CS233, CME251: Geometric and Topological Data Analysis

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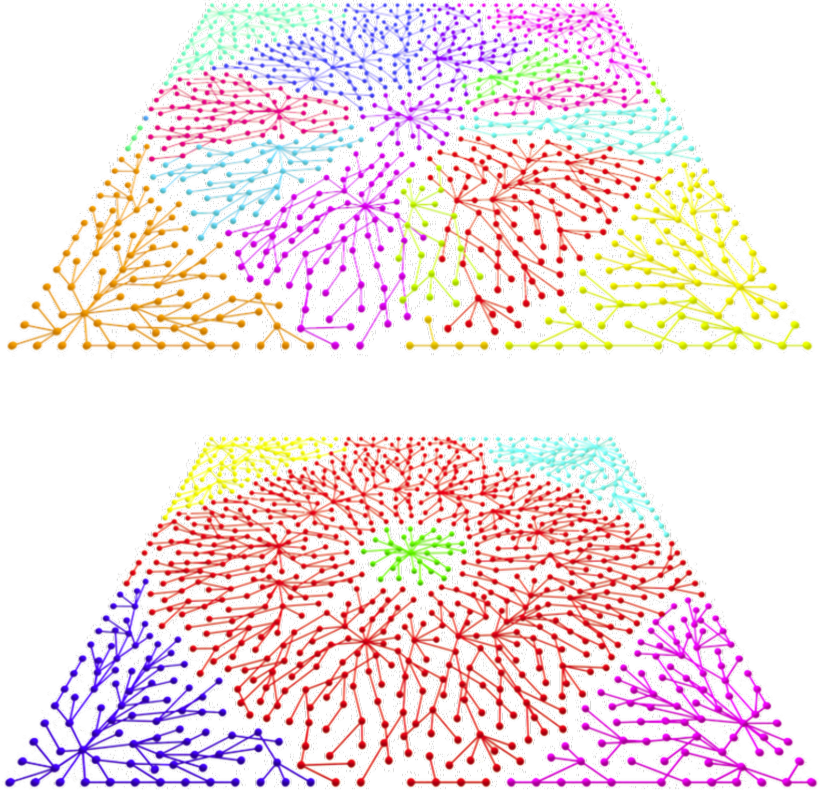
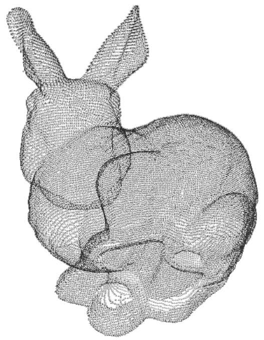


Lecture 11
3 May 2021

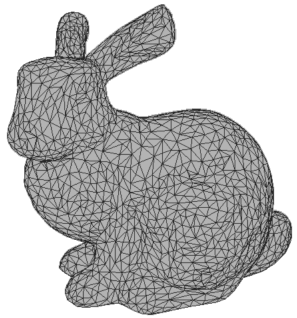
Leonidas Guibas Laboratory
Geometric Computing

Last Time: Geometry Representations in 3D

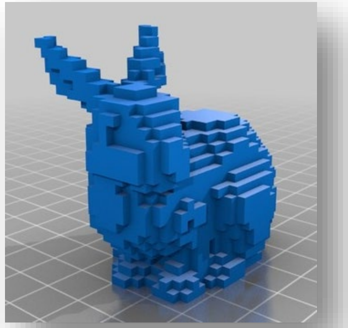
In 3D There is Representation Diversity



Point Cloud



Mesh



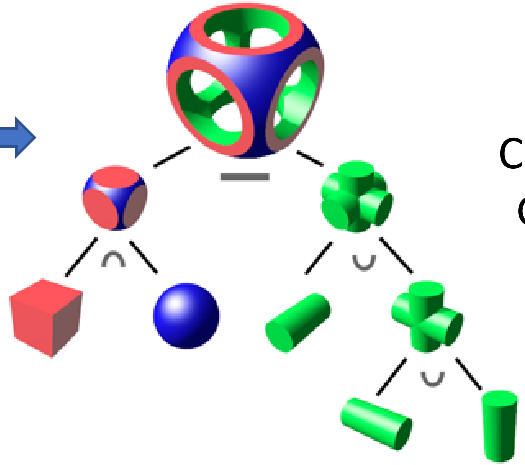
Voxels

...

Multiple Projected Views

RGB(D)

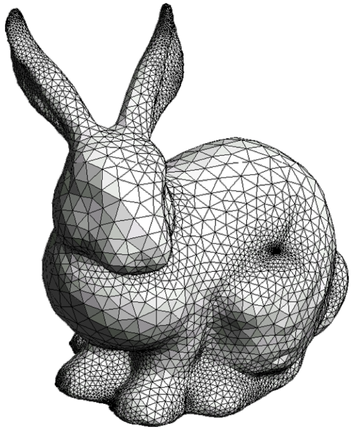
These are irregular representations

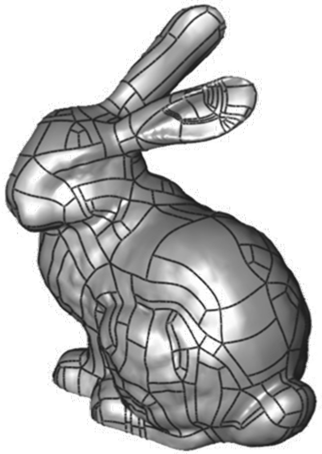


Constructive Solid Geometry (CSG)

3D: Boundary or Volumetric?

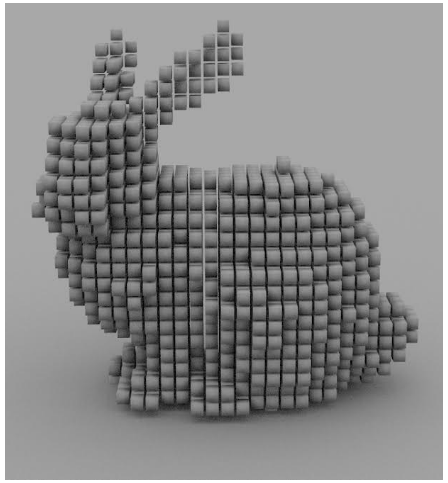
- B(oundary)-Reps

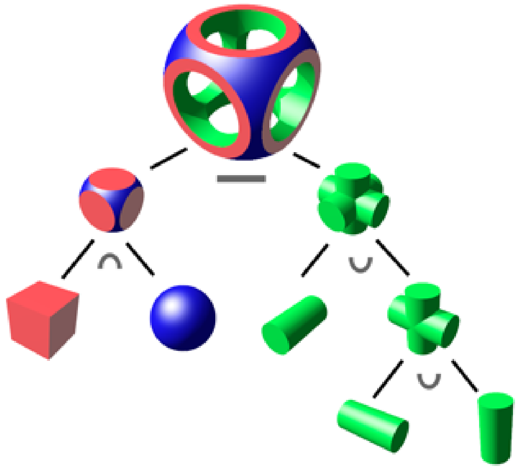




  - more efficient
  - closer to semantics, support local editing

- V(olume)-Reps

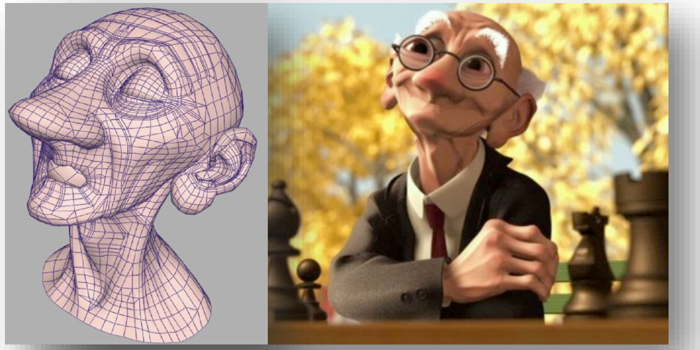


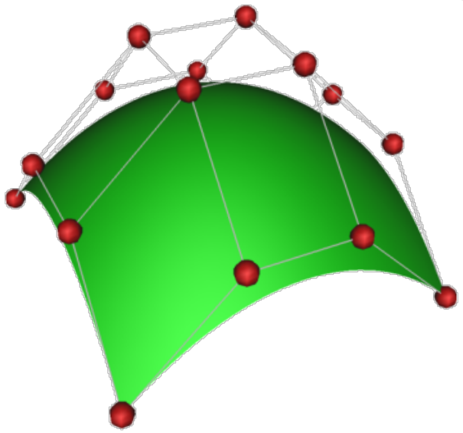


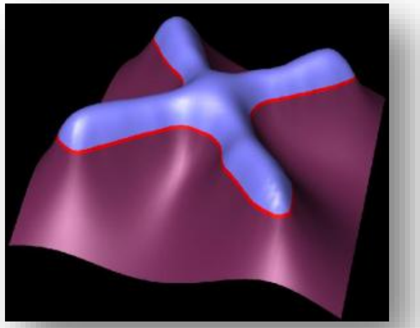
  - more regular
  - support unions and intersections

High-Level: Boundary-Reps

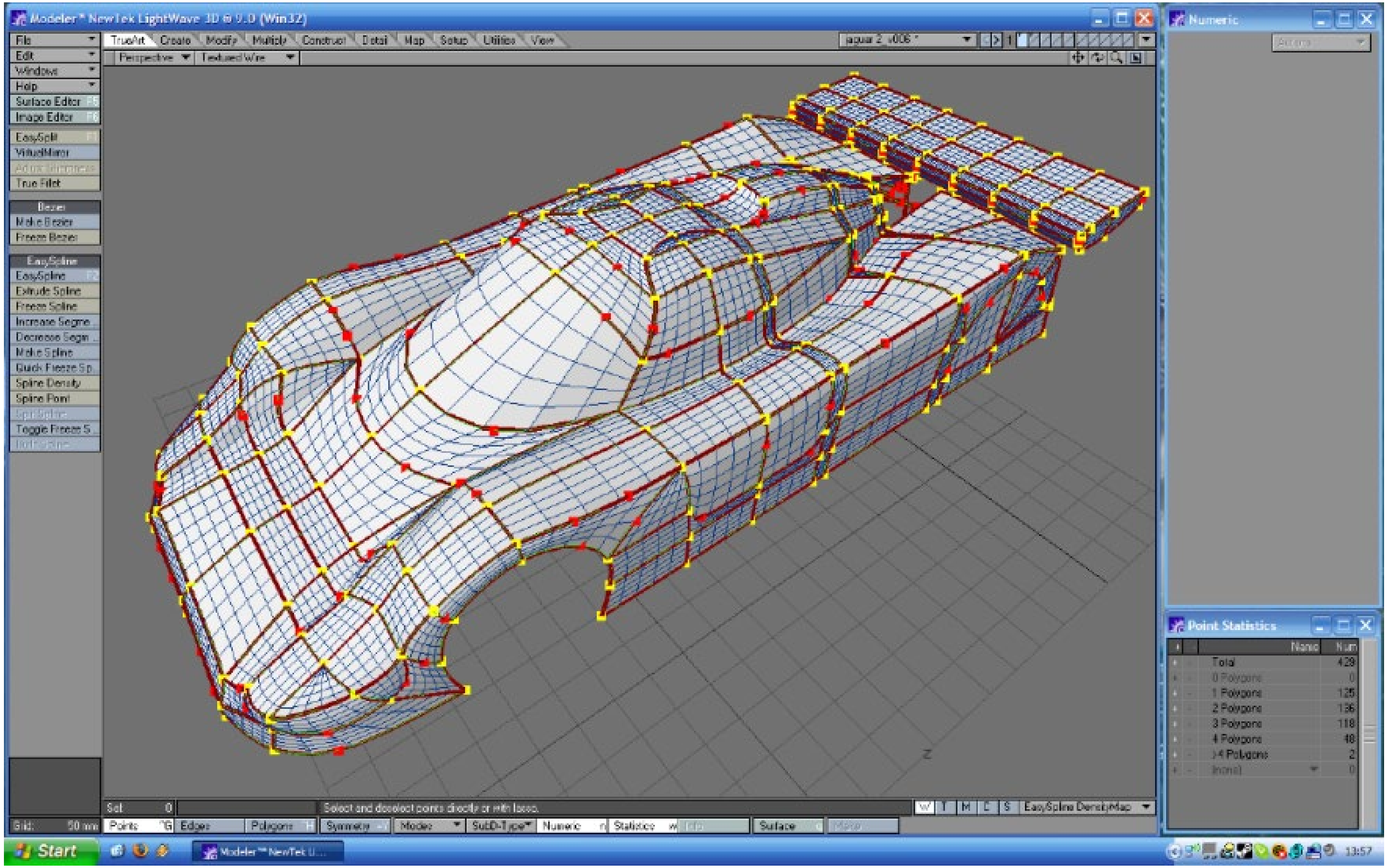
- Parametric surfaces / Splines
- Implicit functions
- Subdivision surfaces





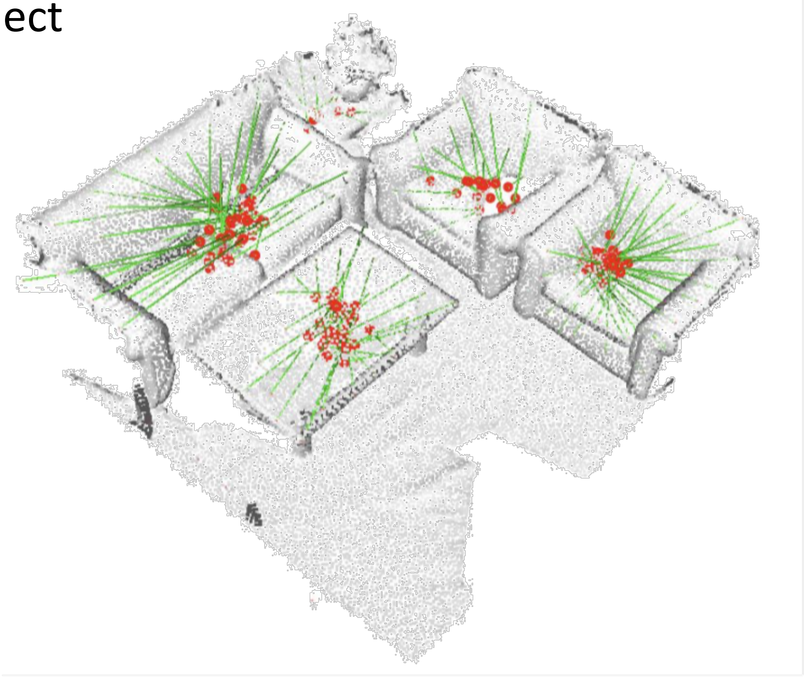


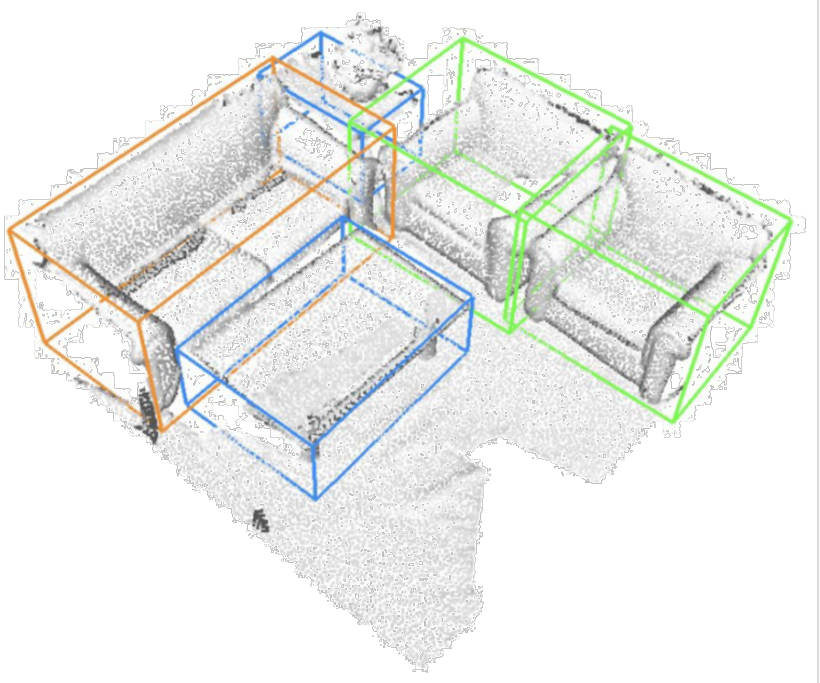
CAGD: Computer-Aided Geometric Design



Low-Level: Point Cloud Data

Microsoft Kinect

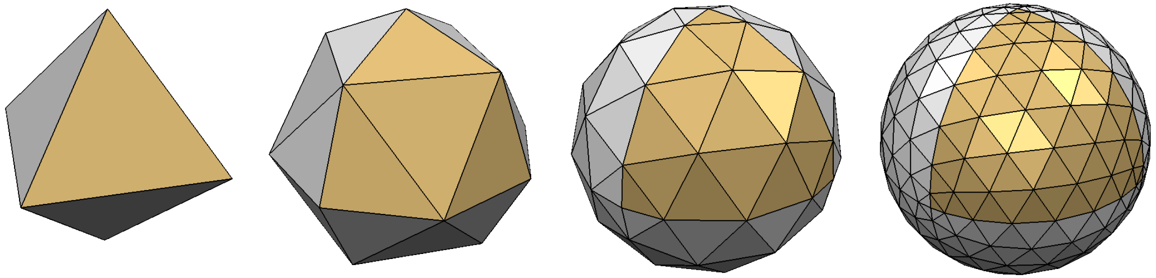


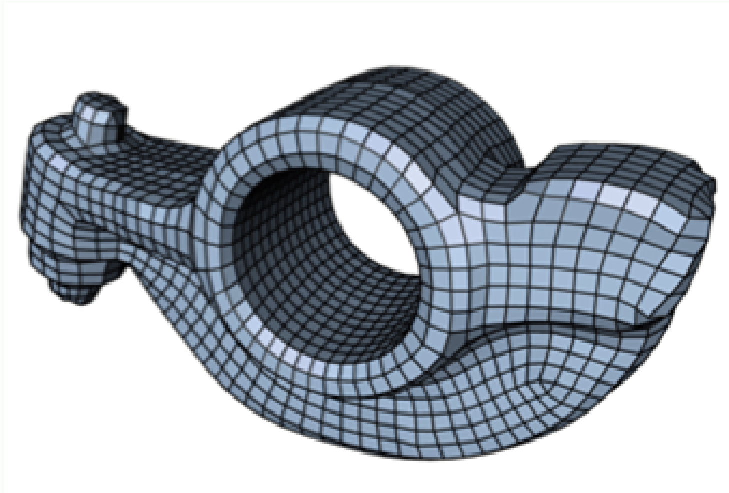


Intel RealSense

Low-Level: Polygonal Meshes

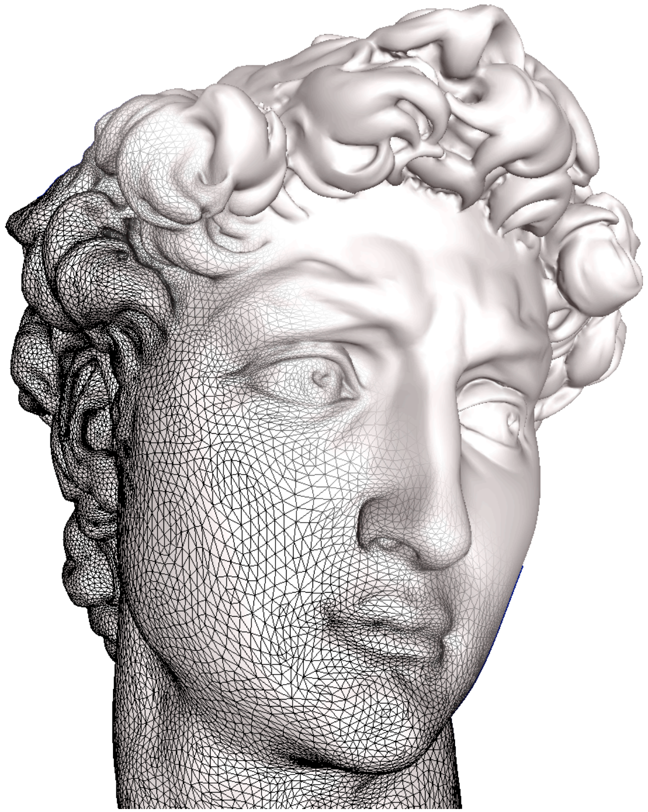
- Boundary representations of objects using polygonal primitives





Data Structures

- What should be stored?
  - Geometry: 3D coordinates
  - Connectivity
    - Adjacency relationships
  - Attributes
    - Normal, color, texture coordinates
    - Per vertex, face, edge



