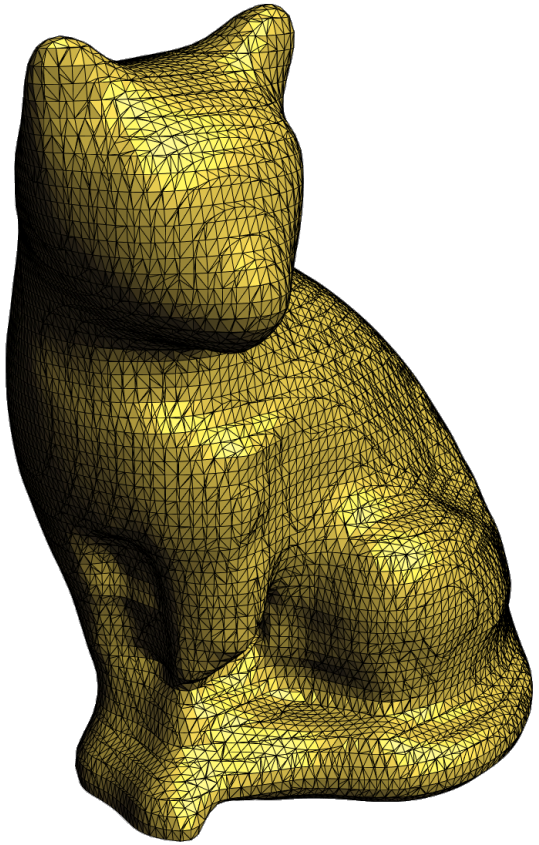


Main Weaknesses:

- Can create badly shaped (skinny) triangles.
- Many special cases (implemented as big lookup tables).
- No sharp features.

Recap: Points→Implicit→Mesh



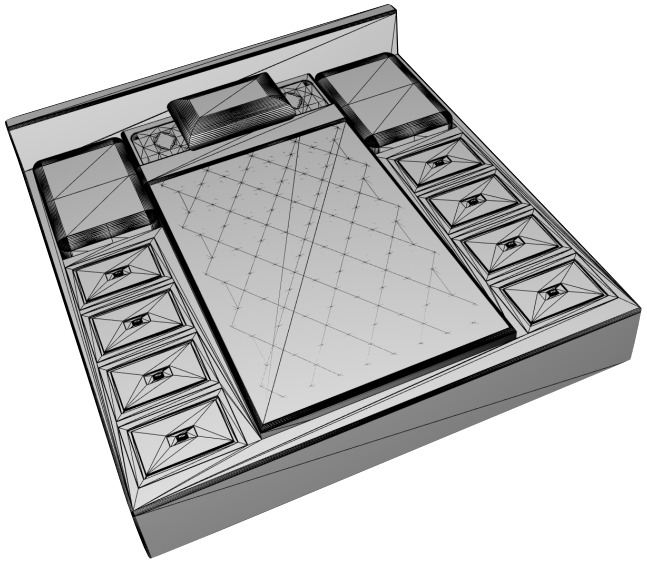


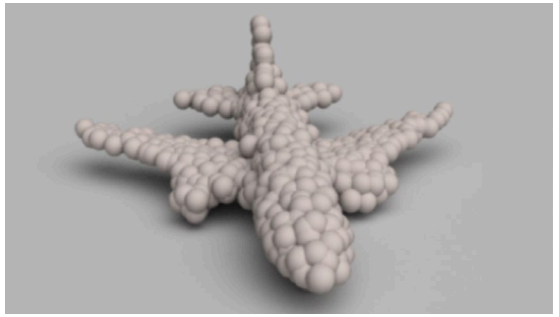
MESH-> POINT CLOUD

Sampling

From Surface to Point Cloud Why?

- Points are simple but expressive!
  - Few points can suffice
- Flexible, unstructured, few constraints
- Also: ML applications!





CAD meshes:
many components
bad triangles
connectivity problems

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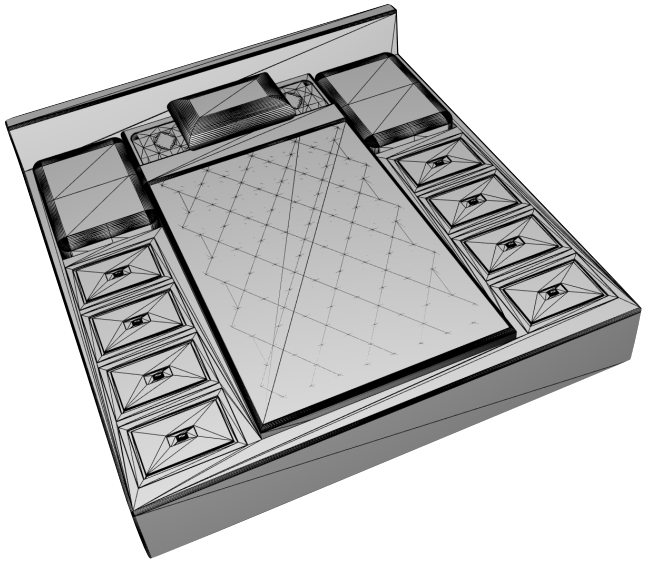