Representation Conversions

Points → Implicit
Implicit → Mesh
Mesh → Points

POINTS → IMPLICIT

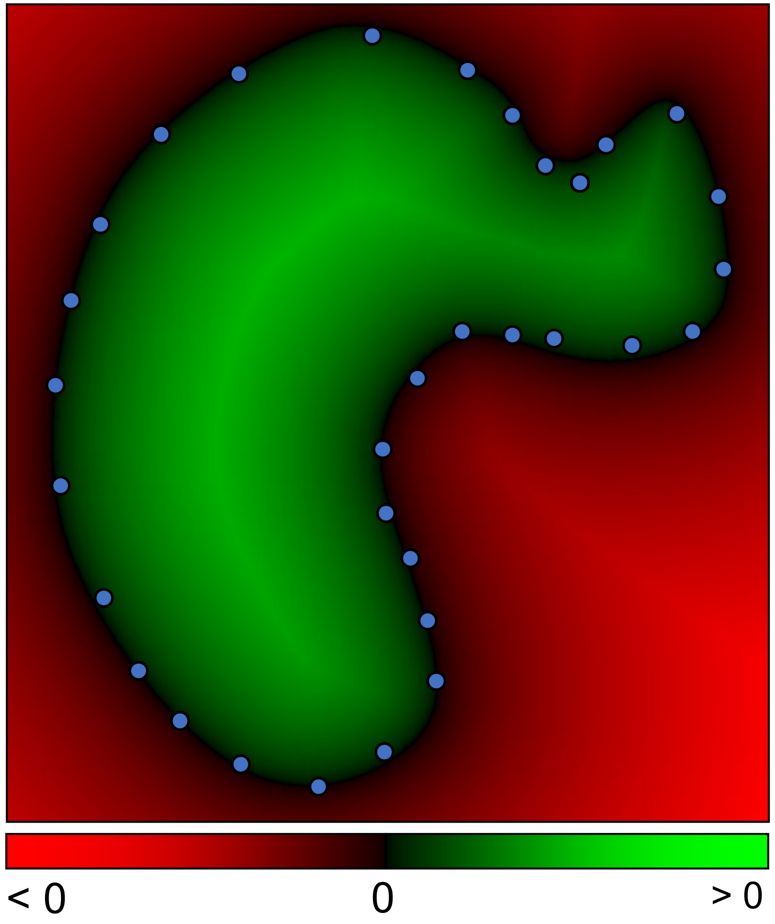
Implicit Surface Reconstruction

Implicit Function Approach

- Given a point cloud

- Define a function $f : R^3 \rightarrow R$

  with value > 0 outside the shape and
  < 0 inside



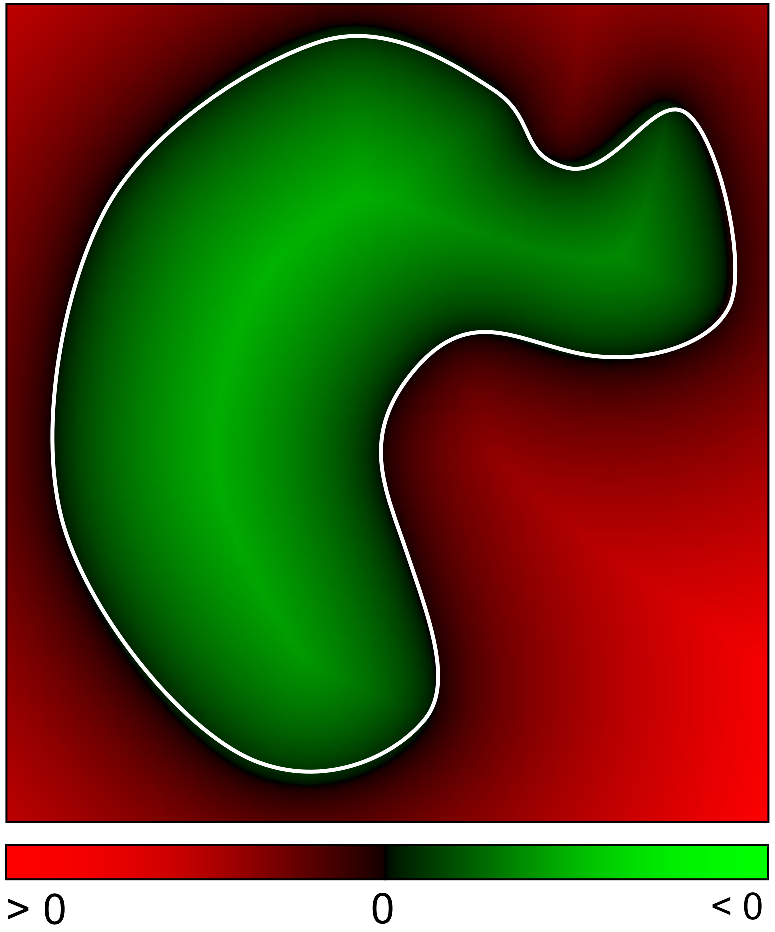
Implicit Function Approach

- Define a function $f : R^3 \rightarrow R$

  with value > 0 outside the shape and < 0 inside

- Extract the zero-set

$$\{x : f(x) = 0\}$$

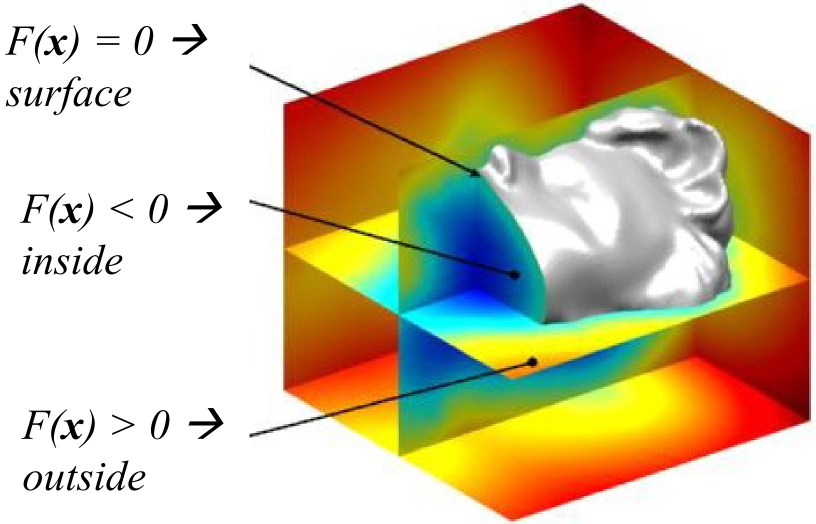


IMPLICT → MESH

Marching Cubes

Extracting the Surface

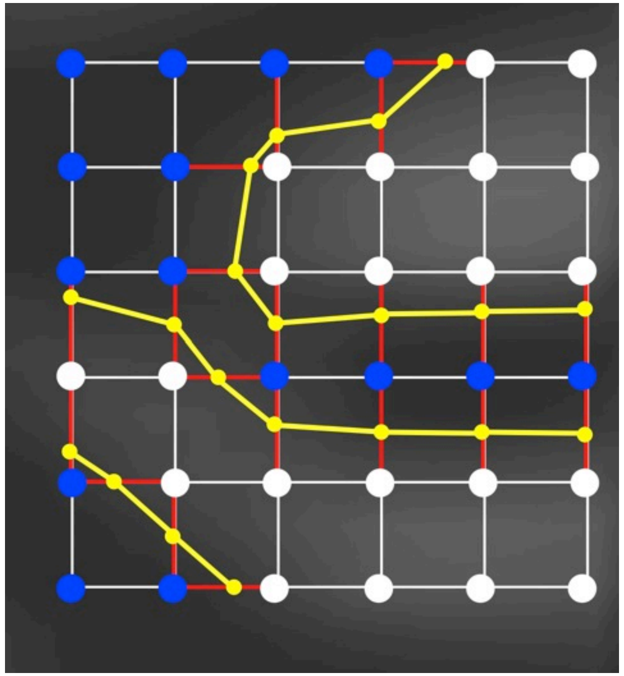
- Wish to compute a manifold mesh of the level set



Marching Squares (2D)

Given a function: $f(x)$

- $f(\mathbf{x}) < 0$ inside
- $f(\mathbf{x}) > 0$ outside



1. Discretize space.
2. Evaluate $f(x)$ on a grid.
3. Classify grid points (+/-)
4. Classify grid edges
5. Compute intersections
6. Connect intersections

Marching Squares (2D)

Given a function: $f(x)$

- $f(\mathbf{x}) < 0$ inside
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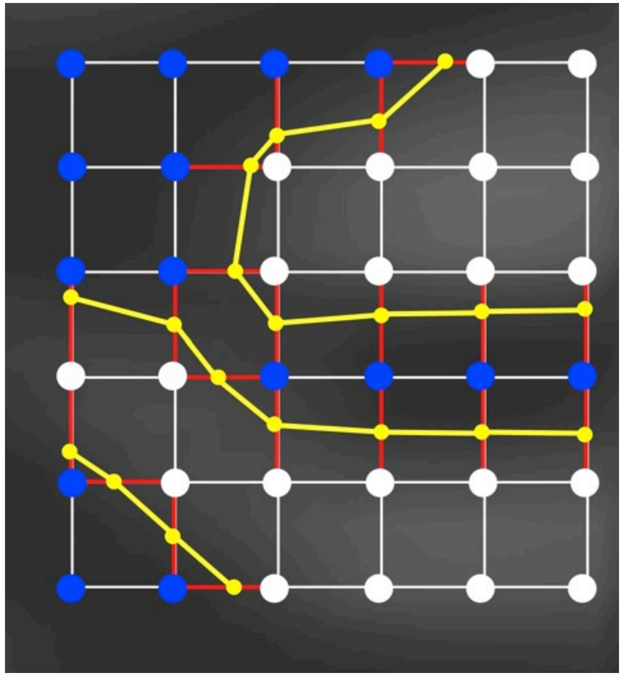

1. Discretize space.
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3. Classify grid points (+/-)
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5. Compute intersections
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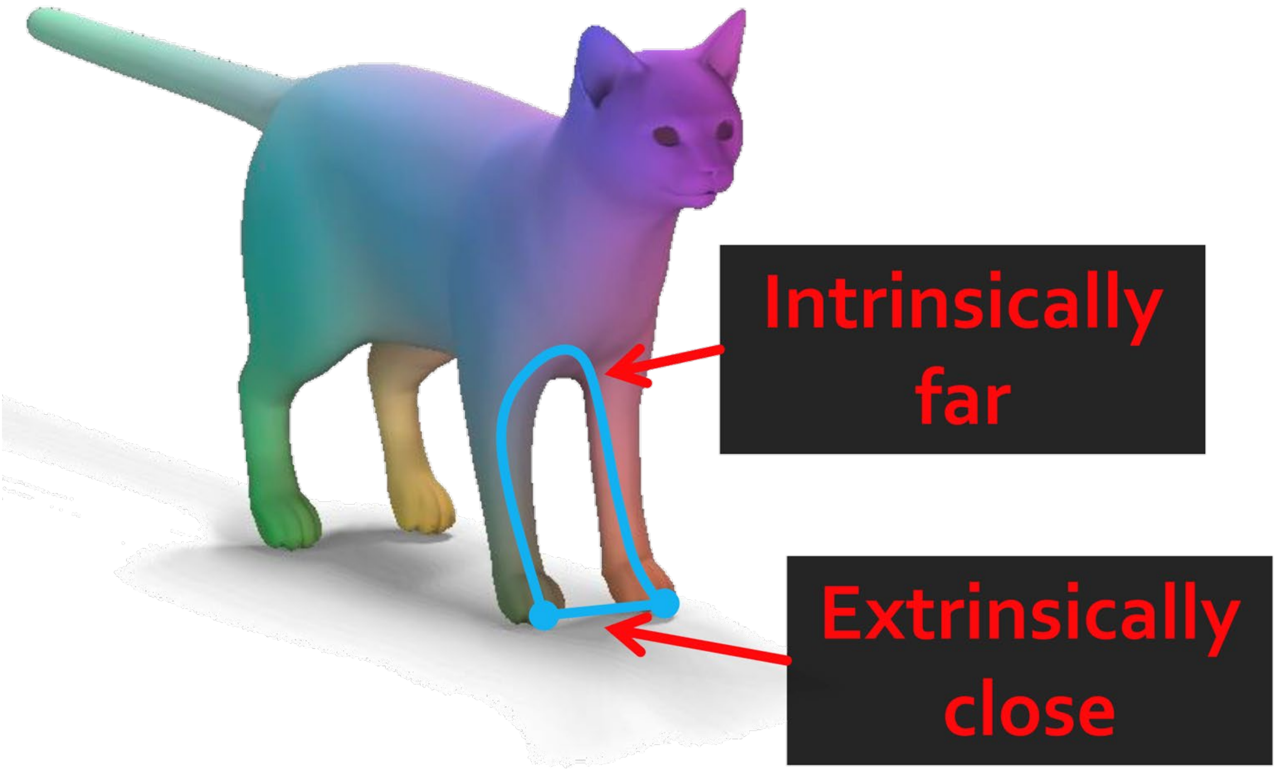
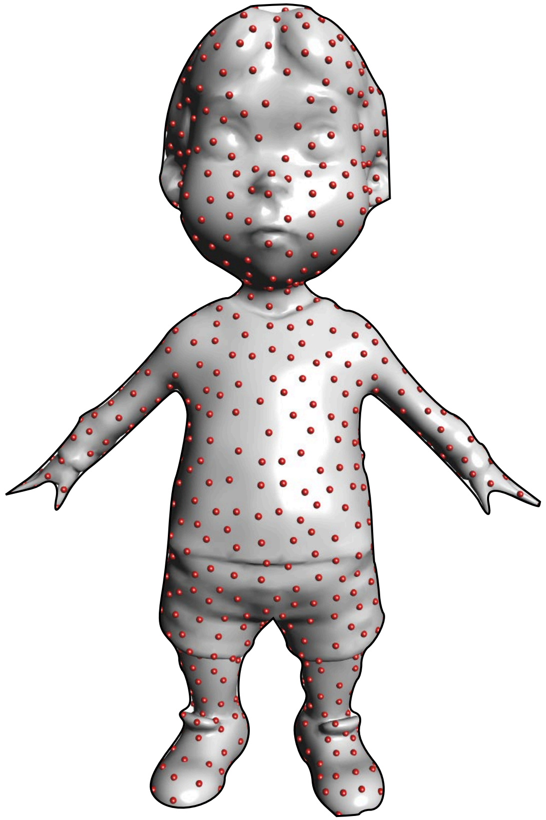
MESH-> POINT CLOUD

Sampling

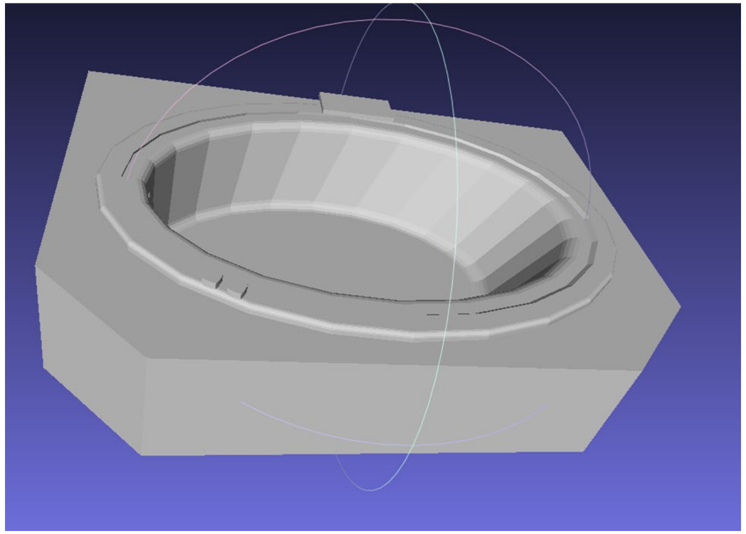
Furthest Point Sampling on surfaces

- What's an appropriate distance?

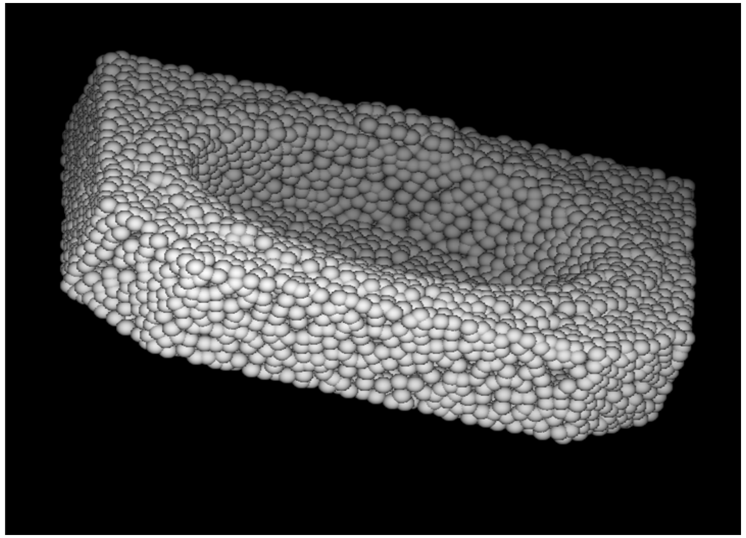




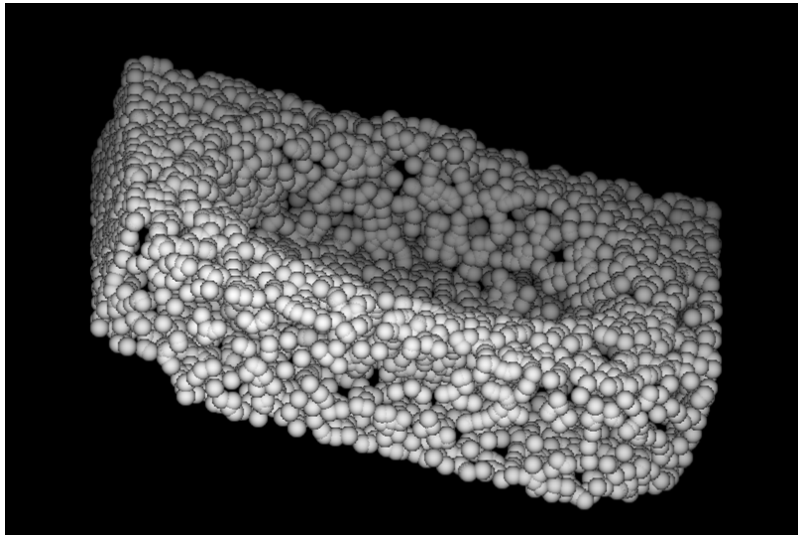
FPS vs. Random



Mesh



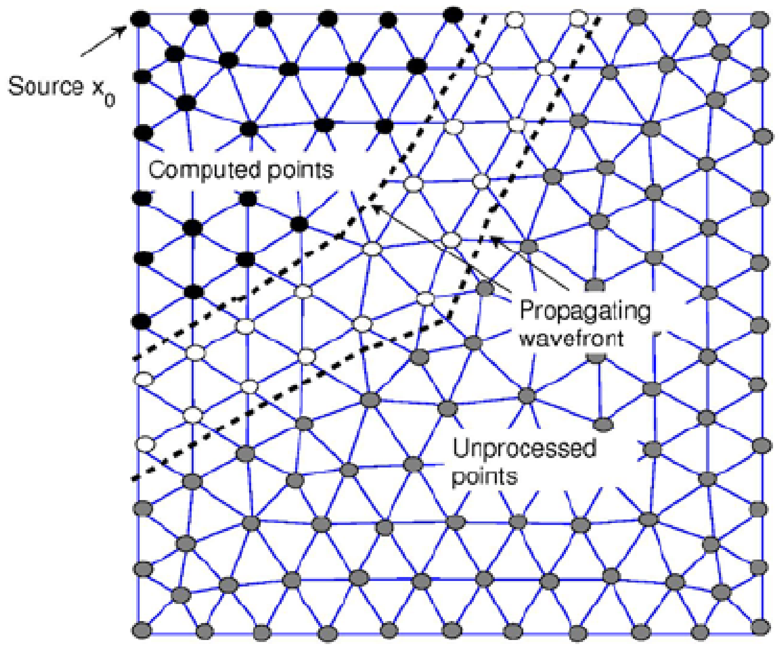
FPS

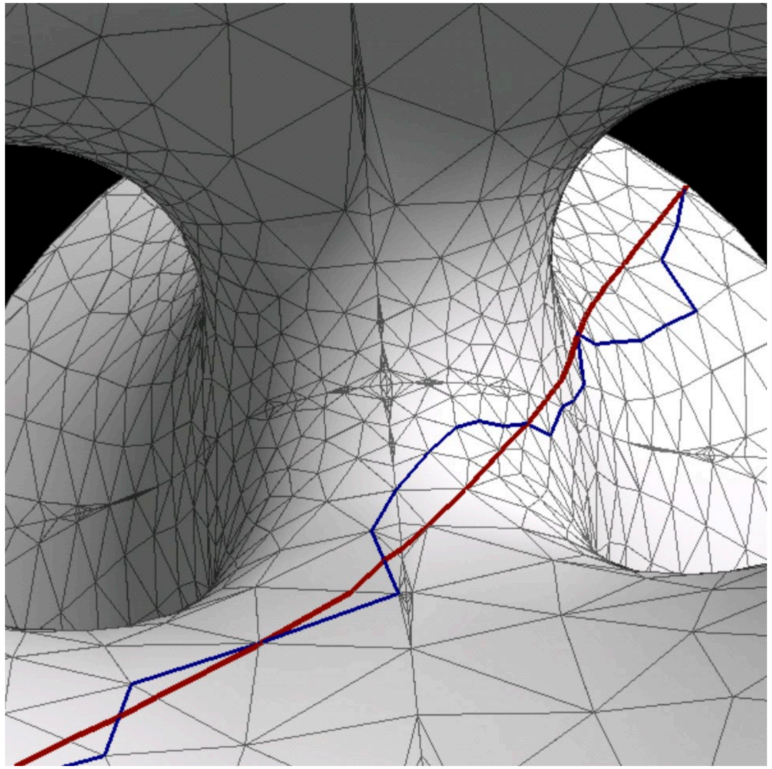


Random

Fast Marching Geodesics

- A better approximation: allow fronts to cross triangles!





Kimmel and Sethian 1997, “Computing Geodesic Paths on Manifolds”

Faster Distance Approximations

Carne, Weischedel, and Wardetzky 2017,
The Heat Method for Distance Computation

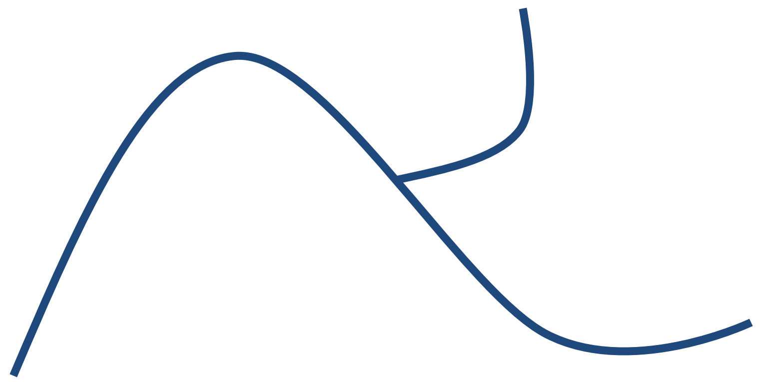
Today: Differential Geometry of Surfaces and the Laplace-Beltrami Operator

Continuous and Discrete

A Primer on Differential Geometry

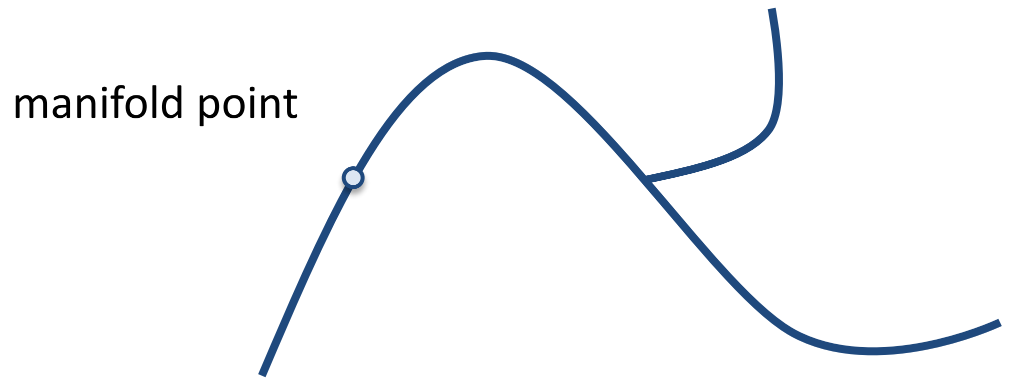
Differential Geometry Basics

- Geometry of manifolds
- Properties that can be discovered by local observation: point + neighborhood



Differential Geometry Basics 2D

- Geometry of manifolds
- Properties that can be discovered by local observation: point + neighborhood



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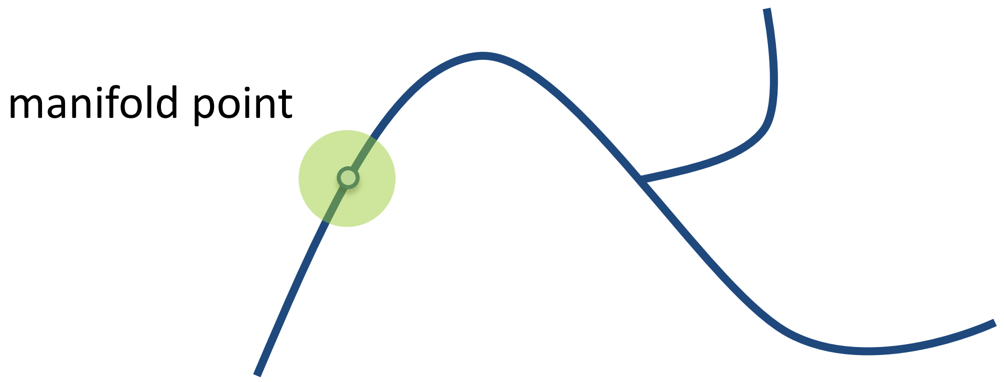