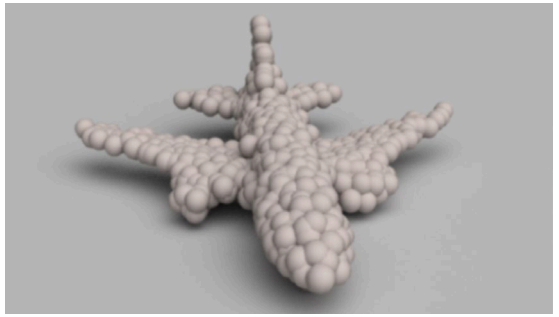

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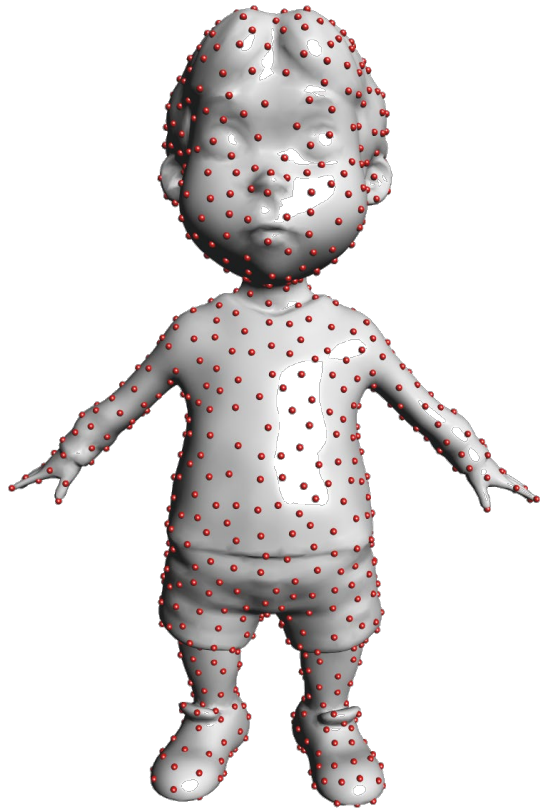
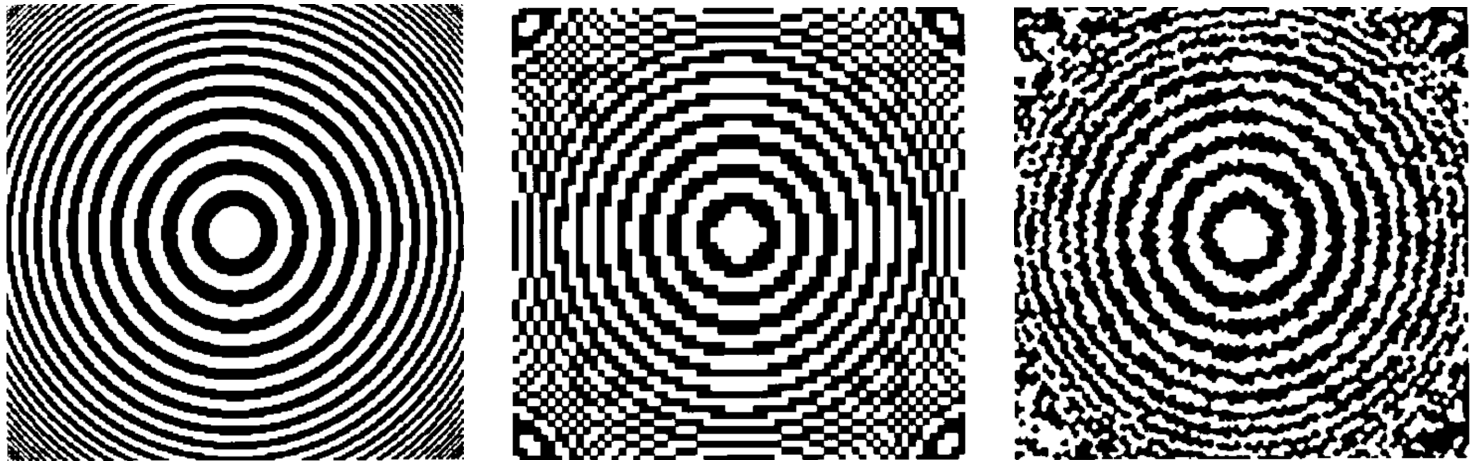


the problem:
sampling the mesh

Farthest Point Sampling

- Introduced for progressive transmission/acquisition of images
- Quality of approximation improves with increasing number of samples
  - as opposed eg. to raster scan
- Key Idea: repeatedly place next sample in the middle of the least-known area of the domain.