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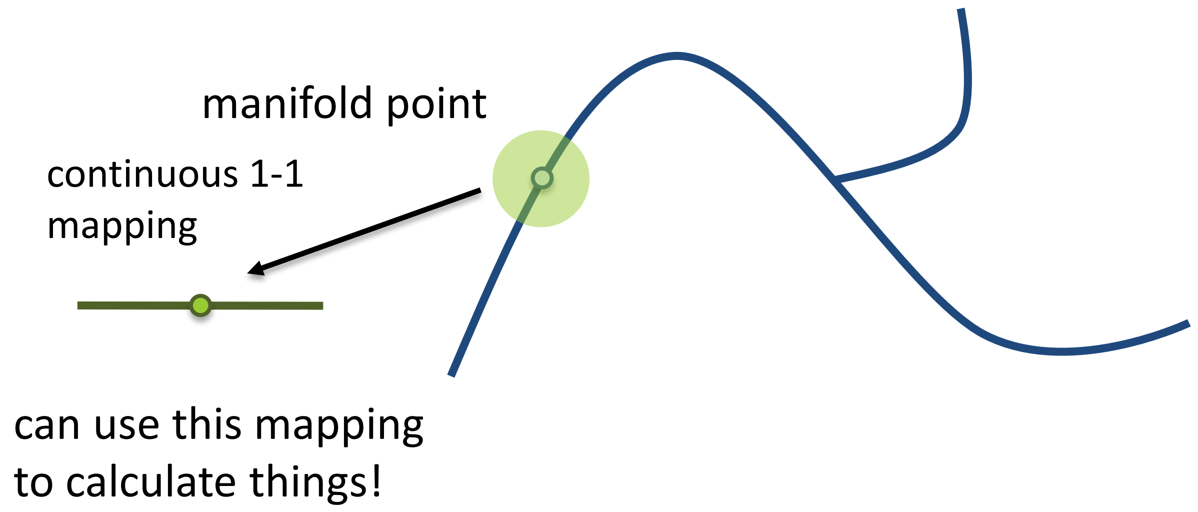

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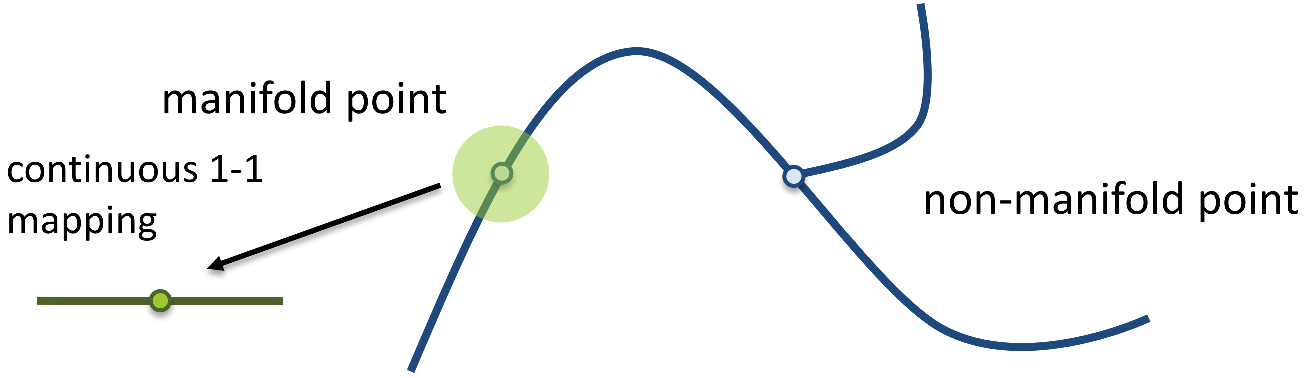
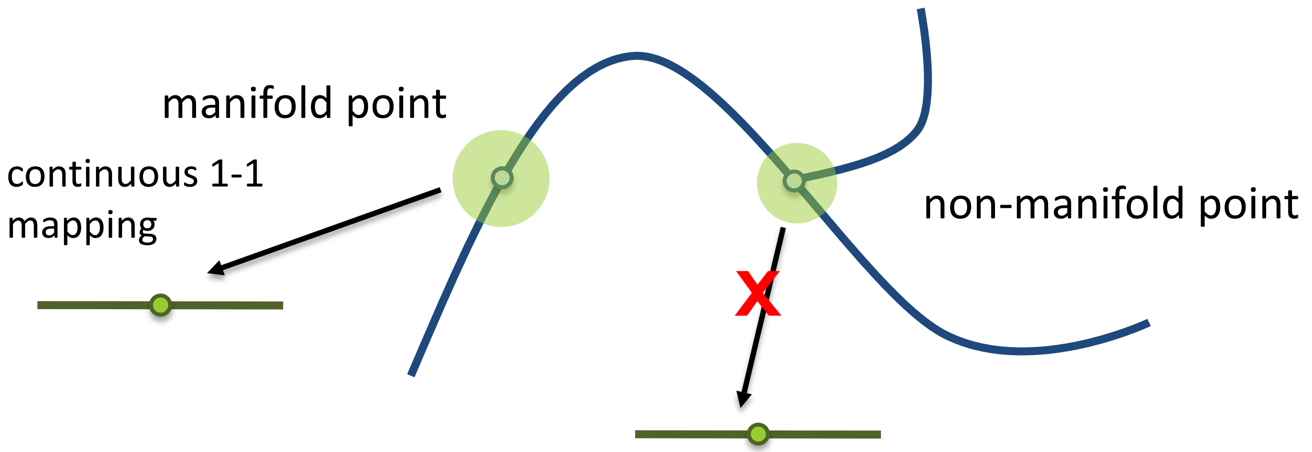


Differential Geometry Basics

- Geometry of manifolds
- Properties that can be discovered by local observation: point + neighborhood



Differential Geometry Basics 3D

- Geometry of manifolds
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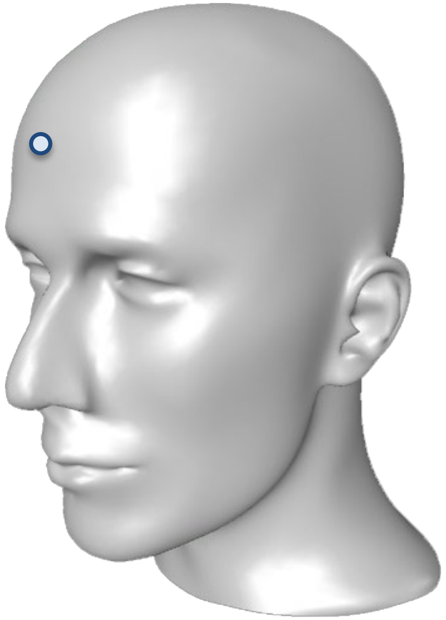
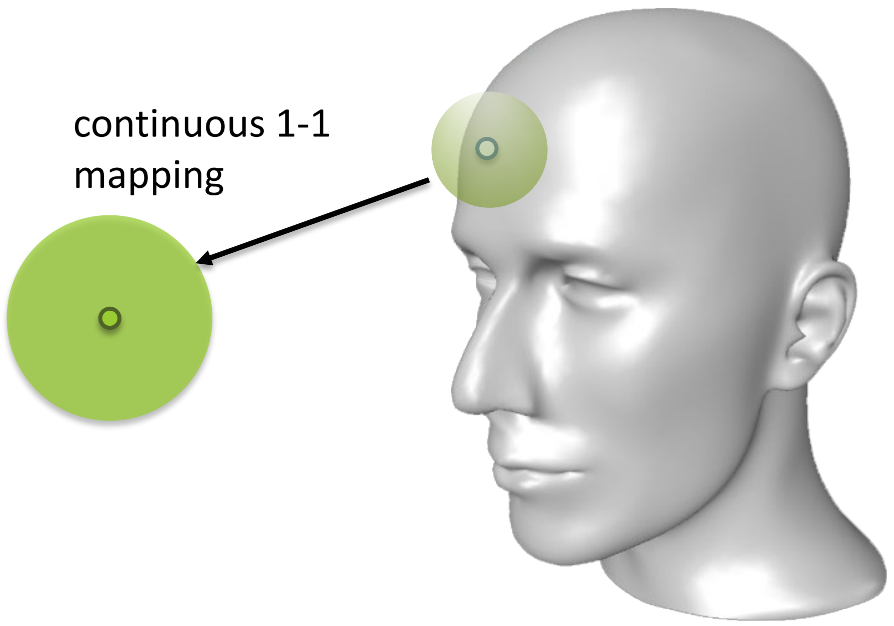


Differential Geometry Basics 3D

- Geometry of manifolds
- Properties that can be discovered by local observation: point + neighborhood



Differential Geometry Basics 3D

- Geometry of manifolds
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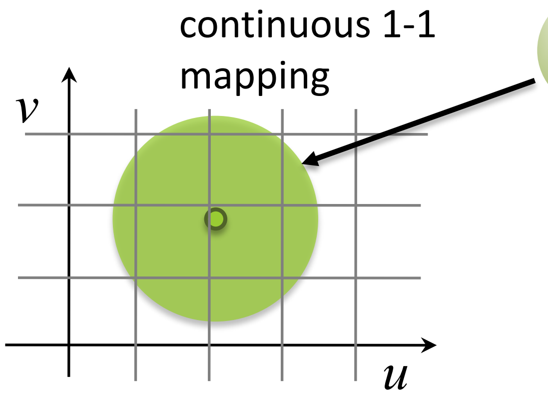
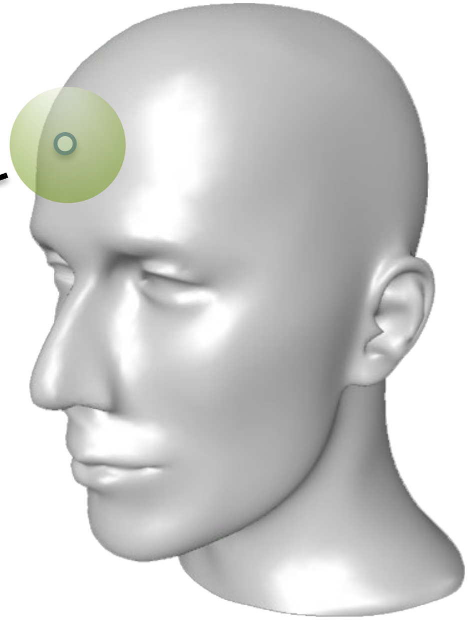


$$\frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial^2}{\partial^2 u} \quad \frac{\partial^2}{\partial u \partial v} \quad \ldots$$

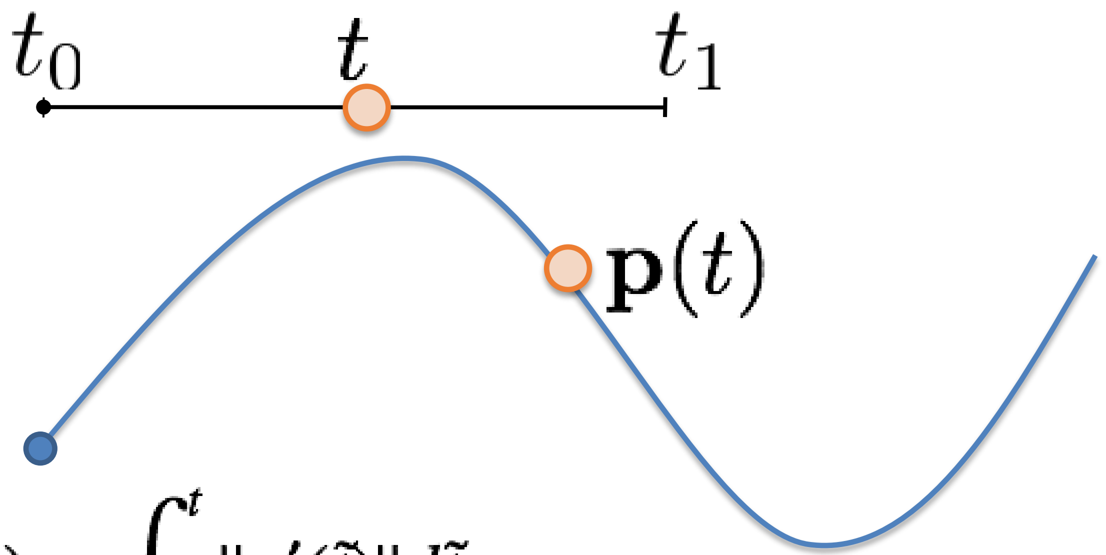


If a sufficiently smooth mapping can be constructed, we can look at its first and second derivatives

Tangents, normals, curvatures, curve angles, distances

Parametric Curves

- 2D: $\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \; t \in [t_0, t_1]$
- $\mathbf{p}(t)$ must be continuous

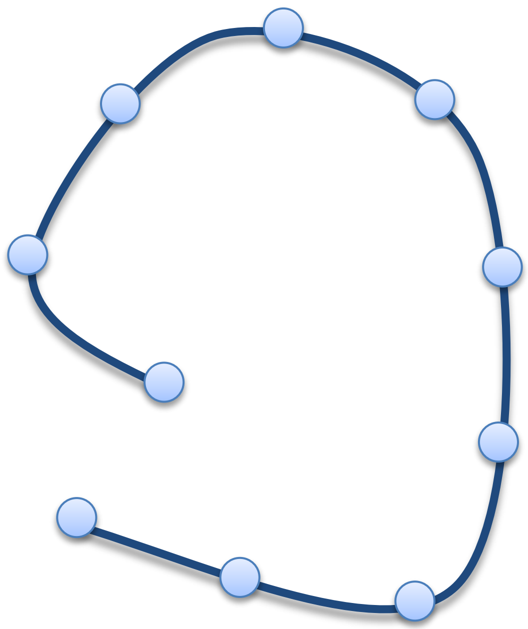


$$len\left(\mathbf{p}(t_0), \mathbf{p}(t)\right) = \int_{t_0}^{t} ||\mathbf{p}'(\tilde{t})|| d\tilde{t}$$

Arc Length Parameterization

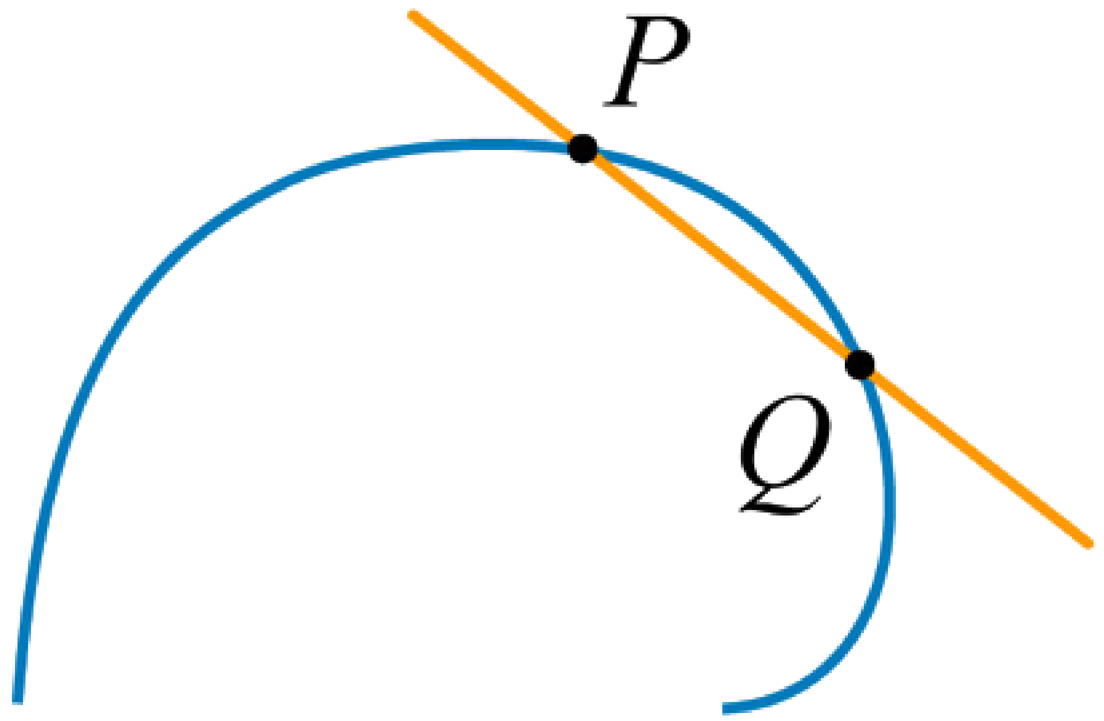
- Equal pace of the parameter along the curve
- $len\ (\mathbf{p}(s_1), \mathbf{p}(s_2)) = |s_1 - s_2|$
- Now parameter goes from 0 to L

$$\|\mathbf{p}'(s)\| = 1$$



Secant

- A line through two points on the curve.



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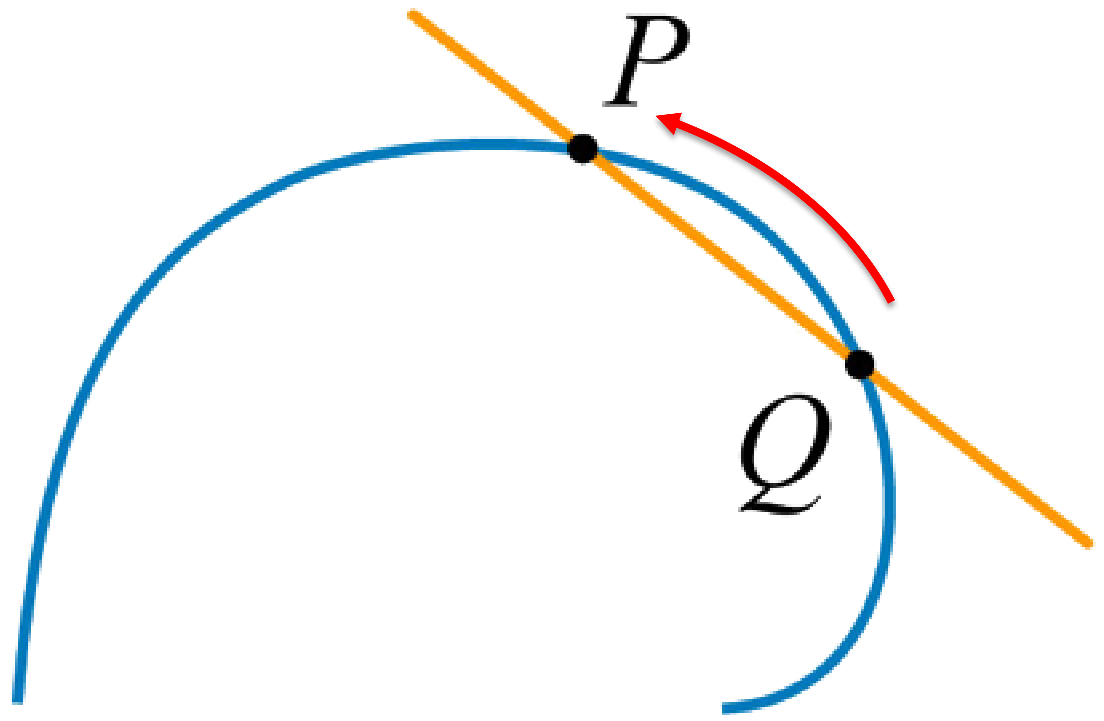
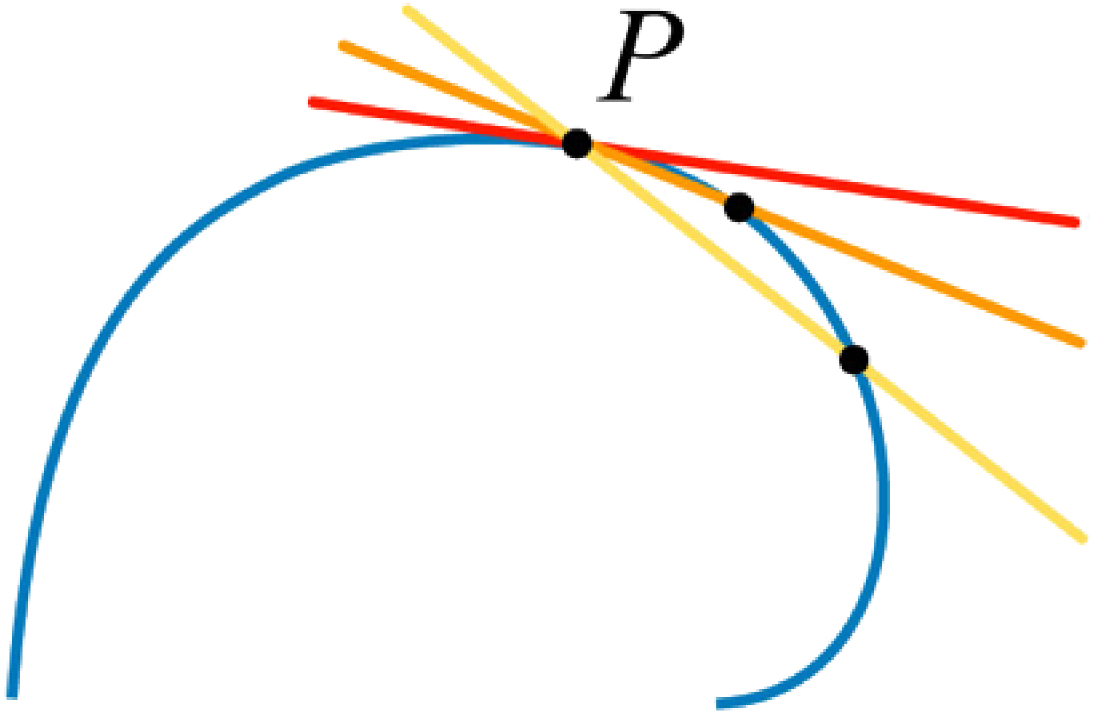


Tangent

- The limiting secant as the two points come together.