Gonzalez 1985, “Clustering to minimize the maximum intercluster distance”
Hochbaum and Shmoys 1985, “A best possible heuristic for the k-center problem”

Pipeline

1.Create an initial sample point set *S*

- Image corners + additional random point.

2. Find the point which is the farthest from all point in *S*

$$d(p,S) = \max_{q \in A}(d(q,S))$$

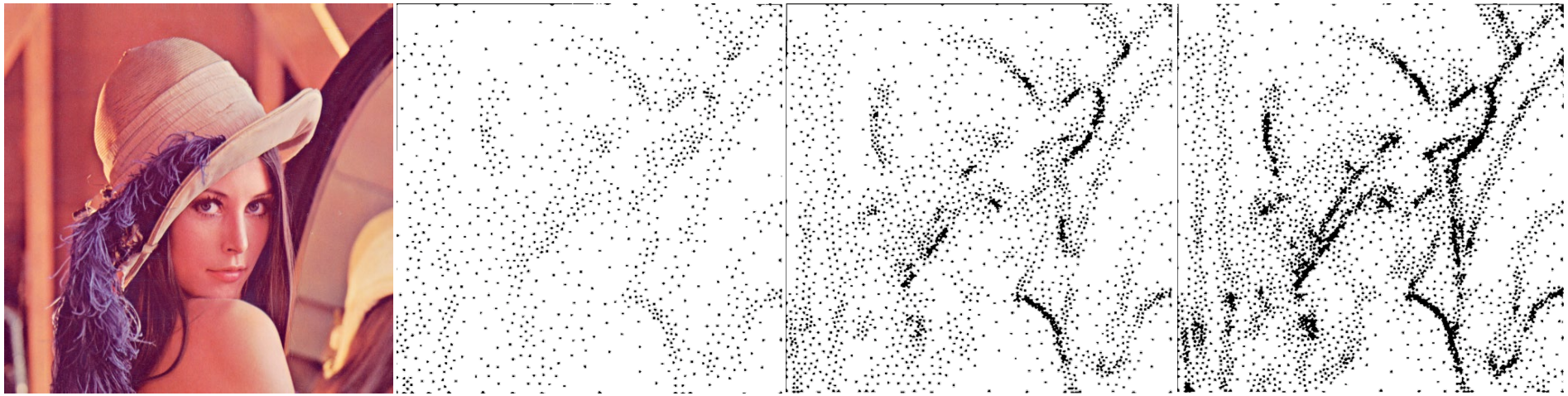
$$= \max_{q \in A}\left(\min_{0 \le i < N}(d(q,s_i))\right)$$

3. Insert the point to *S* and update the distances

4. While more points are needed, iterate

Farthest Point Sampling

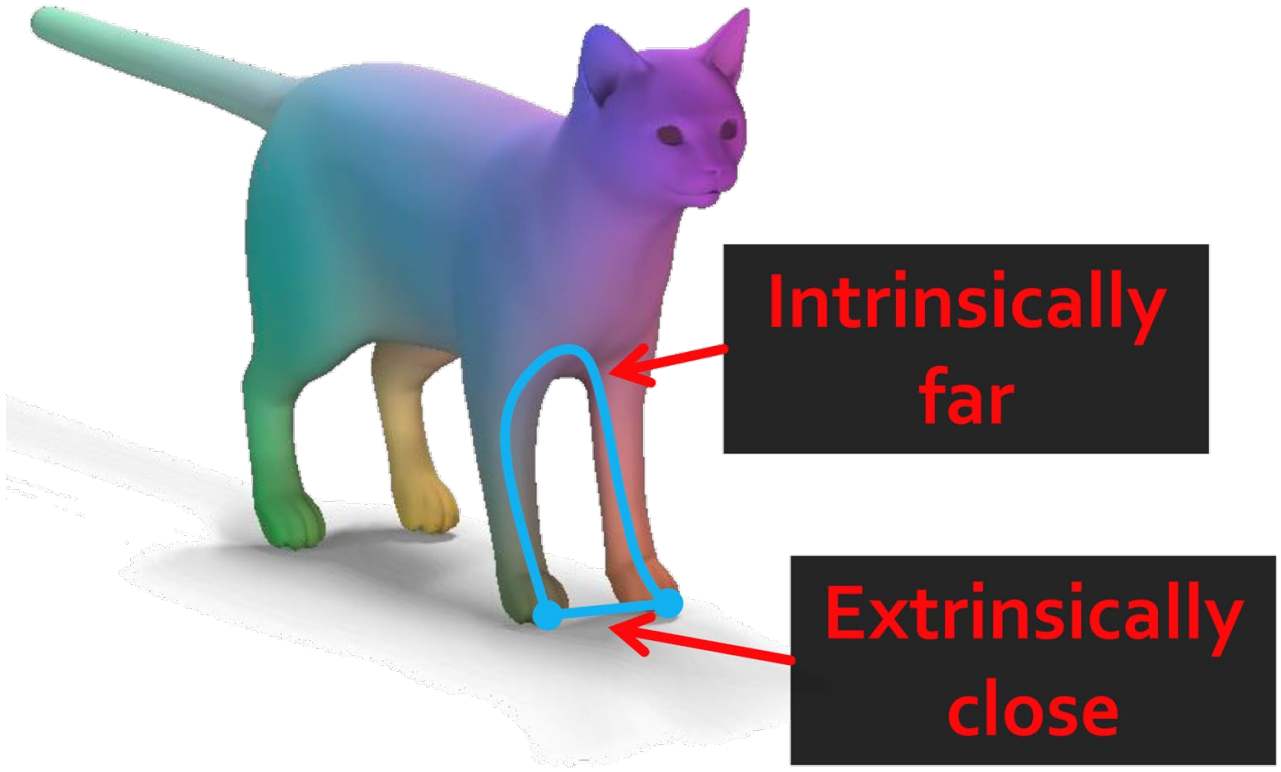
- Depends on a notion of distance on the sampling domain
- Can be made adaptive, via a weighted distance

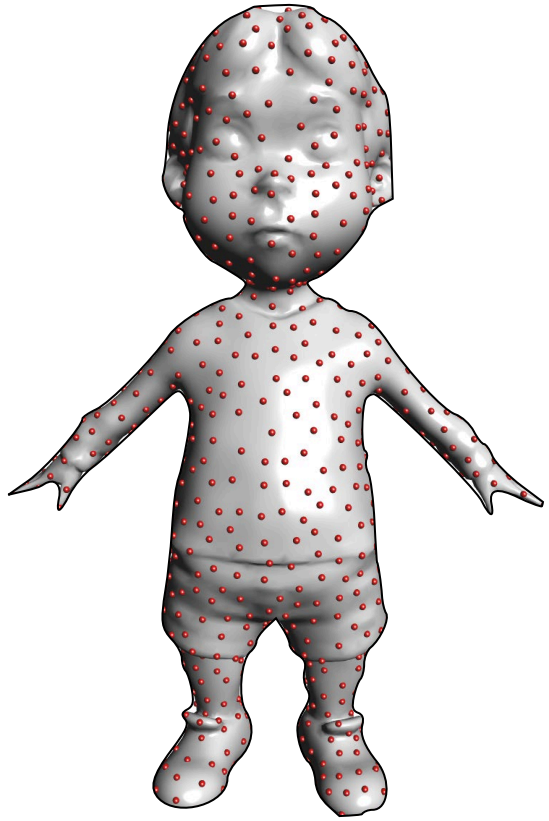


Eldar et al. 1997, “The Farthest Point Strategy for Progressive Image Sampling”

FPS on surfaces

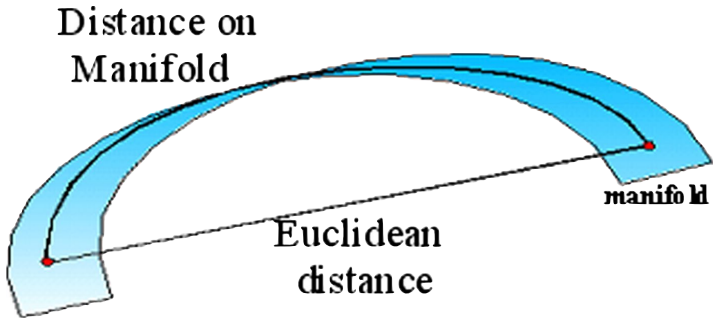
- What's an appropriate distance?