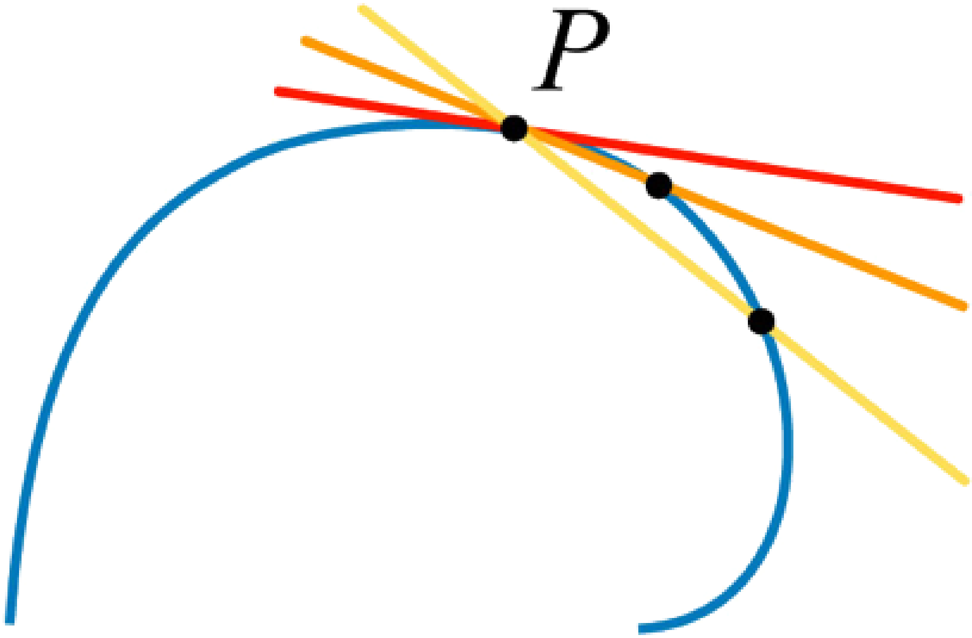
Secant and Tangent – Parametric Form

- Secant: $\mathbf{p}(t) - \mathbf{p}(s)$
- Tangent: $\mathbf{p}'(t) = (x'(t), y'(t), \ldots)^{\mathrm{T}}$
- If t is arc-length:

  $\|\mathbf{p}'(t)\| = 1$

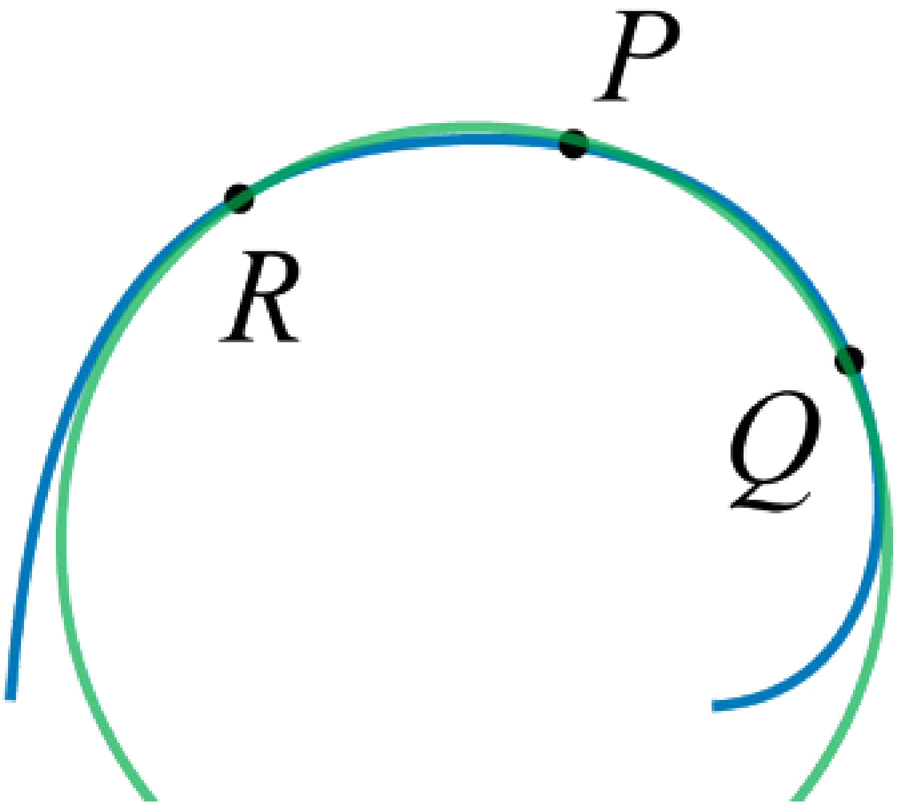
Recall

$$len(\mathbf{p}(t_0), \mathbf{p}(t)) = \int_0^t \|\mathbf{p}'(t)\| dt$$



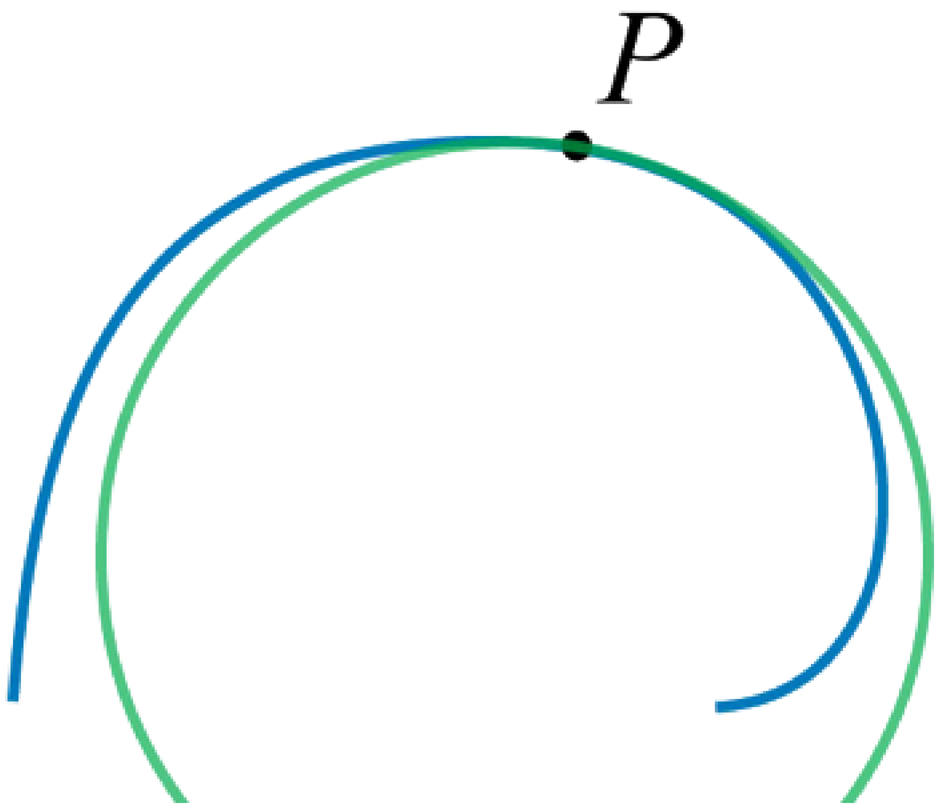
Circle of Curvature

- Consider the circle passing through three points on the curve…

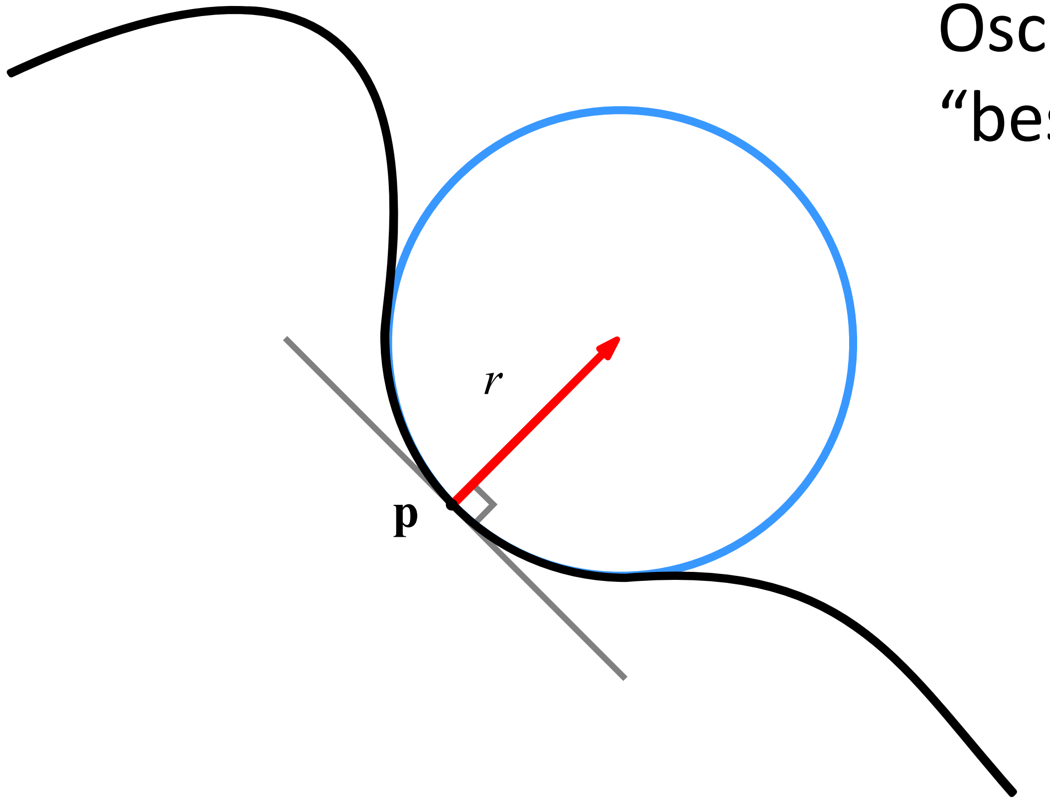


Circle of Curvature

- …the limiting circle as three points come together.



Tangent, normal, radius of curvature

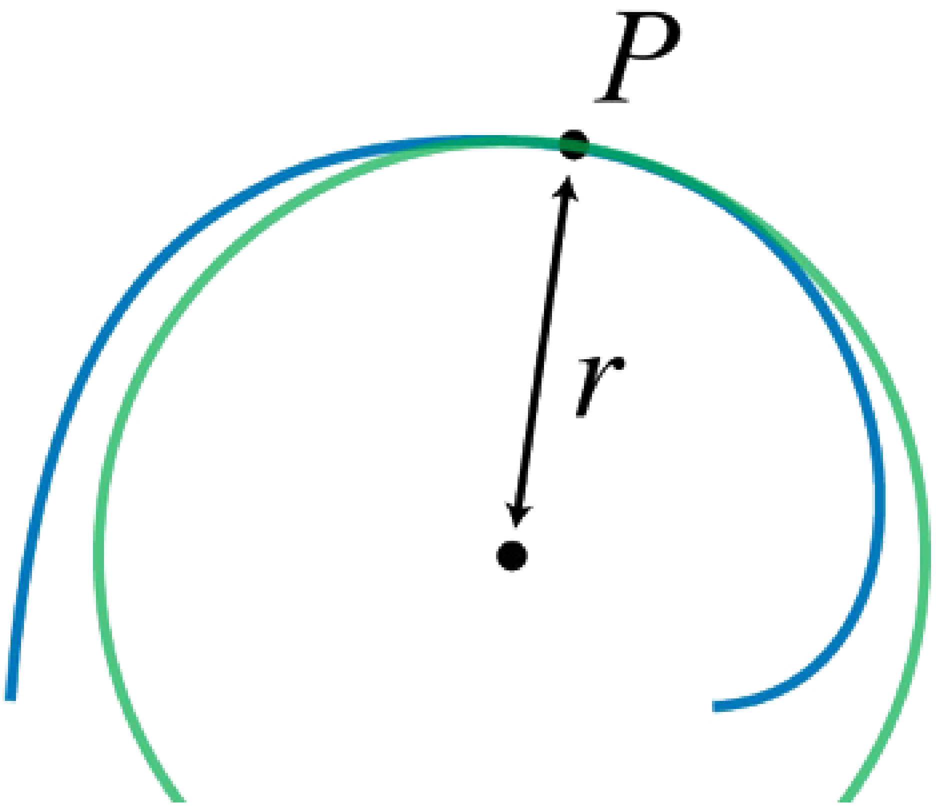


Osculating circle
“best fitting circle”

Radius of Curvature, $r = 1/\kappa$

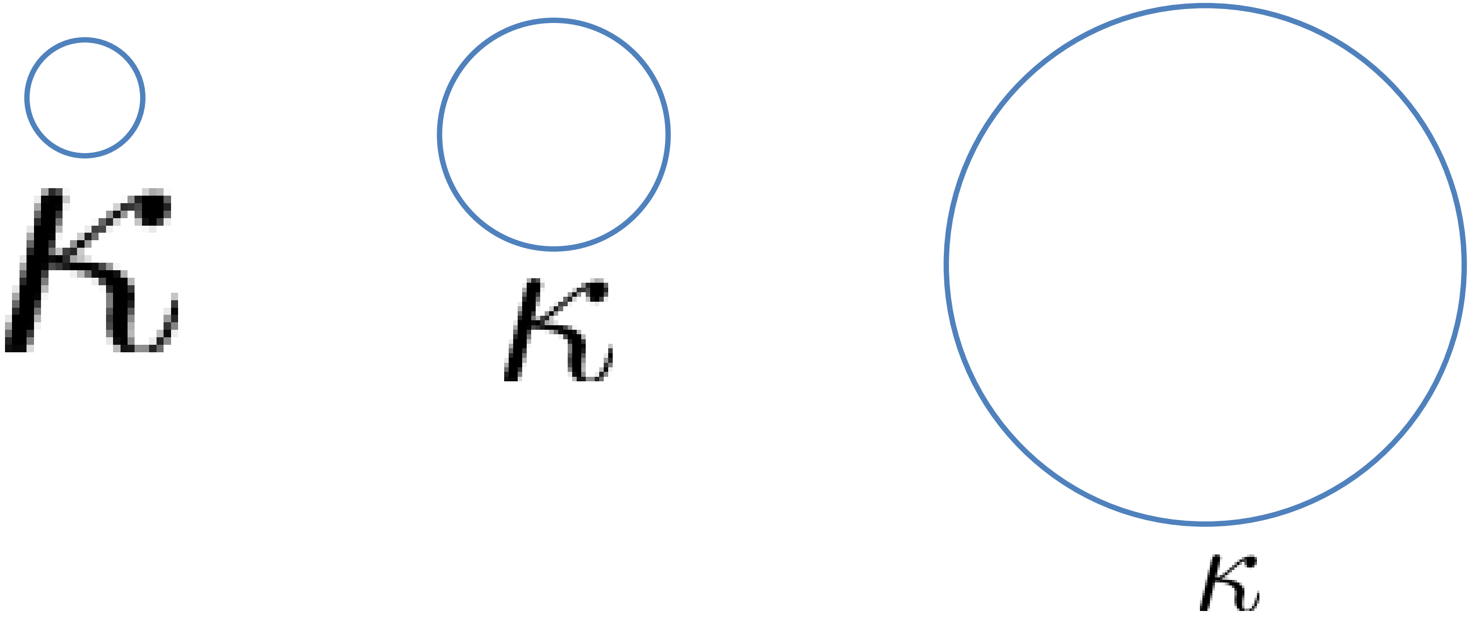
Curvature

$$\kappa = \frac{1}{r}$$



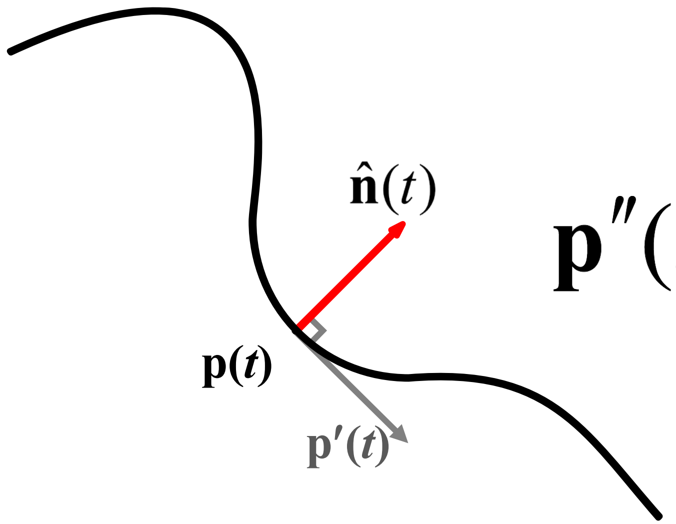
Curvature is Scale Dependent

$$\kappa = \frac{1}{r}$$

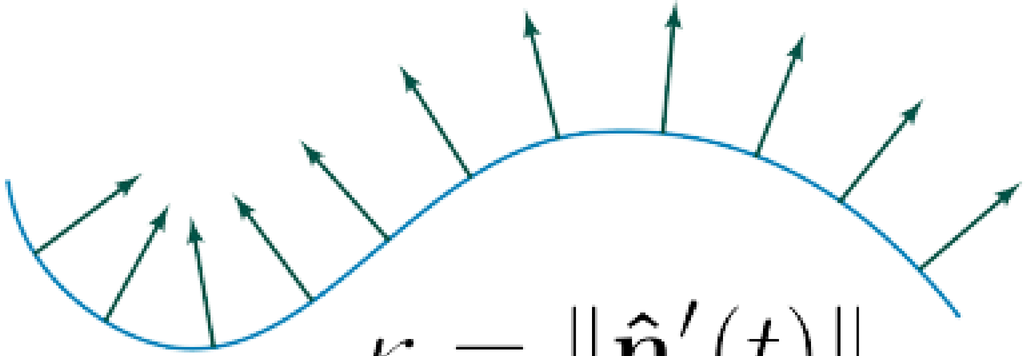


Curvature and Normal

- Assuming t is arc-length parameter:



$$\mathbf{p}''(t) = \kappa \hat{\mathbf{n}}(t)$$ normal to the curve



$$\kappa = \|\hat{\mathbf{n}}'(t)\|$$

Surfaces, Parametric Form

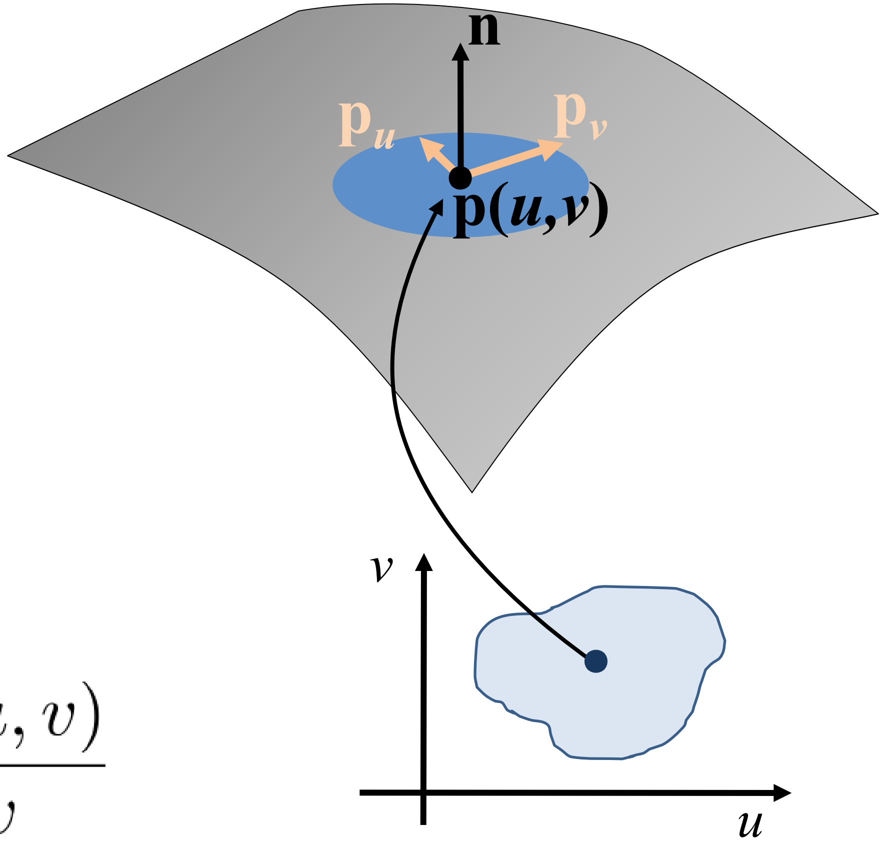
- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \mathbb{R}^2$$

- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

These vectors don't have to be orthogonal



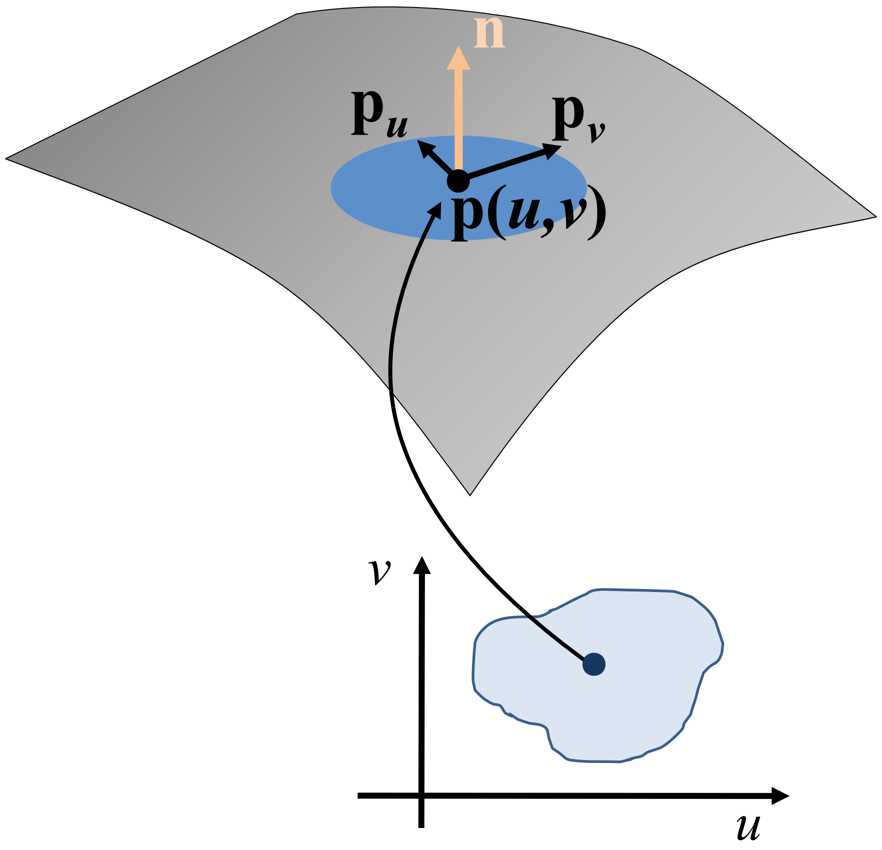
Surface Normals

- Surface normal:

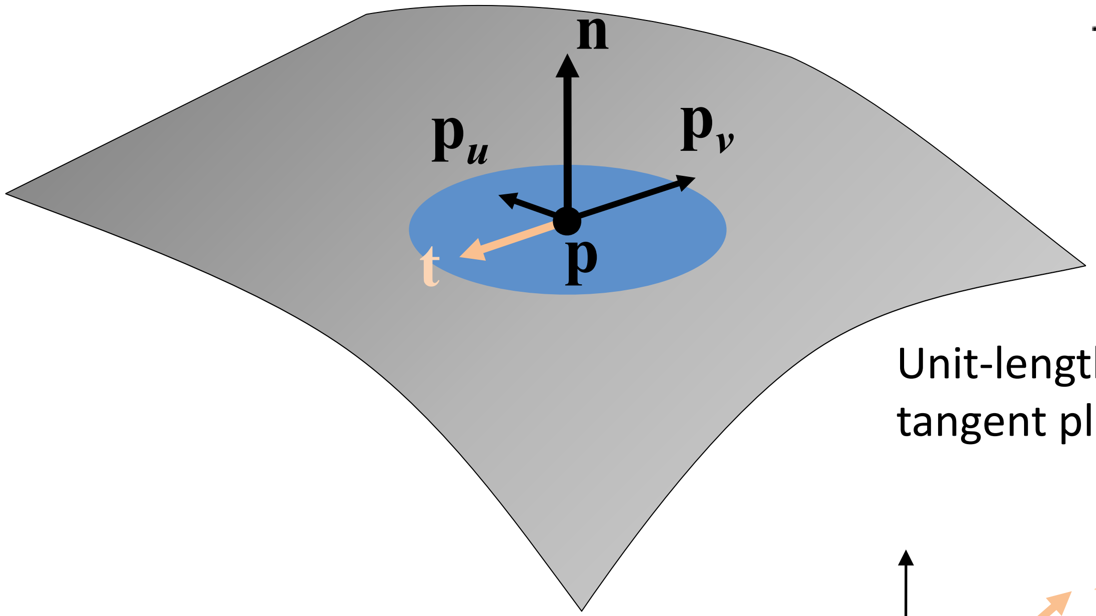
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

  - Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



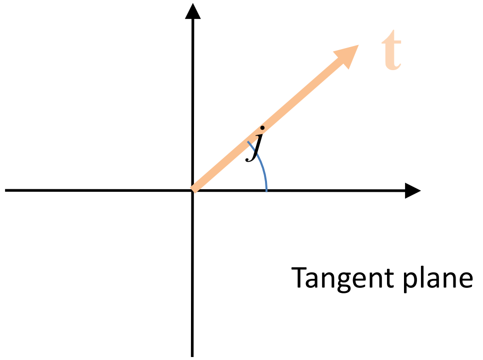
Normal Curvature



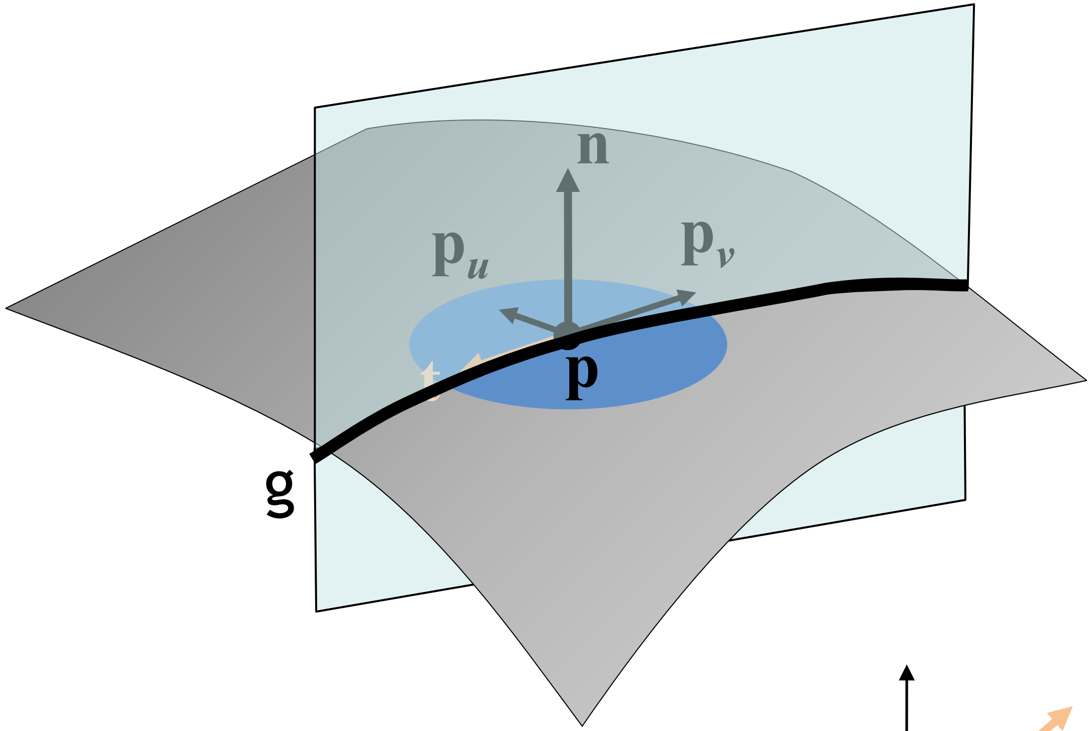
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

Unit-length direction **t** in the tangent plane (if $\mathbf{p}_u$ and $\mathbf{p}_v$ are orthogonal):

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin\varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



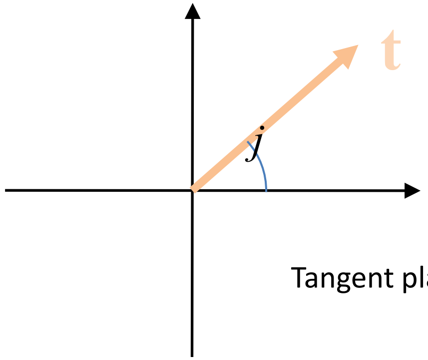
Normal Curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$ - a normal section.

Normal curvature:

$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$



Surface Curvatures

- **Principal curvatures**
  - Minimal curvature
  - Maximal curvature

$$\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$$

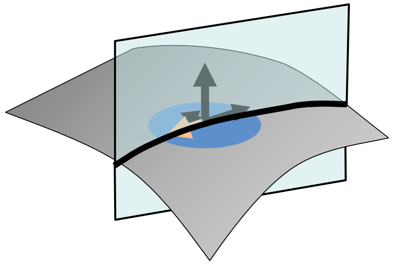
$$\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$$

- **Mean curvature**

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$$

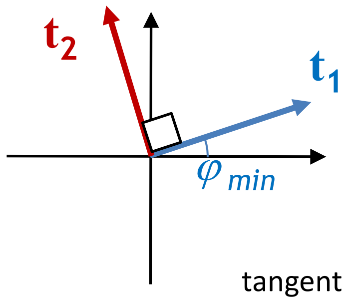
- **Gaussian curvature**

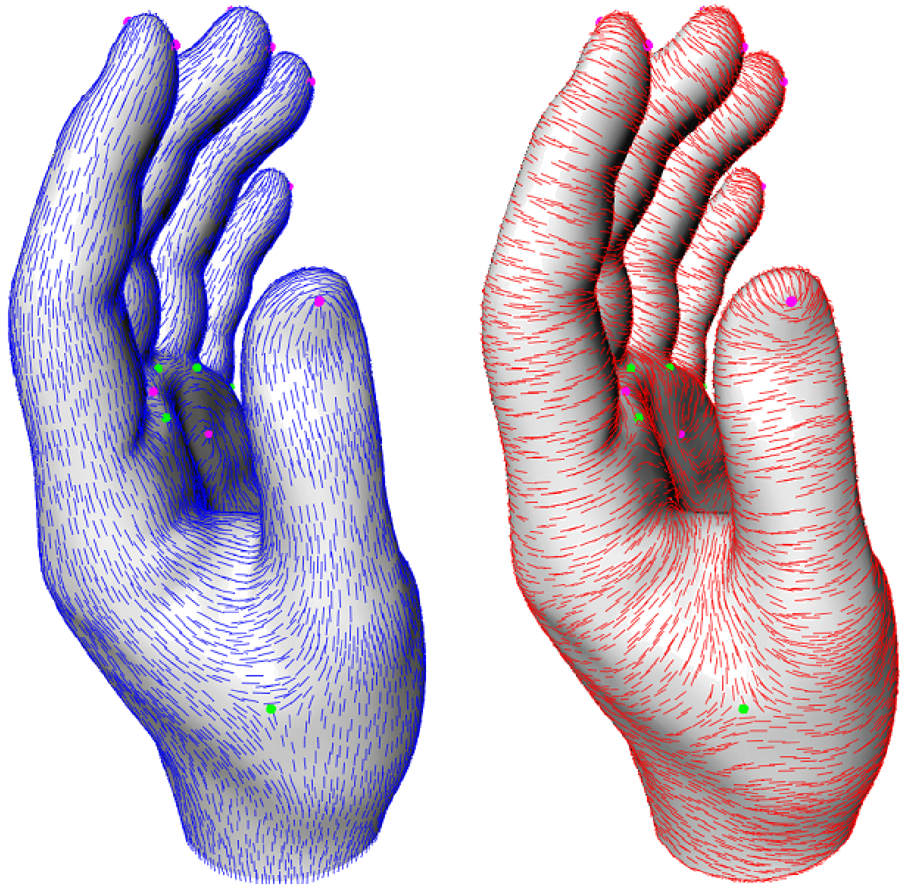
$$K = \kappa_1 \cdot \kappa_2$$



Principal Directions

- Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}

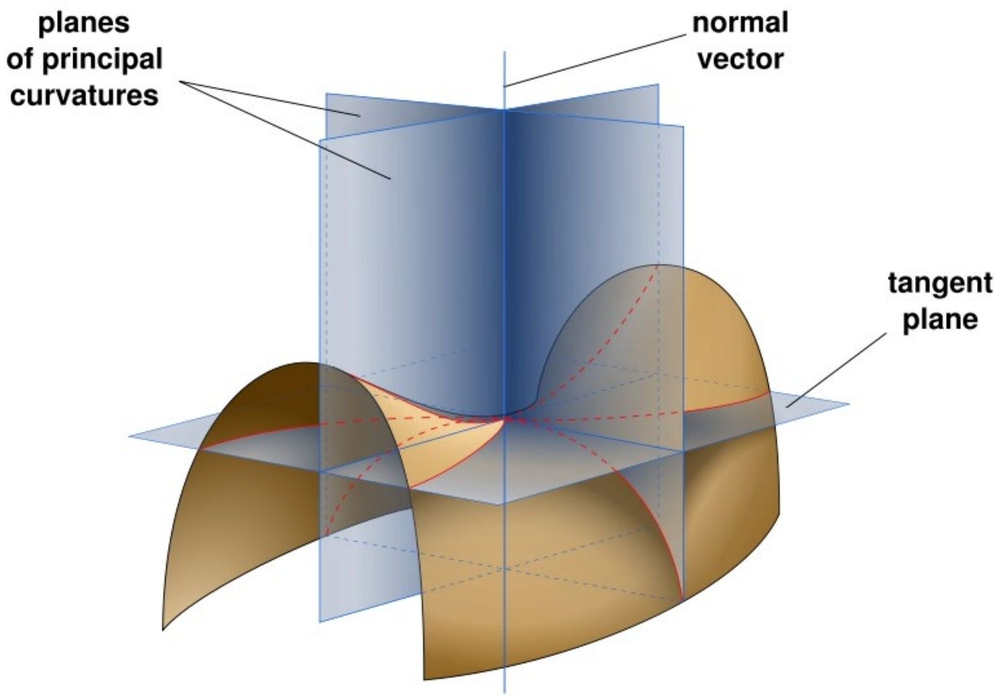




min curvature

max curvature

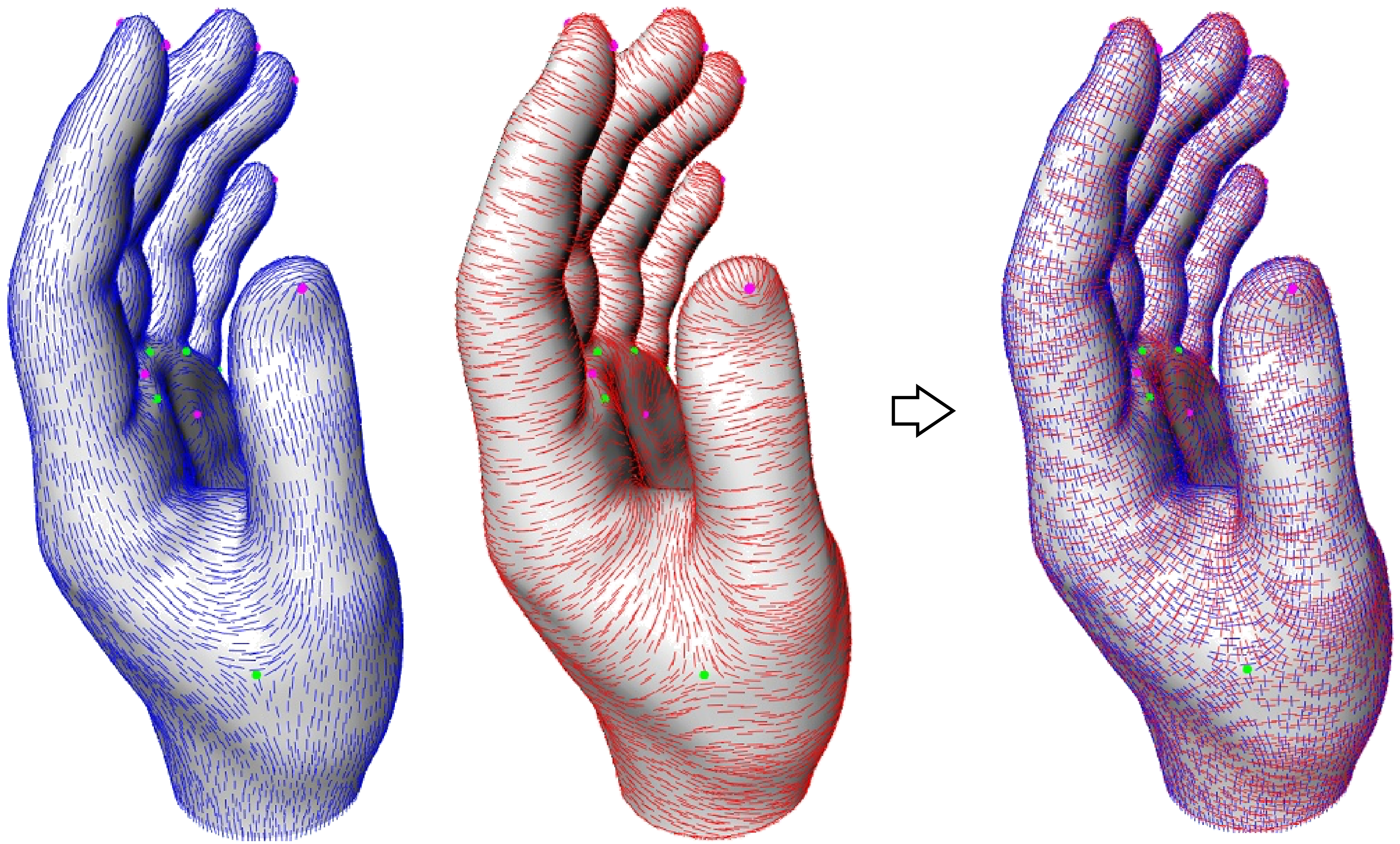
Principal Directions



Euler's Theorem: Planes of principal curvature are orthogonal and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } \mathbf{t}_1$$

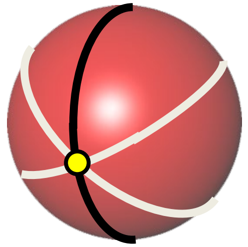
Principal Directions



Local Surface Shape By Curvatures

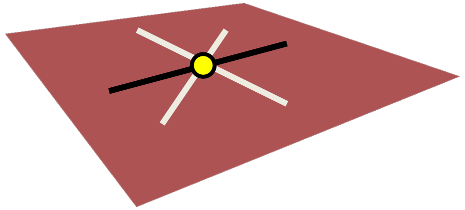
Isotropic:
all directions are principal directions

$K > 0,\ \kappa_1 = \kappa_2$



spherical (umbilical)

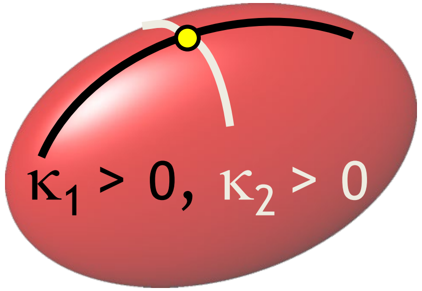
$K = 0$



planar

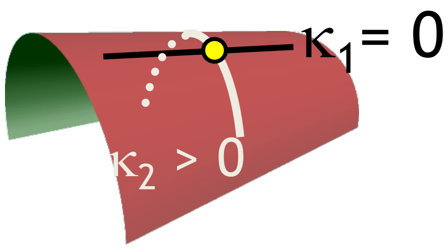
Anisotropic:
2 distinct principal directions

$K > 0$



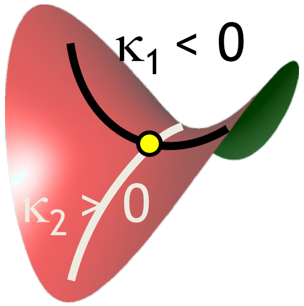
elliptic

$K = 0$



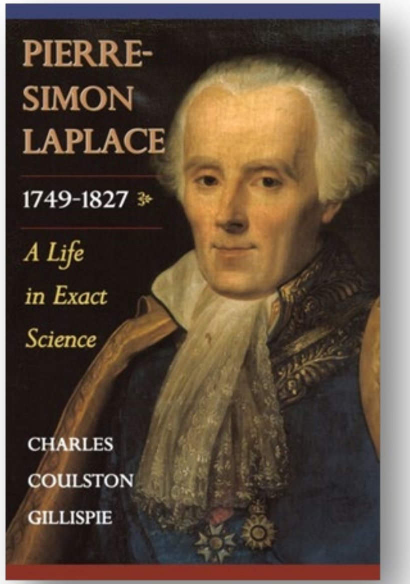
parabolic

$K < 0$



hyperbolic

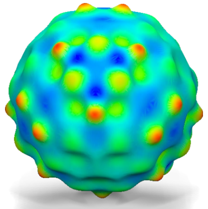
The Laplace-Beltrami Operator



Eugenio Beltrami

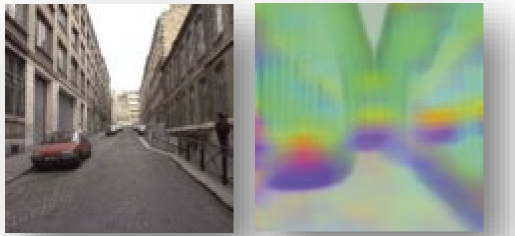
Knowledge as Functions over Data

Knowledge towers over visual data: function spaces



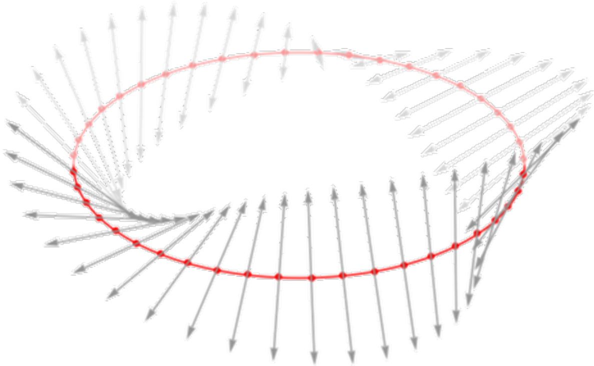
Curvature

Parts



SIFT flow, C. Liu 2011

Vector bundles and sheaves



Motivation: Information Representation

- Various surface properties can be represented as functions "living" on the surface
  - Curvatures (k_1, k_2, K, H)
  - Part indicator functions
  - Vector-valued functions, like surface coordinates, normals and texture
  - …

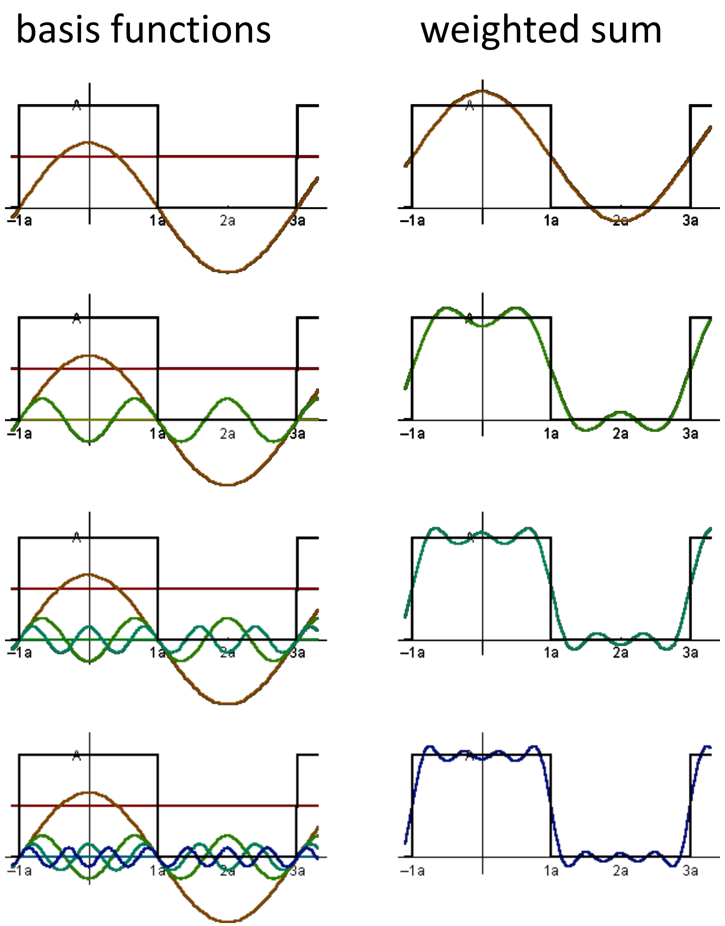
- We need a tool to work with functions on surfaces

- Operators map functions to functions

Reminder: Fourier Analysis on the Real Line

- Represent a function as a weighted sum of sines and cosines (basis functions)

Joseph Fourier 1768 - 1830



$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \ldots$$

Coefficients : co-integrate function with basis

More Generally - Fourier Analysis

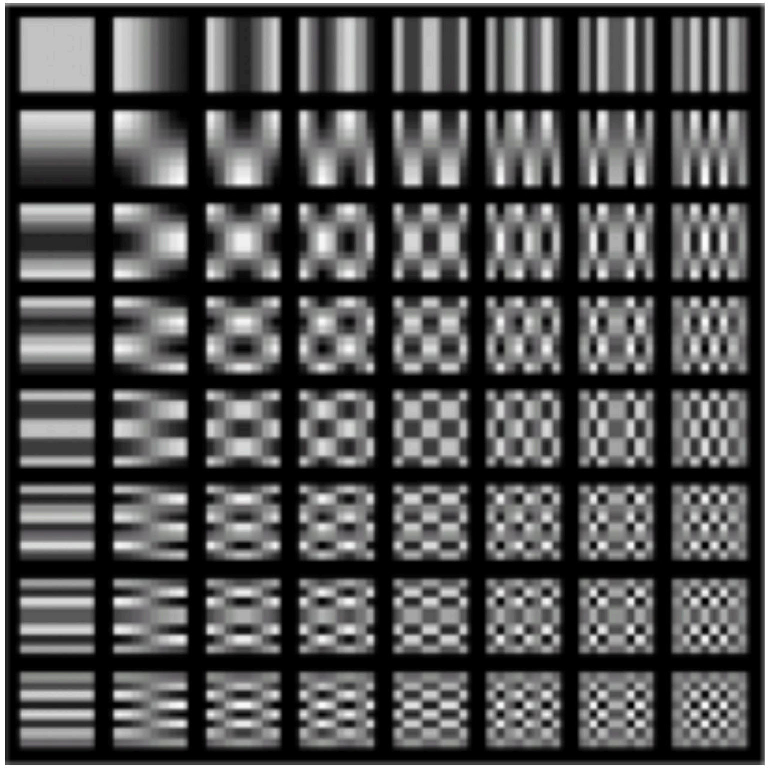
- Inner product for L^2 function space $\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,\mathrm{d}x$
- Orthonormal basis : complex "waves"

$$e_u(x) \;:=\; \mathrm{e}^{\mathrm{i}2\pi u x} \;=\; \cos(2\pi u x) - \mathrm{i}\sin(2\pi u x)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i \omega x}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{2\pi i \omega x}\, d\omega$$

Fourier on Rectangular 2D (or 3D) Domains

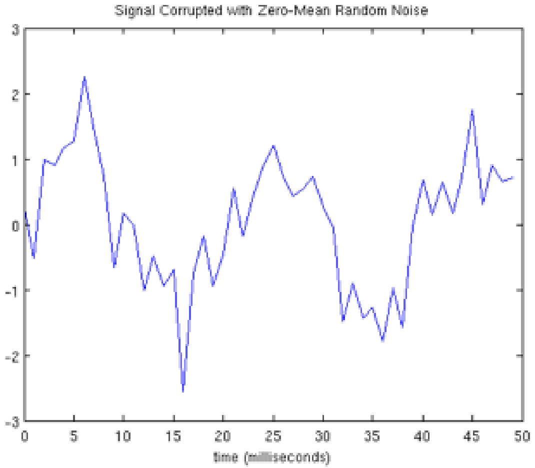


Fourier (DCT) basis functions for 8x8 grayscale images

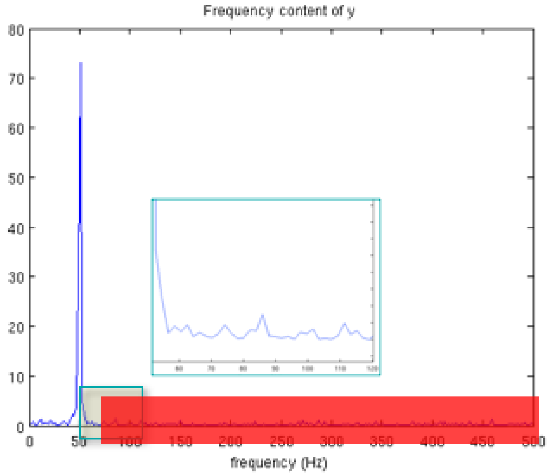
$$\cos(2\pi\omega_h)\cos(2\pi\omega_v)$$

Image Processing: Filtering

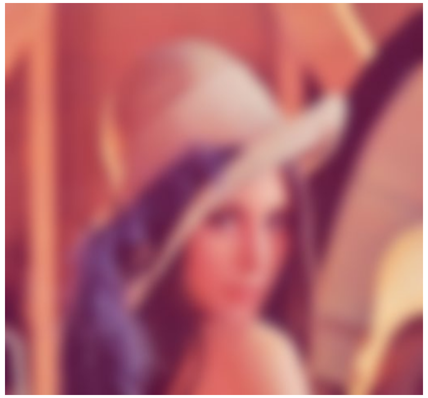
- Freq ai



spatial domain



frequency domain



- Spatial domain $f(x)$ → Frequency domain $F(u)$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}2\pi ux} \mathrm{d}x$$

- Multiply by low-pass filter $G(u)$

$$F(u) \leftarrow F(u) \cdot G(u)$$

$$f(x) = \int_{-\infty}^{\infty} F(u)\, \mathrm{e}^{\mathrm{i}2\pi ux}\, \mathrm{d}u$$

Extend Fourier to (Meshed) Surfaces?