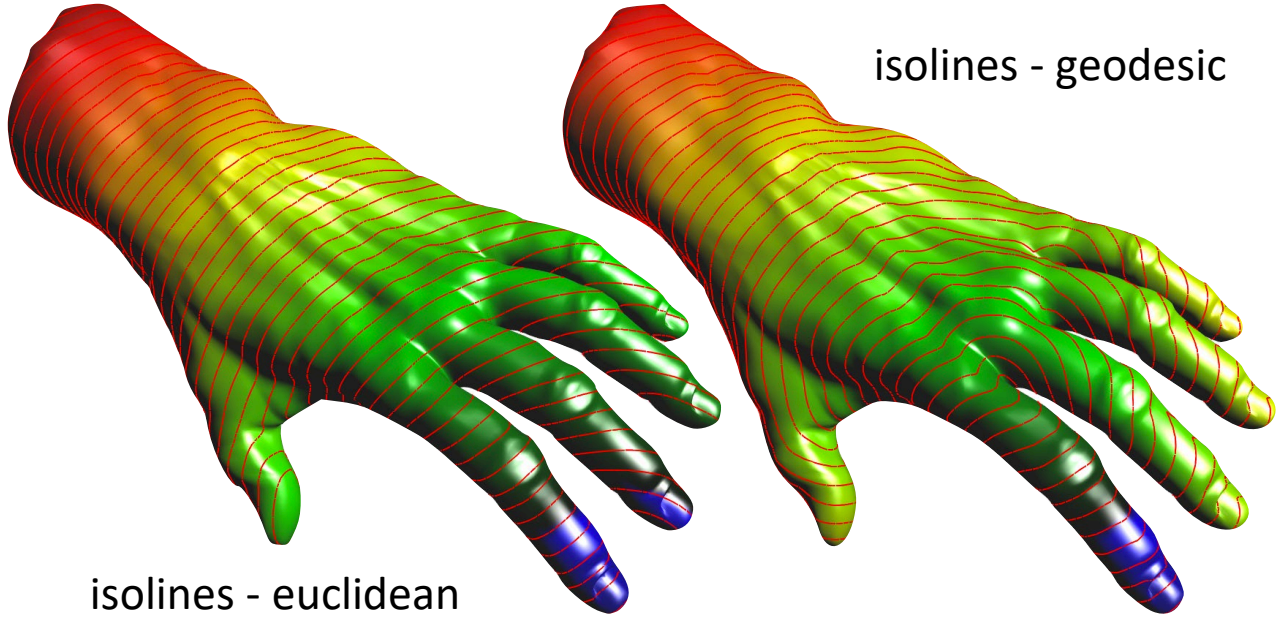




On-Surface Distances

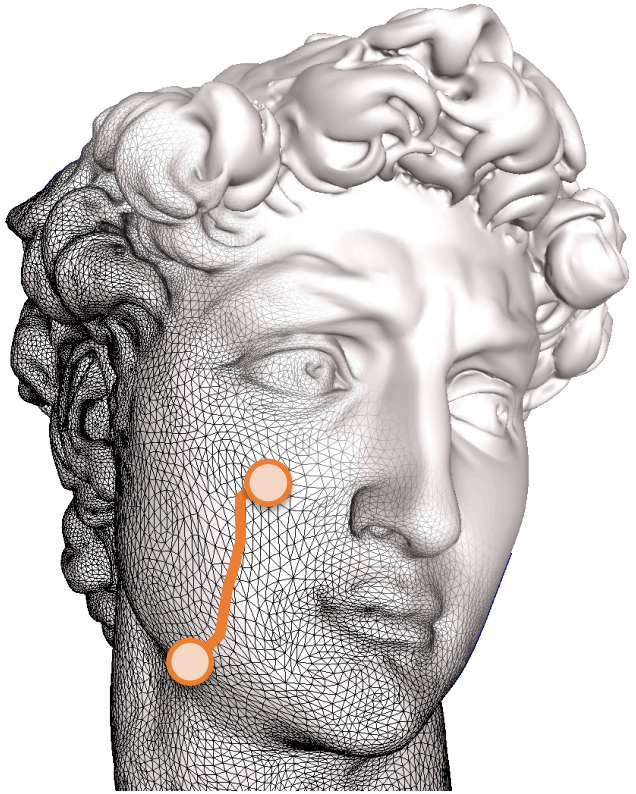
- Geodesics: Straightest and **locally shortest** curves





Discrete Geodesics

- Recall: a mesh is a graph!
- Approximate geodesics as paths along edges

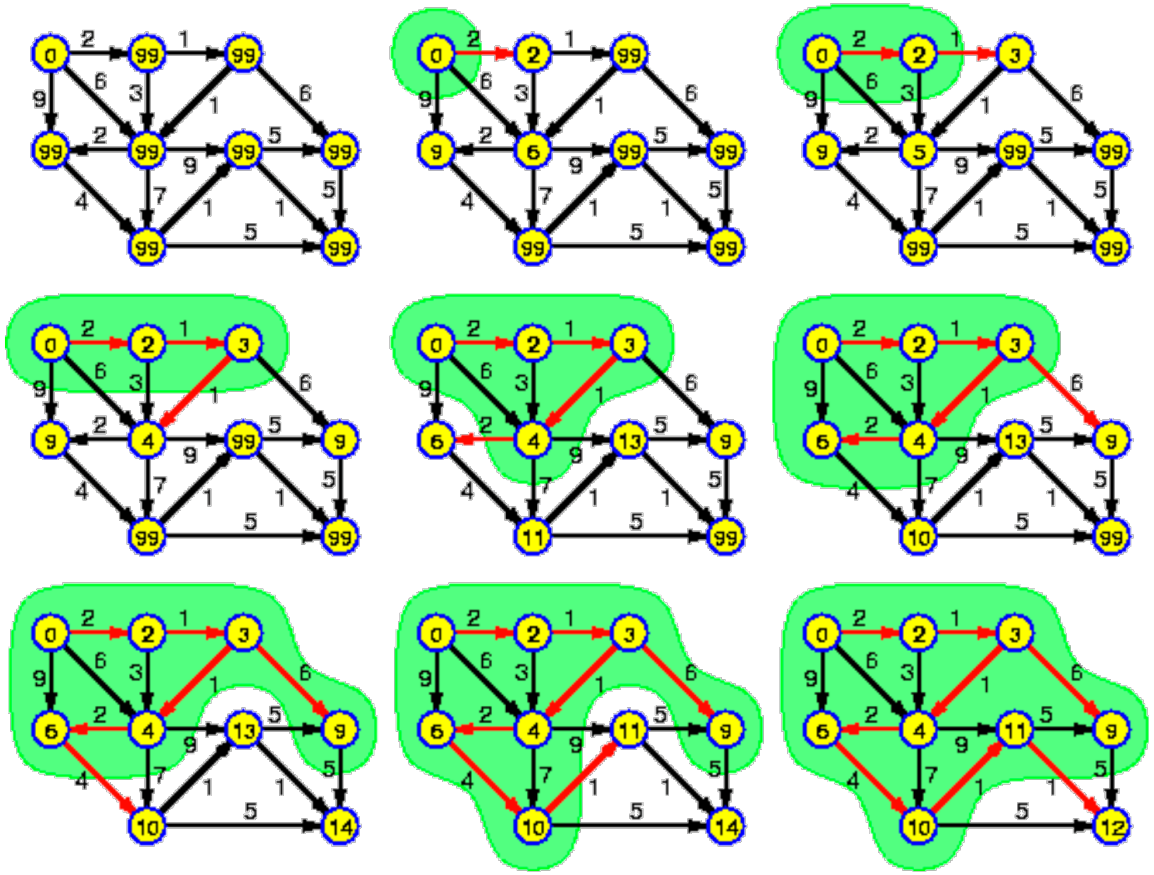


```
v0= initial vertex
di = current distance to vertex i
S = verticies with known optimal distance

# initialize
d0 = 0
di = [inf for d in di]
S = {}

for each iteration k:
    # update
    k = argmin(dk), for vk not in S
    S.append(vk)
    for neighbors index vl of vk:
        dl = min([dl,dk + dkl])
```

Dijkstra's algorithm!