- Fourier basis functions are eigenfunctions of the (standard) Laplace operator Δ: $L^2 \longrightarrow L^2$

$$\Delta f = \mathrm{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\Delta \left(e^{2\pi i \omega x} \right) = \frac{\partial^2}{\partial x^2} e^{2\pi i \omega x} = -(2\pi\omega)^2 e^{2\pi i \omega x}$$

- We need
  - A version of this operator for 2D manifolds, and
  - A discrete (mesh-based) version!

Laplace-Beltrami on 2-Manifolds

Fourier analysis on manifolds

Continuous Laplace Operator in R^3

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \Delta f : \mathbb{R}^3 \to \mathbb{R}$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \dots$$

function in Euclidean space

divergence operator

Cartesian coordinates

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Continuous Laplace-Beltrami Operator

- Extension of Laplace operator to functions on manifolds

$$f : \mathcal{M} \to \mathbb{R} \qquad \Delta f : \mathcal{M} \to \mathbb{R}$$

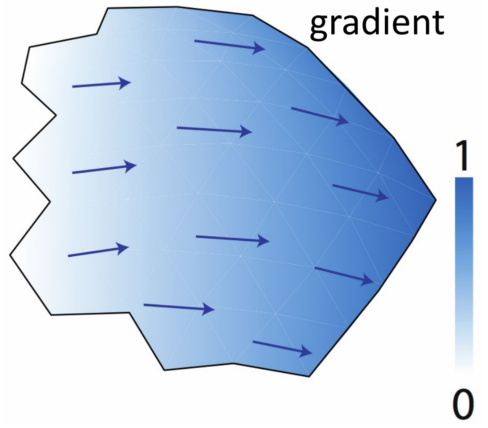
gradient operator

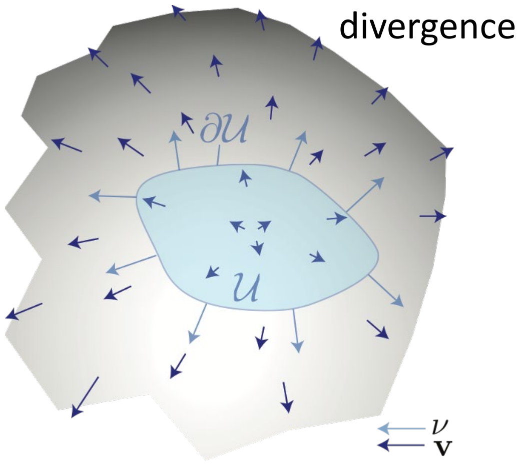
Laplace-Beltrami

$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

function on surface M

divergence operator





Continuous Setting: Laplace-Beltrami

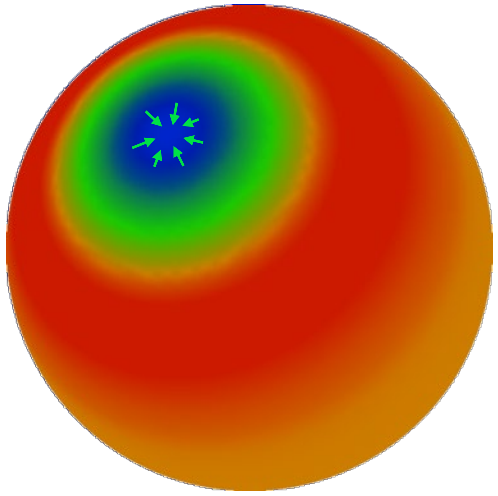
Definition:

Given a surface without boundary $\mathcal{M}$, define the Laplace-Beltrami operator on $\mathcal{M}$

$$\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \ \Delta f = \text{div}\,\nabla f$$

divergence of the gradient of f.

$\text{div}\nabla f$



Continuous Setting: Laplace-Beltrami

Definition:

Given a surface without boundary, define the Laplace-Beltrami operator

$$\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),\ \Delta f = \text{div}\, \nabla f$$

divergence of the gradient of f.

Divergence key property for vector field X: for any f

$$\int_{\mathcal{M}} f\, \text{div} X\, d\mu = \int_{\mathcal{M}} \langle X, \nabla f \rangle\, d\mu.$$

Thus, on a surface we also have, for any pair of smooth functions:

$$\int_{\mathcal{M}} g \Delta f d\mu = \int_{\mathcal{M}} < \nabla g, \nabla f > d\mu = \int_{\mathcal{M}} f \Delta g d\mu$$

Continuous Setting: Laplace-Beltrami

Definition:

Given a surface, define the inner product between two functions as:

$$\langle f, g \rangle = \int_{\mathcal{M}} f(x) g(x) d\mu(x)$$

Then: Δ is a symmetric positive semi-definite operator:

$$\langle f, \Delta g \rangle = \langle g, \Delta f \rangle$$

And:

$$\langle f, \Delta f \rangle = \int_{\mathcal{M}} \|\nabla f\|^2 d\mu$$

As such, Δ has a countable set of eigen*functions* with real positive, eigenvalues .

Continuous Setting: Laplace-Beltrami

Basic Properties:

For a compact surface, $\mathcal{M}$, the Laplace-Beltrami operator Δ has a countable set of eigenfunctions with real positive, eigenvalues

- Each eigenvalue has finite mulitplicity.
- Multiplicity of 0 is the number of connected components

- Eigenfunctions are orthonormal:

$$\left\langle \phi_i, \phi_j \right\rangle = 0, \quad \langle \phi_i, \phi_i \rangle = 1$$

- For any square integrable function:

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x)$$

$$a_i = \langle f, \phi_i \rangle$$

Continuous Setting: Laplace-Beltrami

Multiscale nature of the spectrum:

Intuitively, eigenfunctions corresponding to larger eigenvalues, capture *smaller details* (higher frequency) of the geometry.

$\lambda_0 = 0$ $\quad\lambda_1 = 2.6$ $\quad\lambda_2 = 3.4$ $\quad\lambda_3 = 5.1$ $\quad\lambda_4 = 7.6$

- *n*-th eigenfunction has at most *n* nodal domains.
- Integral of the gradient squared increases.

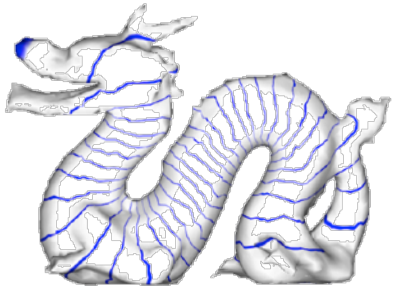
$$\lambda_i = \int_{\mathcal{M}} \phi_i \Delta \phi_i d\mu = \int_{\mathcal{M}} \|\nabla \phi_i\|^2 d\mu$$

Continuous Setting: Laplace-Beltrami

Multiscale nature of the spectrum:

- Fiedler *function*: minimizes the gradient, being orthogonal to the constant.

$$\phi_1 = \underset{f}{\text{argmin}} \frac{\langle f, \Delta f \rangle}{\|f\|^2} = \frac{\int_{\mathcal{M}} \|\nabla f\|^2 d\mu}{\|f\|^2} \text{ s.t. } \int_{\mathcal{M}} f d\mu = 0$$



Continuous Setting: Laplace-Beltrami

Signal Processing on a manifold (generalizing Fourier analysis):

Given a function $f : \mathcal{M} \rightarrow \mathbb{R}$

$$f(x) = \sum_{i=0}^{\infty} \phi_i(x) \langle \phi_i, f \rangle$$

Filter out high frequency "noise", by truncating the series early:

$$f'(x) = \sum_{i=0}^{N} \phi_i(x) \langle \phi_i, f \rangle$$

New function will preserve the "global" properties of f.

Laplace-Beltrami and Curvature

- Apply operator to coordinate functions

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{M}}\mathbf{p} = \mathrm{div}_{\mathcal{M}}\nabla_{\mathcal{M}}\mathbf{p}\ \boxed{= -2H\mathbf{n} \ \in \mathbb{R}^3}$$

coordinate functions on surface M

$$\mathbf{p} = (x, y, z)$$

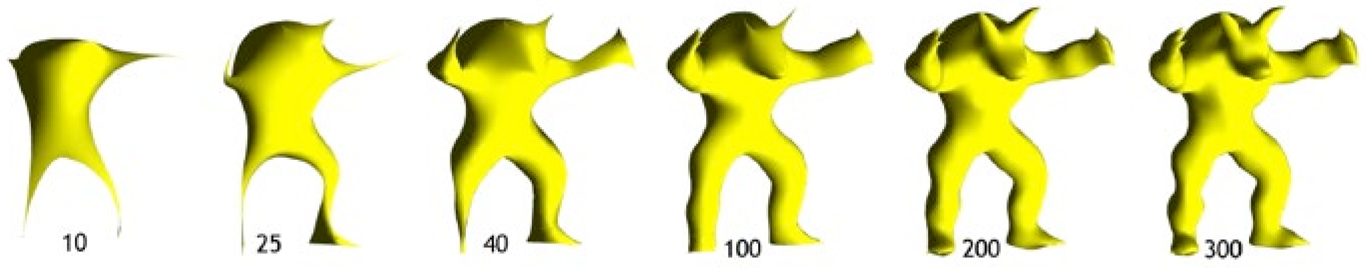
divergence operator

unit surface normal

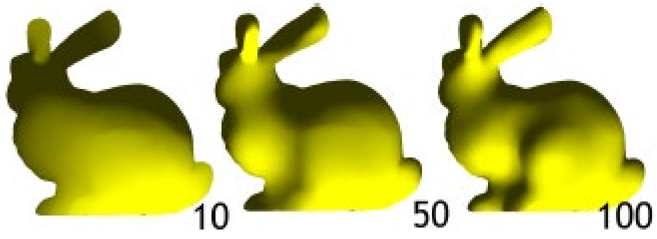
Continuous Setting: Laplace-Beltrami

Signal Processing on a manifold (generalizing Fourier analysis):

Reconstructing Geometry:



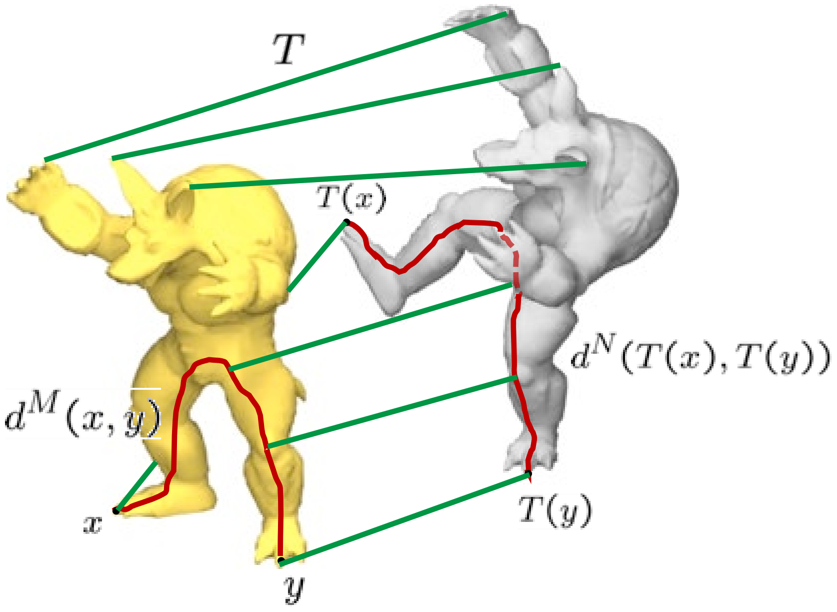
Reconstructing Normals:



Laplace-Beltrami – Isometry Invariant

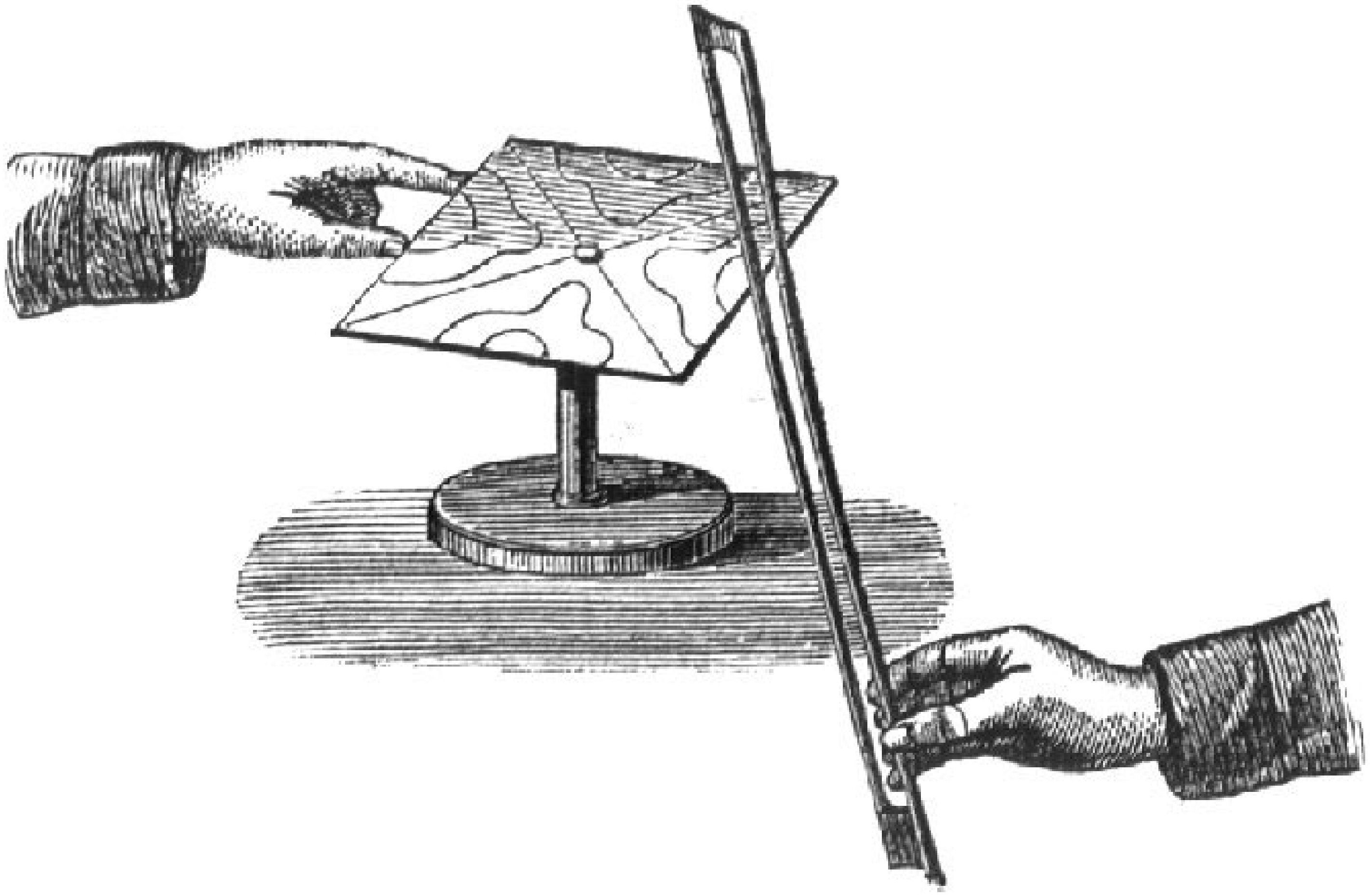
Isometry invariance (LB is intrinsic)

If two shapes are isometric then their LB operators agree.



Any quantity derived from the LB operator has to be invariant to isometries.

Digression: Seeing Sound

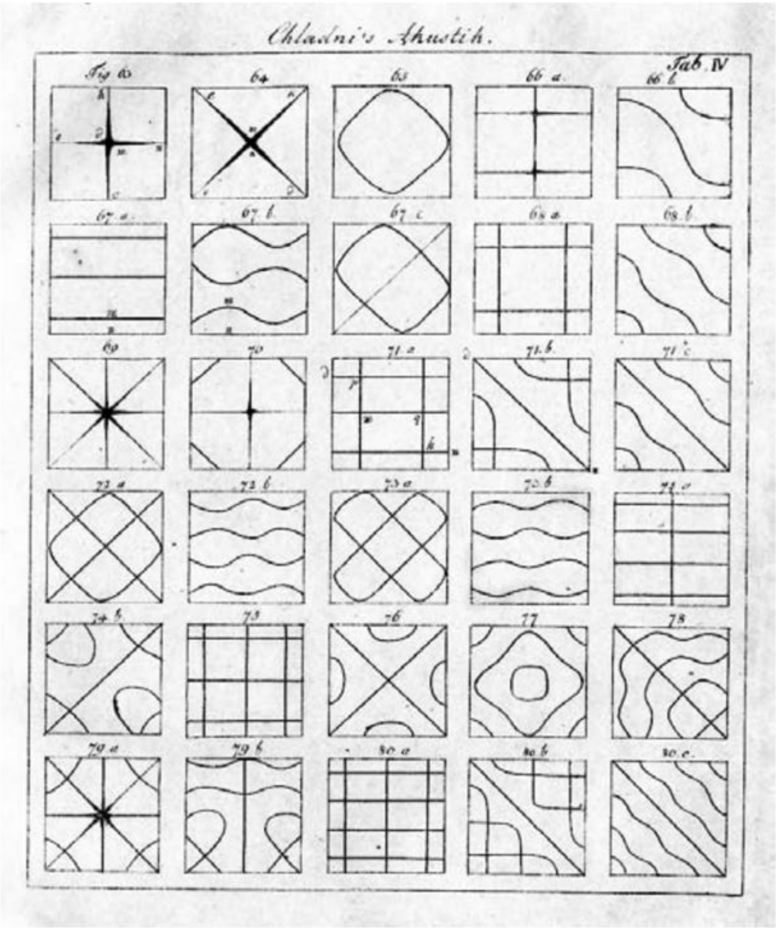


Chladni's experimental setup allowing to visualize acoustic waves

Ernst Chladni ['kladnɪ]
(1715-1782)

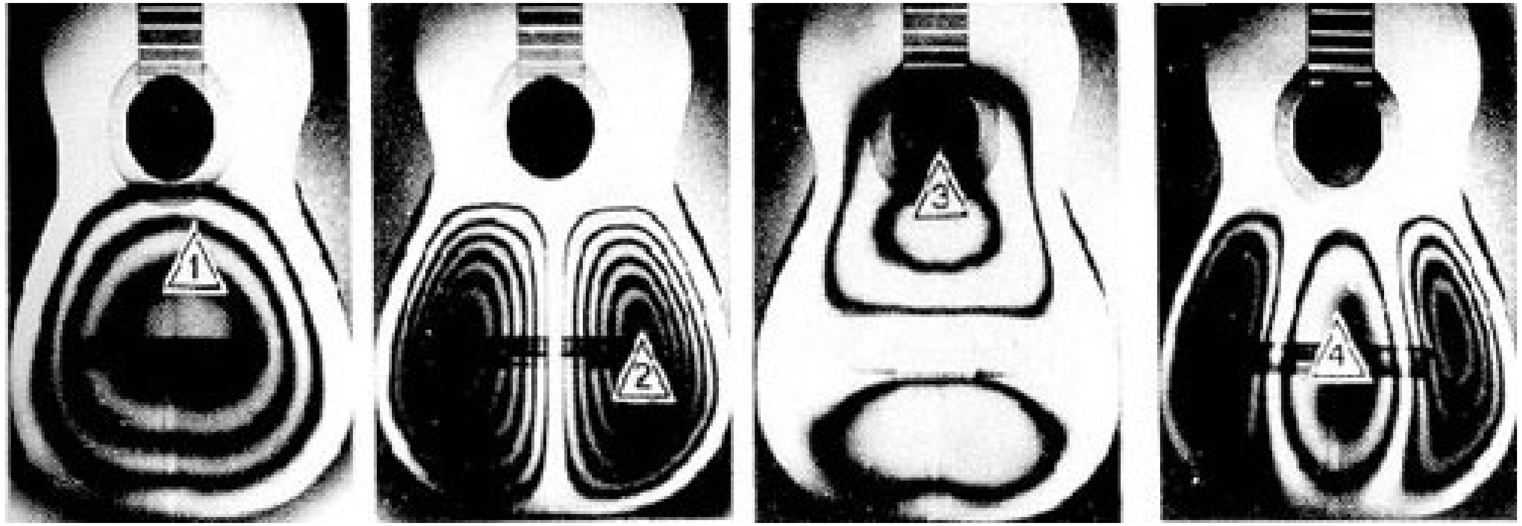
E. Chladni, *Entdeckungen über die Theorie des Klanges*

Seeing Sound – Chladni Plates



Gander, Martin J., and Gerhard Wanner. "From Euler, Ritz, and Galerkin to modern computing." *Siam Review* 54.4 (2012): 627-666.