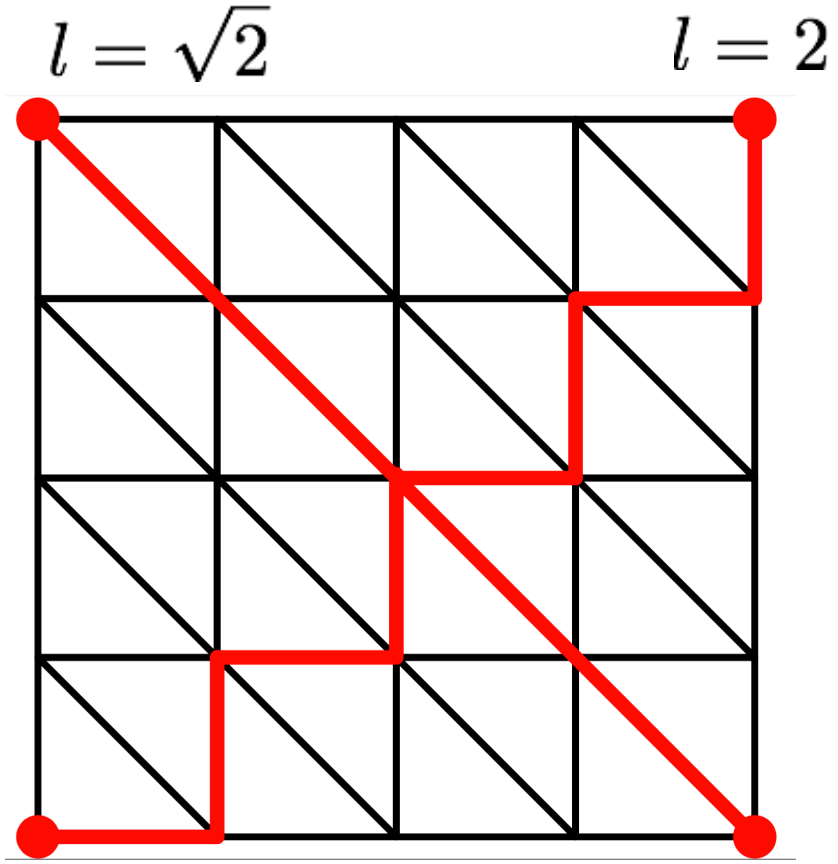
Dijkstra Geodesics

- Dikjstra as wave front propagation



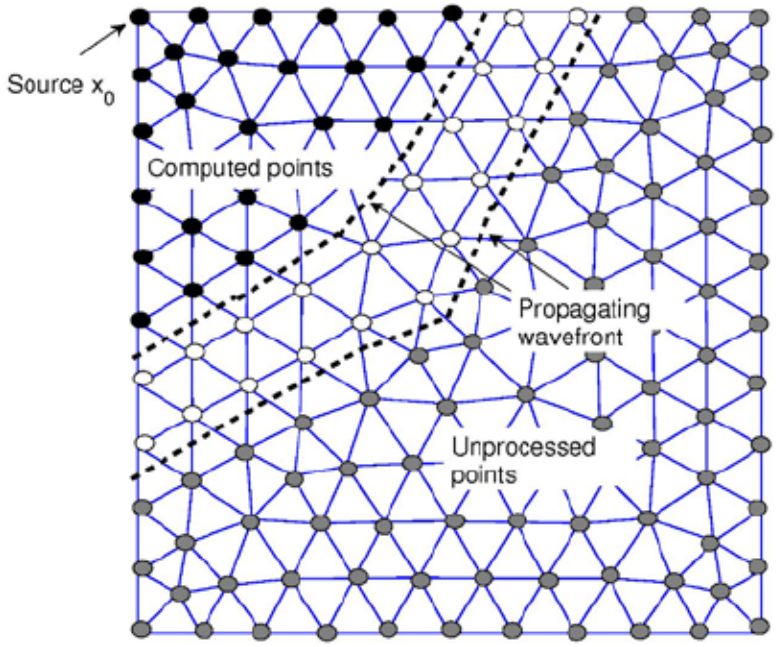
Dijkstra Geodesics

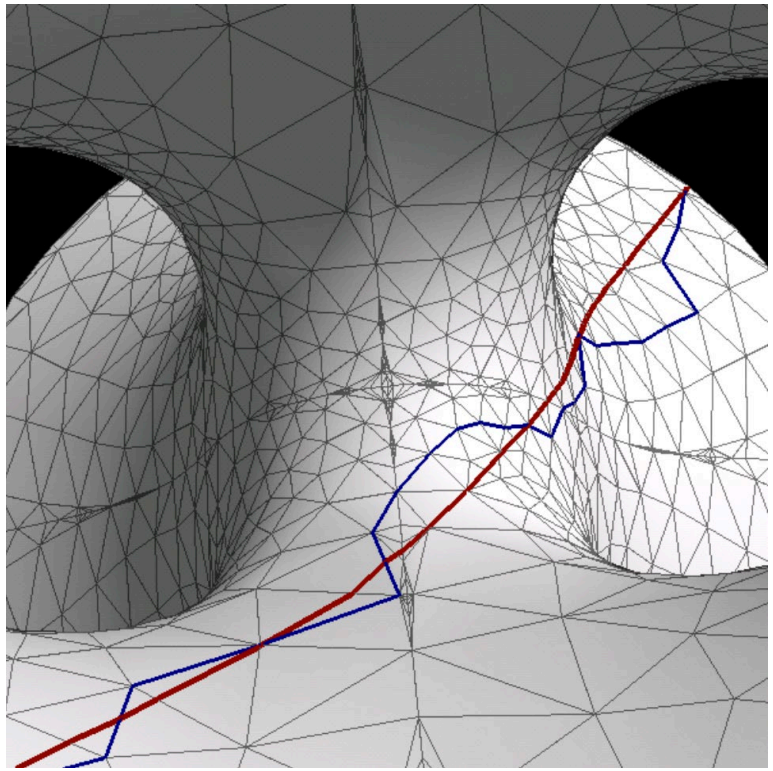
Can be asymmetric - no matter how fine the mesh!