Chladni Plates

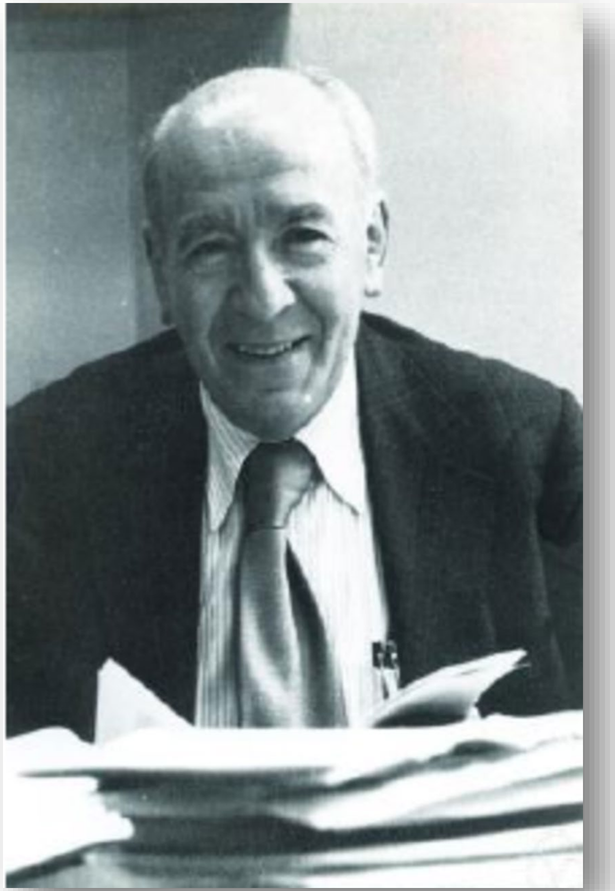


Patterns seen by Chladni are solutions to **stationary Helmholtz equation**

$$\Delta_X f = \lambda f$$

Solutions of this equation are **eigenfunctions** of Laplace-Beltrami operator

Can One Hear the Shape of a Drum?



Mark Kac
(1914-1984)

More prosaically: can one reconstruct the shape (up to an isometry) from its Laplace-Beltrami spectrum?

Can One Hear the Shape of a Drum?

In Chladni's experiments, the spectrum describes acoustic characteristics of the plates ("modes" of vibrations)

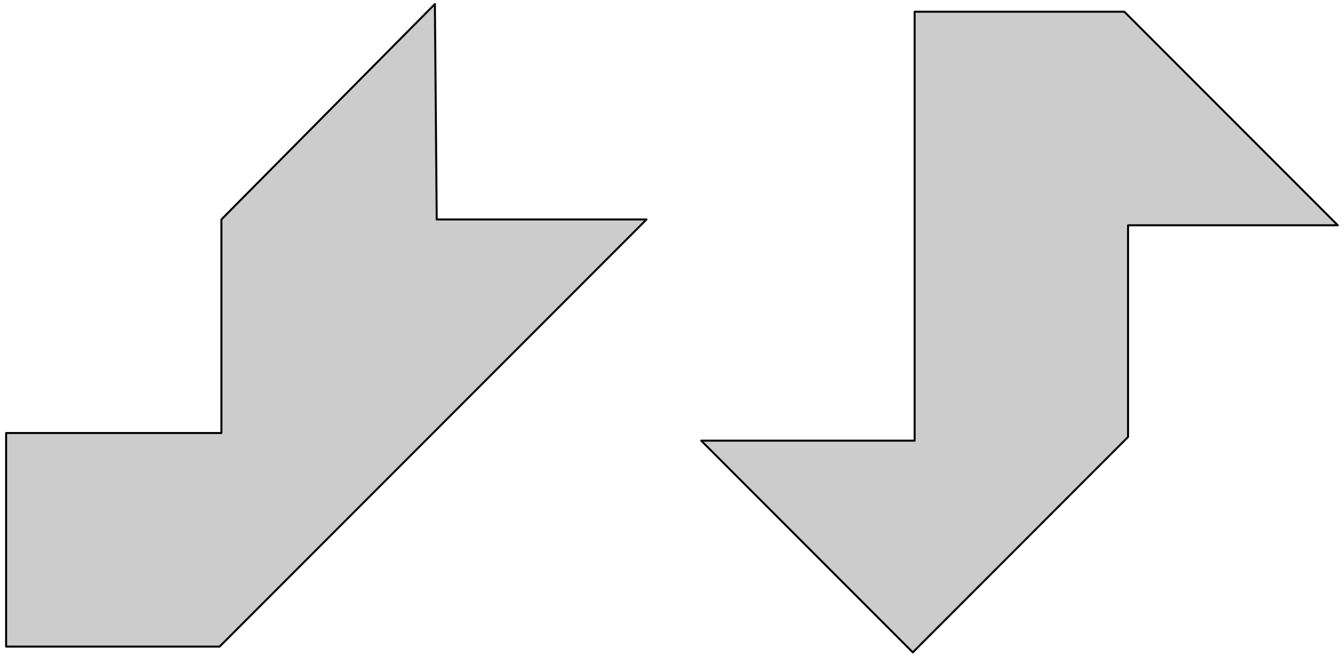
What can be "heard" from the spectrum:

- Total Gaussian curvature
- Euler characteristic
- Area

Can we "hear" the metric (shape)?

One Cannot Hear the Shape of a Drum

[Gordon et al. 1991]:

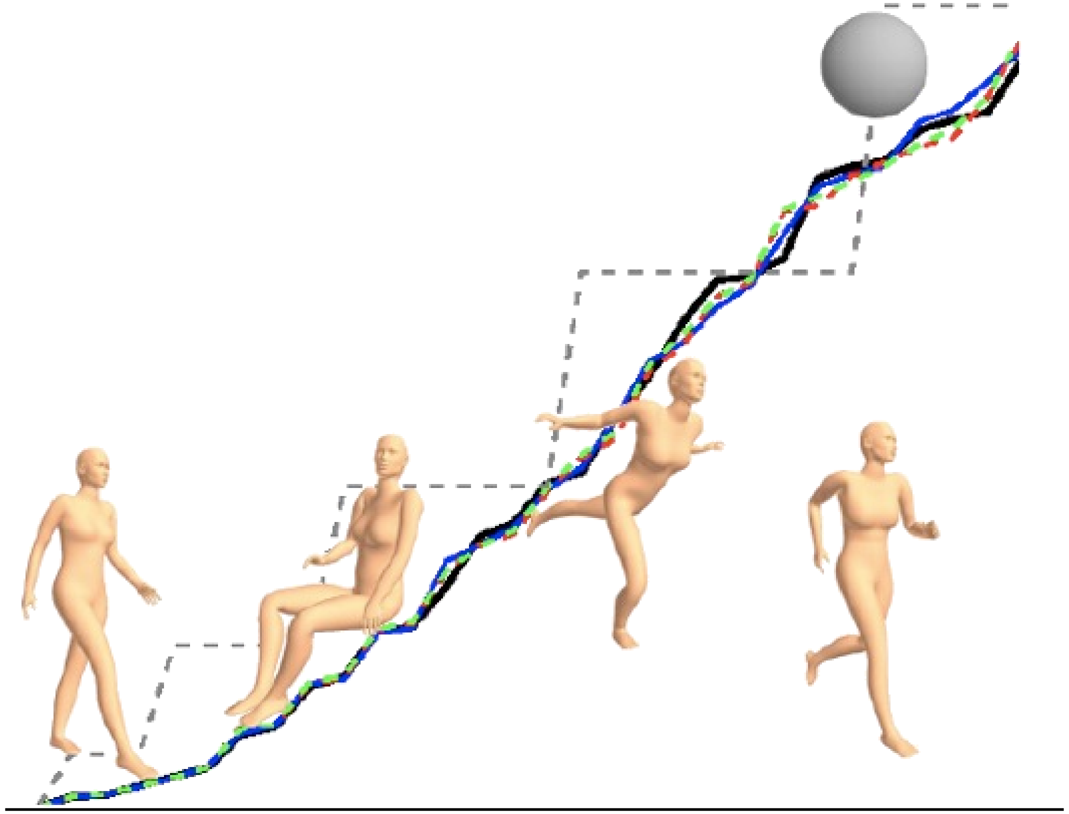


Counter-example of isospectral but not isometric shapes

Carolyn Gordon, David Webb, and Scott Wolpert, 1992

Shape DNA

[Reuter et al. 2006]: use the Laplace-Beltrami spectrum $\{\lambda_i\}_{i\geq 1}$ as an isometry-invariant shape descriptor (**"shape DNA"**)

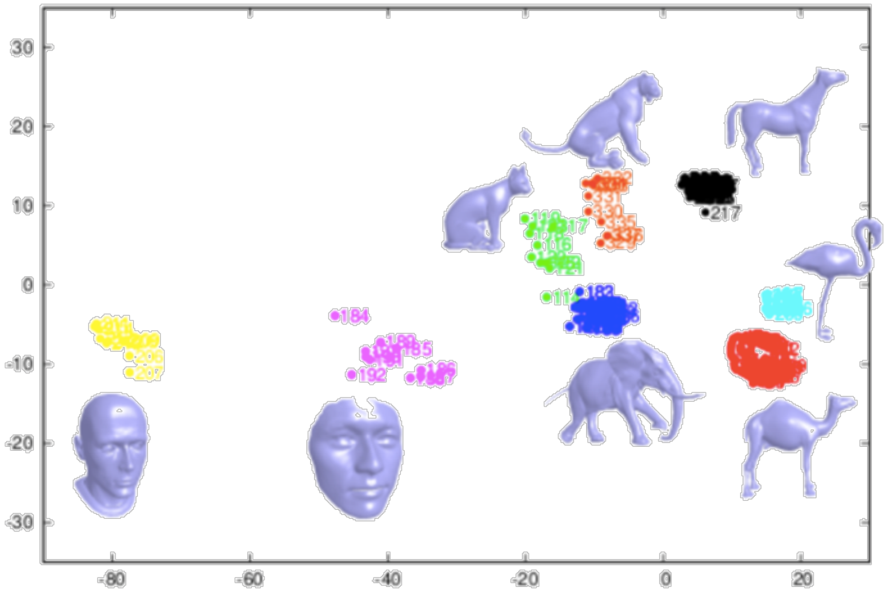


Laplace-Beltrami spectrum

Images: Reuter *et al.*

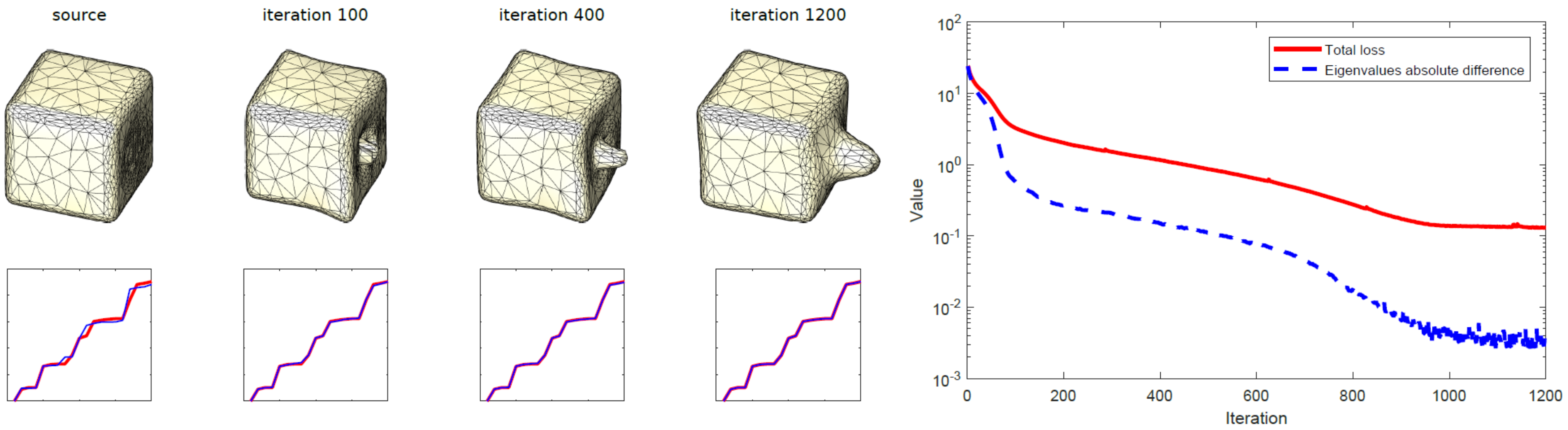
Shape DNA

1. For each shape in the collection, compute its LB operator.
2. Find the k smallest eigenvalues and store them in a vector.
3. Compare the shapes by comparing the corresponding vectors.



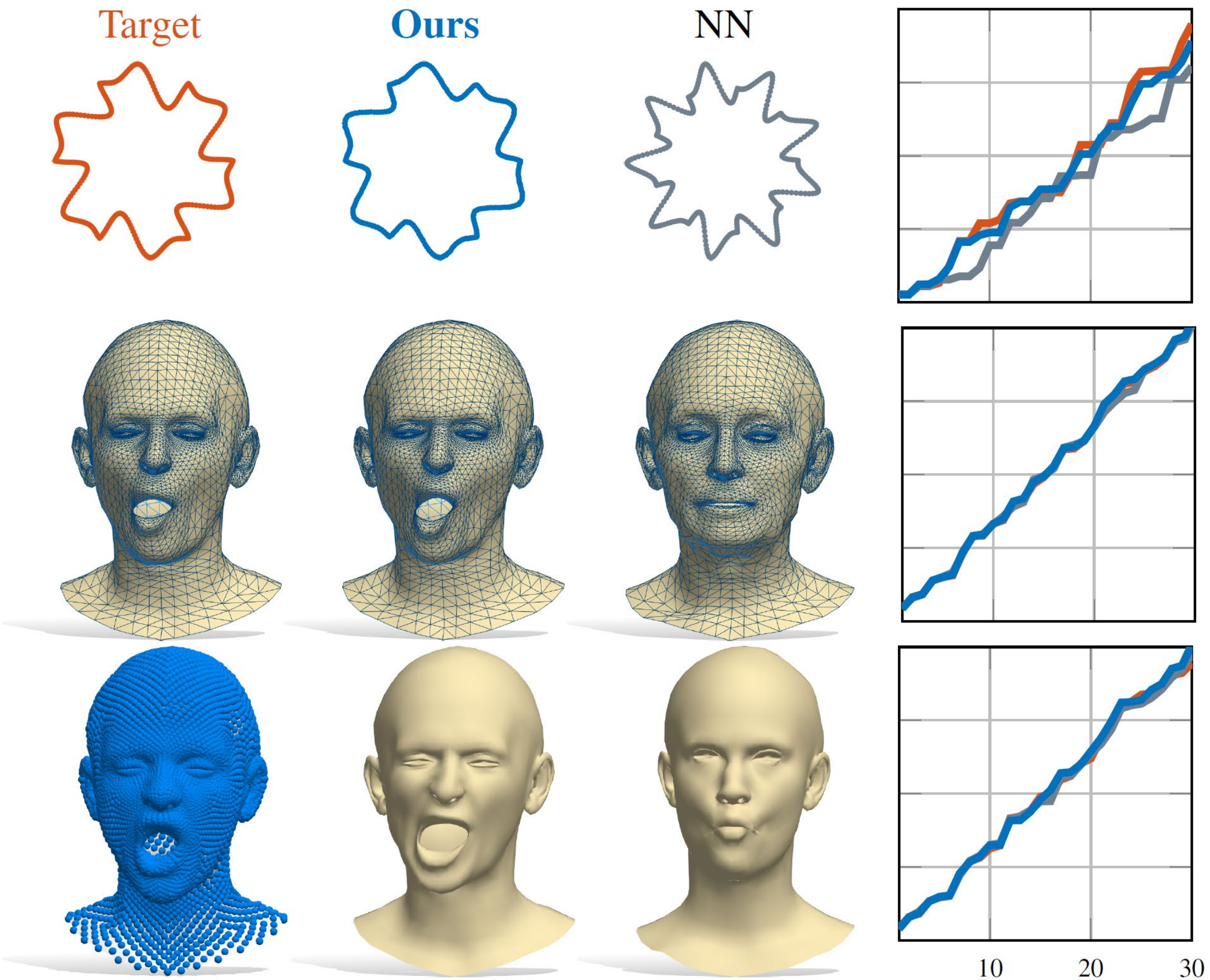
Reuter et al., Laplace–Beltrami spectra as 'Shape-DNA', 2006

Reconstruction from LB Spectrum: Isospectralization



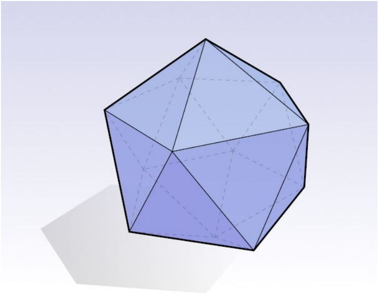
Luca Cosmo, Mikhail Panine, Arianna Rampini, Maks Ovsjanikov, Michael M. Bronstein, Emanuele Rodolà
Isospectralization, or How to Hear Shape, Style, and Correspondence, 2019

Examples

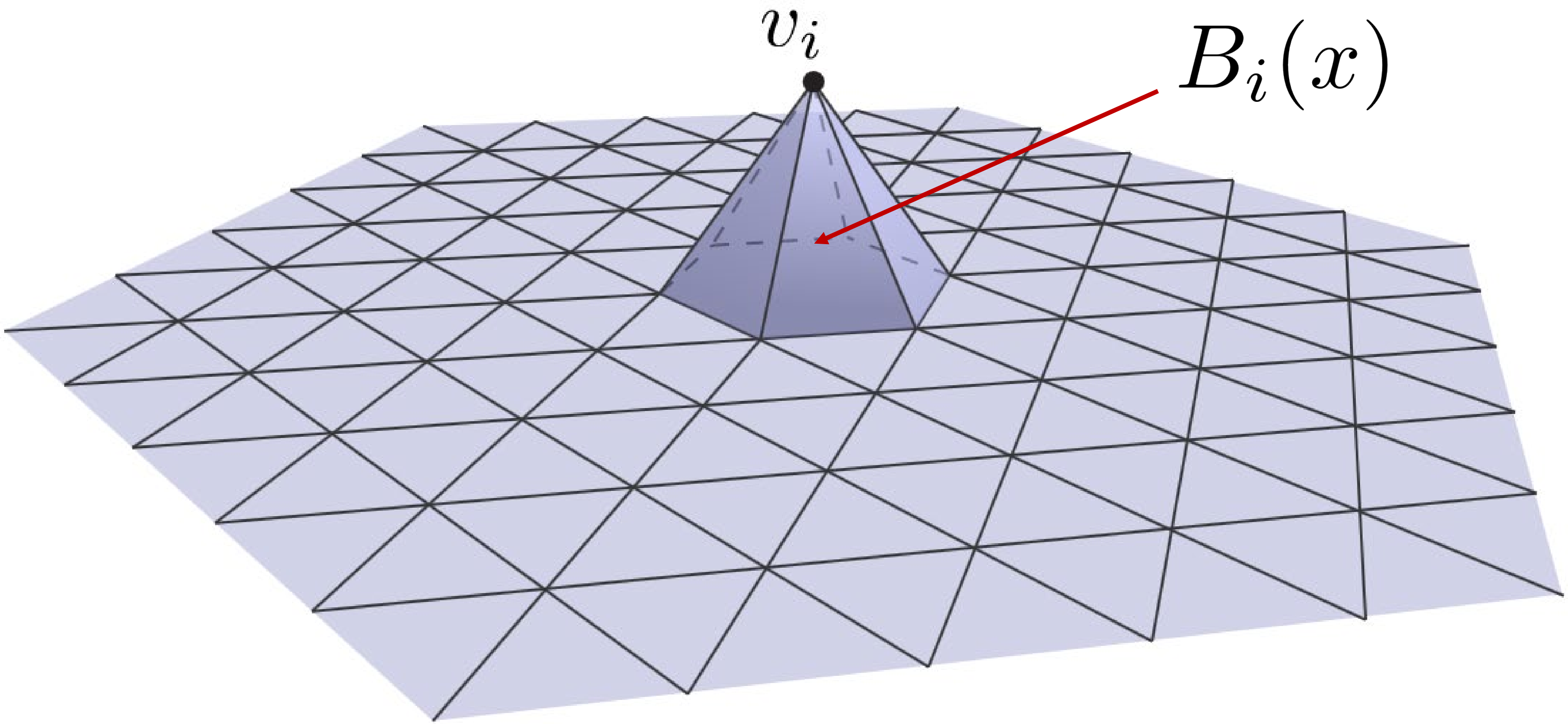


Discretization: Laplace-Beltrami on Meshes

Fourier analysis on meshes



Discretized Functions on Meshes

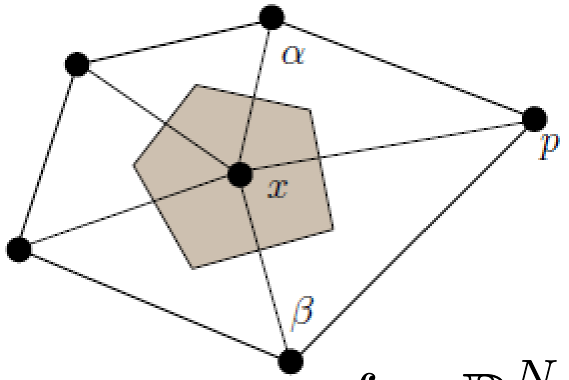


Finite element “hat” functions

Mesh Setting: Discrete Laplace Beltrami

Computing Laplace-Beltrami operator on a mesh:

Functions are real-values defined on the triangles



$f \in \mathbb{R}^N$

Strategy: find an operator L that will satisfy

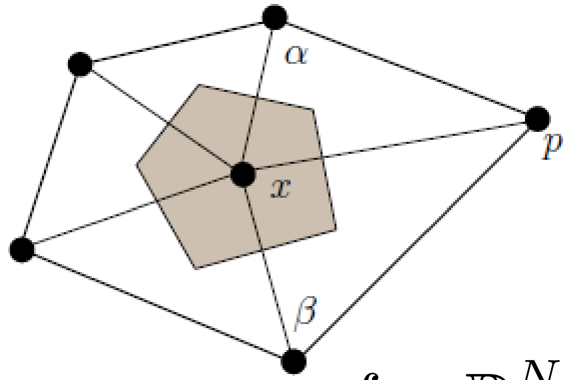
The discrete version of divergence theorem:

$$\int f L(g) dx = \int < \nabla f, \nabla g > dx \ \forall f, g$$

Mesh Setting: Laplace Beltrami

Computing Laplace-Beltrami operator on a mesh:

Functions are real-values defined on the triangles



$f \in \mathbb{R}^N$

To discretize the integral, we use:

$$\int f dx = \sum_i f_i A_i$$

Here, A_i is the local area element associated with vertex *i.*

The simplest choice:

$$A_i = \frac{1}{3} \sum_{t \sim i} A(t)$$

Sum of the areas of adjacent triangles.

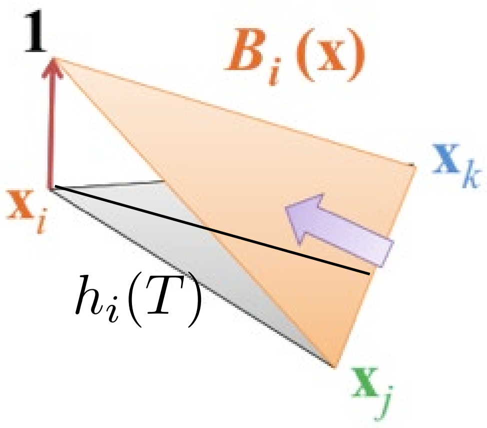
Mesh Setting: Laplace Beltrami

Computing the gradient of a function

Inside a single triangle, use piecewise-linear interpolation:

$$f(x) = f_i B_i(x) + f_j B_j(x) + f_k B_k(x)$$

$$\nabla f(x) = f_i \nabla B_i(x) + f_j \nabla B_j(x) + f_k \nabla B_k(x)$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(x) = \frac{1}{h_i(T)}$$

Gradient is constant on a triangle.