Fast Marching Geodesics

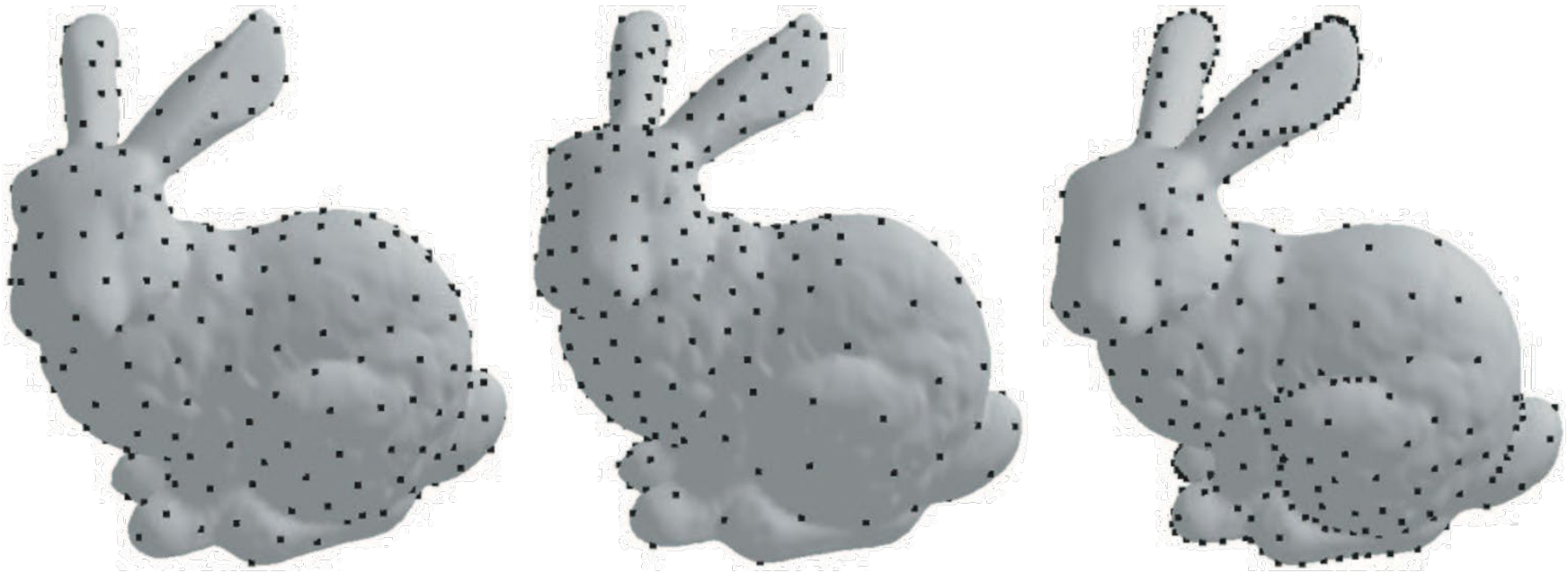
- A better approximation: allow fronts to cross triangles!





Kimmel and Sethian 1997, “Computing Geodesic Paths on Manifolds”

FPS on a Mesh



Peyré and Cohen 2003, Geodesic Remeshing Using Front Propagation

Faster Distance Approximations

Carne, Weischedel, and Wardetzky 2017,
The Heat Method for Distance Computation

Software

- Libigl http://libigl.github.io/libigl/tutorial/tutorial.html
  - MATLAB-style (flat) C++ library, based on indexed face set structure
- OpenMesh www.openmesh.org
  - Mesh processing, based on half-edge data structure
- CGAL www.cgal.org
  - Computational geometry
- MeshLab http://www.meshlab.net/
  - Viewing and processing meshes

Software

- Alec Jacobson's GP toolbox
  - https://github.com/alecjacobson/gptoolbox
  - MATLAB, various mesh and matrix routines
- Gabriel Peyre's Fast Marching Toolbox
  - https://www.mathworks.com/matlabcentral/fileexchange/6110-toolbox-fast-marching
  - On-surface distances (more next time!)
- OpenFlipper https://www.openflipper.org/
  - Various GP algorithms + Viewer

That’s All