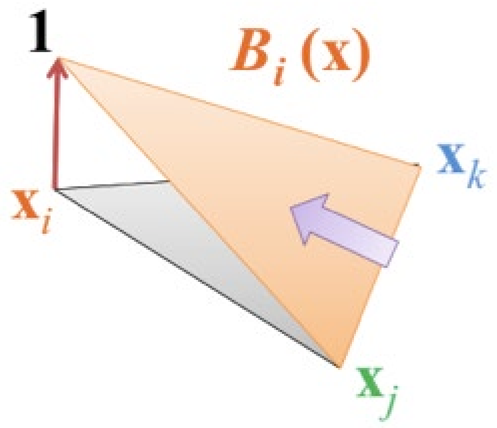
Mesh Setting: Laplace Beltrami

Computing the gradient of a function

Inside a single triangle, use piecewise-linear interpolation:

$$f(x) = f_i B_i(x) + f_j B_j(x) + f_k B_k(x)$$

$$\nabla f(x) = f_i \nabla B_i(x) + f_j \nabla B_j(x) + f_k \nabla B_k(x)$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(x) = \nabla B_i = \frac{(x_k - x_j)^\perp}{2A_T}$$

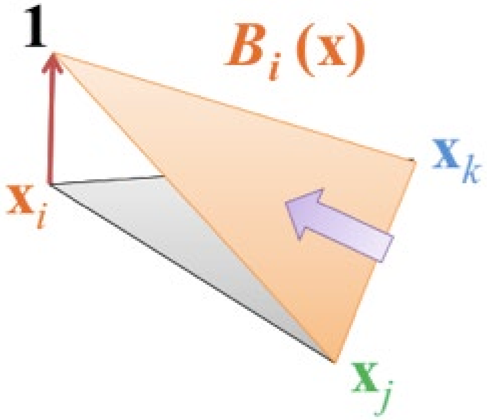
Gradient is constant on a triangle.

Mesh Setting: Laplace Beltrami

Computing the gradient of a function

Inside a single triangle, use piecewise-linear interpolation:

$$\nabla f(x) = f_i \nabla B_i(x) + f_j \nabla B_j(x) + f_k \nabla B_k(x)$$
$$= \frac{f_i}{2A_T}(x_k - x_j)^{\perp} + \frac{f_j}{2A_T}(x_i - x_k)^{\perp} + \frac{f_k}{2A_T}(x_j - x_i)^{\perp}$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(x) = \nabla B_i = \frac{(x_k - x_j)^{\perp}}{2A_T}$$

Gradient is constant on a triangle.

Mesh Setting: Laplace Beltrami

Back to our strategy:

$$\int f L(g) dx = \int < \nabla f, \nabla g > dx \ \forall f, g$$

In the discrete case:

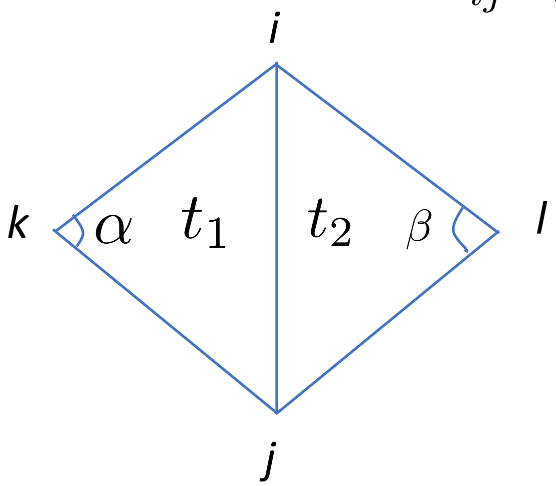
$$\sum_i f_i L(g)_i A(i) = \sum_t < \nabla f(t), \nabla g(t) > A(t)$$

Mesh Setting: Laplace Beltrami

In the discrete case:

$$\sum_i f_i L(g)_i A(i) = \sum_t < \nabla f(t), \nabla g(t) > A(t)$$

Plugging in indicator functions of individual vertices we get:



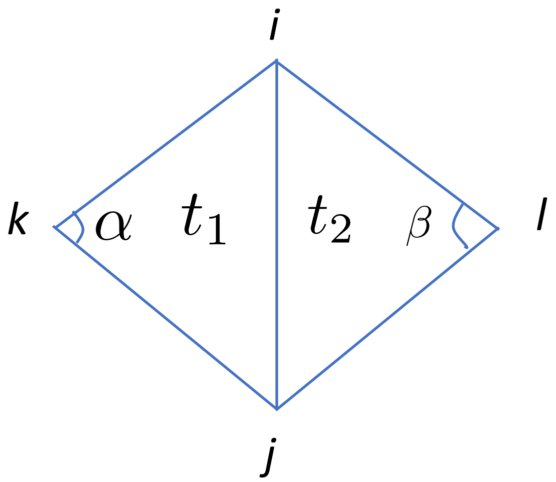
$$L_{ij} A(j) =< \frac{1}{2A(t_1)}(x_k - x_j)^{\perp}, \frac{1}{2A(t_1)}(x_i - x_k)^{\perp} > A(t_1)$$
$$+ < \frac{1}{2A(t_2)}(x_l - x_j)^{\perp}, \frac{1}{2A(t_2)}(x_i - x_l)^{\perp} > A(t_2)$$

Mesh Setting: Laplace Beltrami

In the discrete case:

$$\sum_i f_i L(g)_i A(i) = \sum_t < \nabla f(t), \nabla g(t) > A(t)$$

Plugging in indicator functions of individual vertices we get:



$$= \frac{1}{4A(t_1)} \cos(\alpha) \|e_{ki}\| \|e_{kj}\| + \frac{1}{4A(t_2)} \cos(\beta) \|e_{li}\| \|e_{lj}\|$$

$$A(t_1) = \frac{1}{2} \sin(\alpha) \|e_{ki}\| \|e_{kj}\|$$

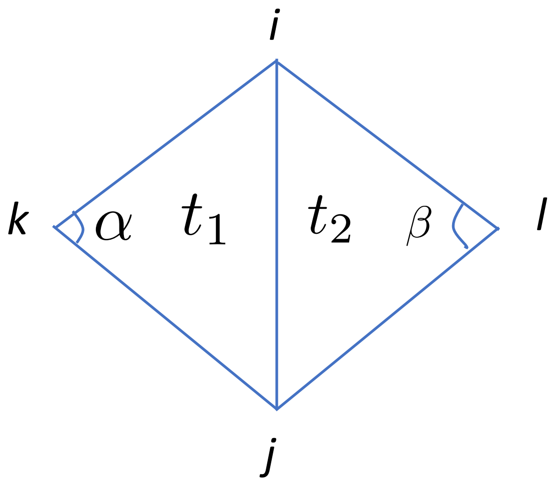
$$L_{ij} A(j) = \frac{1}{2} \cot(\alpha) + \frac{1}{2} \cot(\beta)$$

Mesh Setting: Laplace Beltrami

In the discrete case:

$$L_{ij}A(j) = \frac{1}{2}\cot(\alpha) + \frac{1}{2}\cot(\beta)$$

In matrix notation: $L = A^{-1}W$



$$W_{ij} = \begin{cases} -\frac{1}{2}\left(\cot(\alpha) + \cot(\beta)\right) & \text{if } i \sim j \\ -\sum_j W_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

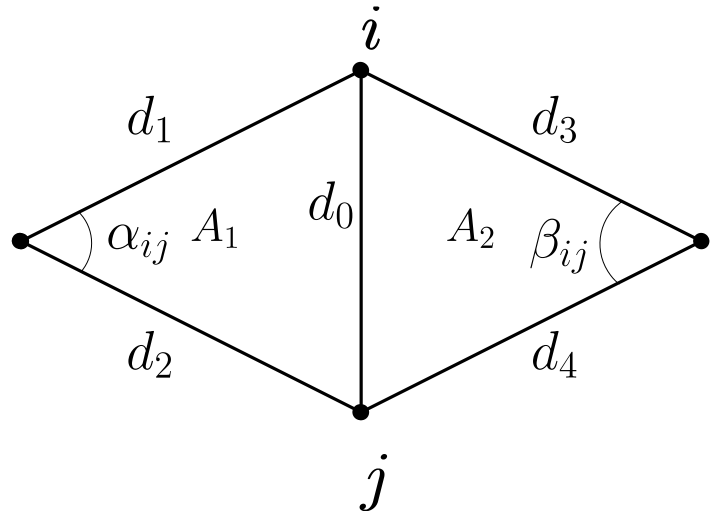
$$A_{ij} = \begin{cases} A(j) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Mesh Setting: Laplace Beltrami

In the discrete case:

$$L_{ij}A(j) = \frac{1}{2}\cot(\alpha) + \frac{1}{2}\cot(\beta)$$

In matrix notation: $L = A^{-1}W$



Small computational trick:

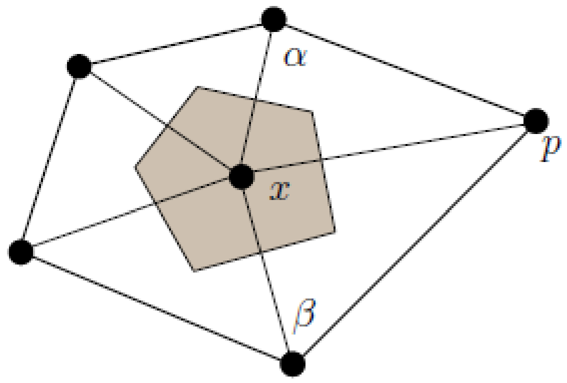
$$\begin{aligned} -W_{ij} &= \frac{1}{2}\cot(\alpha_{ij}) + \frac{1}{2}\cot(\beta_{ij}) \\ &= \frac{1}{8A_1}(d_0^2 - d_1^2 - d_2^2) \\ &+ \frac{1}{8A_2}(d_0^2 - d_3^2 - d_4^2) \end{aligned}$$

Mesh Setting: Laplace Beltrami

Computing Laplace-Beltrami from a mesh:

Classical discretization

$$L = A^{-1}W$$



Very simple to compute, produces sparse matrices.

Using this discretization, for any two real-valued functions f,g we obtain

$$\int fL(g)dx = \int < \nabla f, \nabla g > dx \ \forall f, g$$

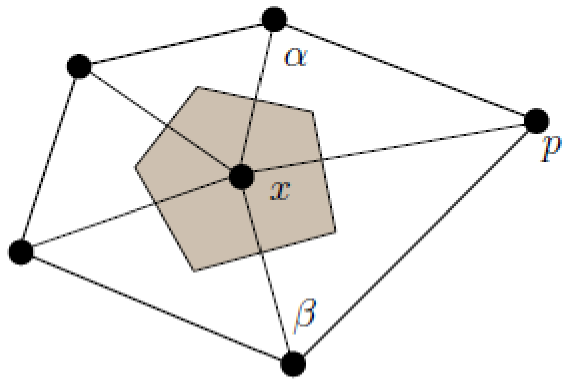
Mesh Setting: Laplace Beltrami

Computing Laplace-Beltrami from a mesh:

Classical discretization

$$L = A^{-1}W$$

Very simple to compute, produces sparse matrices.



To find eigenvalues, solve *generalized* eigenproblem:

$$W\phi_i = \lambda_i A\phi_i$$

Guarantee real solution since W is symmetric and A is SPD.

Mesh Setting: Laplace Beltrami

Good news:

Cotangent Laplacian converges weakly under mesh refinement for "nice enough" meshes

M. Wardetzky. Convergence of the cotangent formula: An overview. 2008

Bad news: No free lunch

There is no discrete LB operator that is

- Symmetric
- Local
- Has Linear Precision
- Always has positive weights.

M. Wardetzky. Discrete Laplace operators: No free lunch, 2007

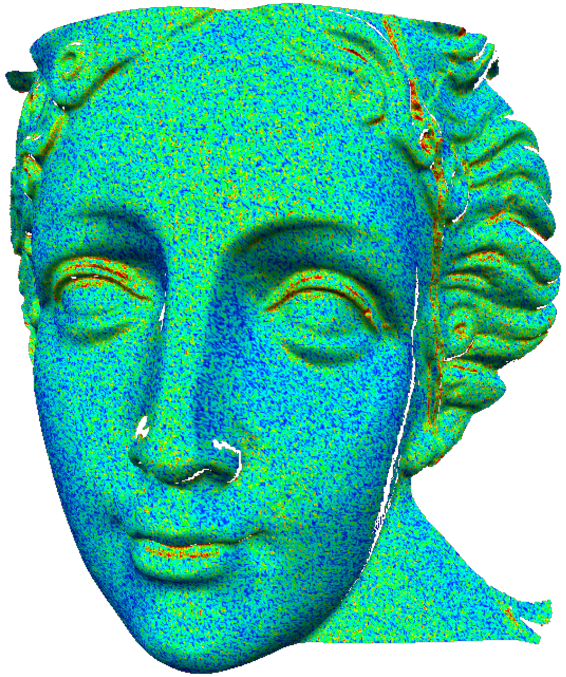
Laplace-Beltrami – Some Applications

Input: Noisy mesh (scanned or other)

Output: Smooth mesh

How: Filter out high frequency noise

Mesh Smoothing



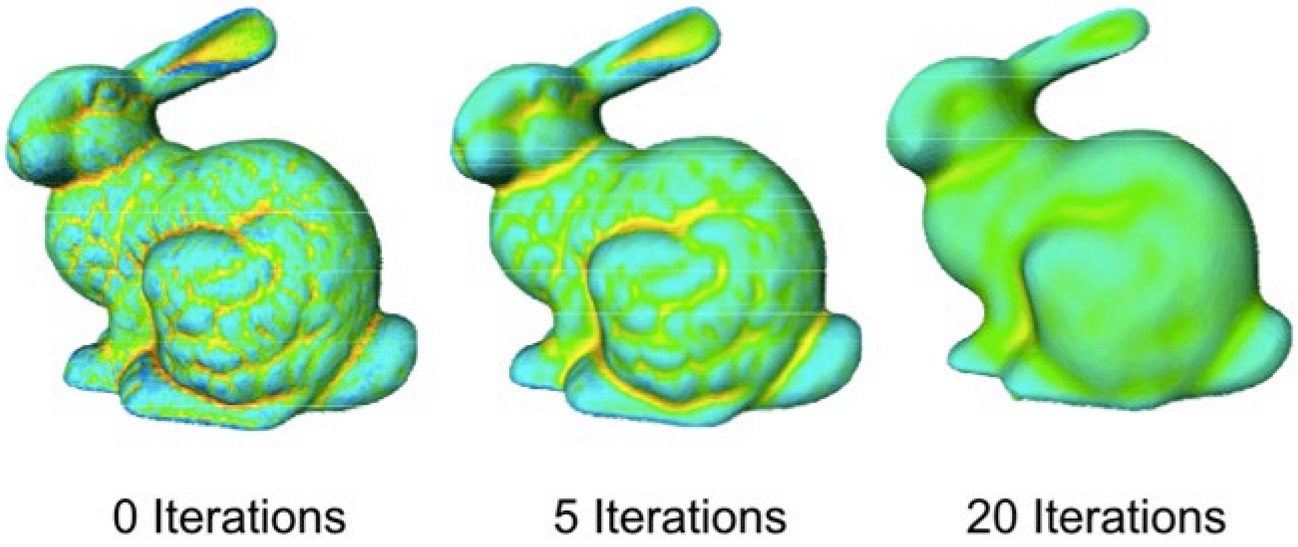
Laplace-Beltrami – Some Applications

Laplacian Smoothing:

Given the matrix of 3D coordinates, apply a few iterations of

$$\mathbf{P}^{(t)} = \mathbf{P}^{(t-1)} - \lambda L \mathbf{P}^{(t-1)}$$

Move each vertex towards the mean of its neighbors



Spectral Analysis

Then: $$\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L}\mathbf{P}^{(t)} = (\mathbf{I} - \lambda \mathbf{L})\mathbf{P}^{(t)}$$

After m iterations: $$\mathbf{P}^{(m)} = (\mathbf{I} - \lambda \mathbf{L})^m \mathbf{P}^{(0)}$$

Can be described using eigen-decomposition of $\mathbf{L}$

Filtering high frequencies

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T$$

$$\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \cdots & \\ & & & k_n \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{P}^{(m)} = \mathbf{V}\boxed{(\mathbf{I} - \lambda \mathbf{D})^m}\mathbf{V}^T\mathbf{P}^{(0)}$$

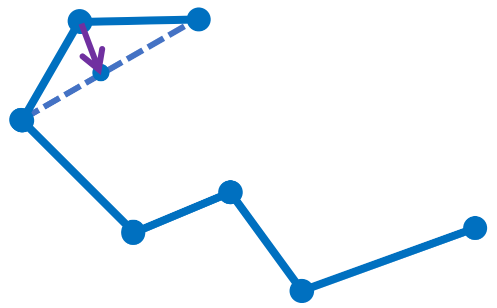
Laplacian Smoothing on Meshes

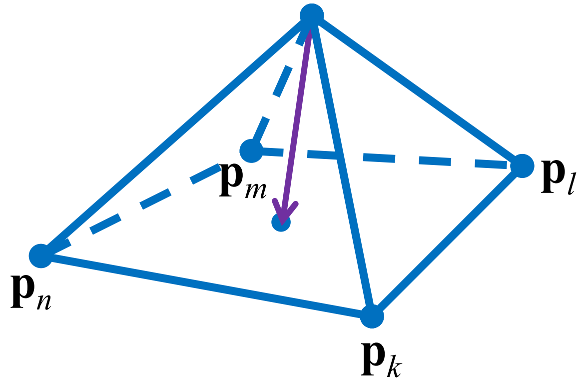
$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} - \lambda \Delta \mathbf{p}_i^{(t)}$$

What is $\Delta \mathbf{p}_i$?

$N_i = \{k, l, m, n\}$

$\mathbf{p}_i = (x_i, y_i, z_i)$





$$\frac{1}{2}\left(\mathbf{p}_{i+1} + \mathbf{p}_{i-1}\right) - \mathbf{p}_i \qquad \frac{1}{|N_i|}\left(\sum_{j \in N_i} \mathbf{p}_j\right) - \mathbf{p}_i$$

Laplacian Smoothing

$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} - \lambda \Delta \mathbf{p}_i^{(t)}$$

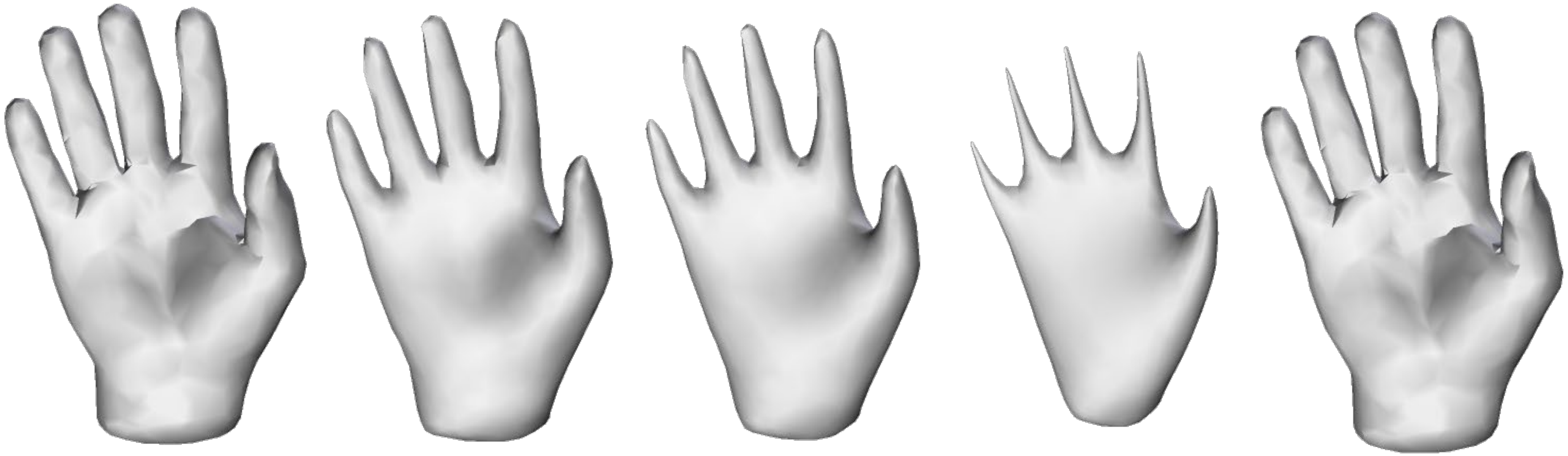
$\Delta \mathbf{p}_i$ = mean curvature normal

mean curvature flow

$$\Delta_{\mathcal{M}} \mathbf{p} = \operatorname{div}_{\mathcal{M}} \nabla_{\mathcal{M}} \mathbf{p} \boxed{= -2H\mathbf{n} \;\; \in \mathbb{R}^3}$$

Problem - Shrinkage

Repeated iterations of Laplacian smoothing shrinks the mesh



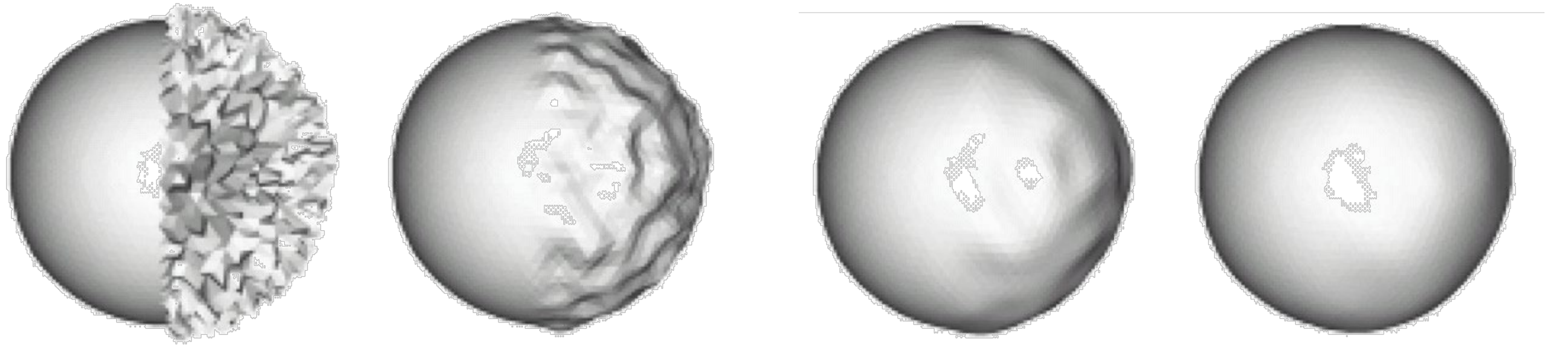
original 3 steps 6 steps 18 steps original

Taubin Smoothing

Iterate: $\mathbf{p}_i \leftarrow \mathbf{p}_i - \lambda \Delta \mathbf{p}_i$ Shrink

$\mathbf{p}_i \leftarrow \mathbf{p}_i - \mu \Delta \mathbf{p}_i$ Inflate

with $\lambda > 0$ and $\mu < 0$



original 10 steps 50 steps 200 steps

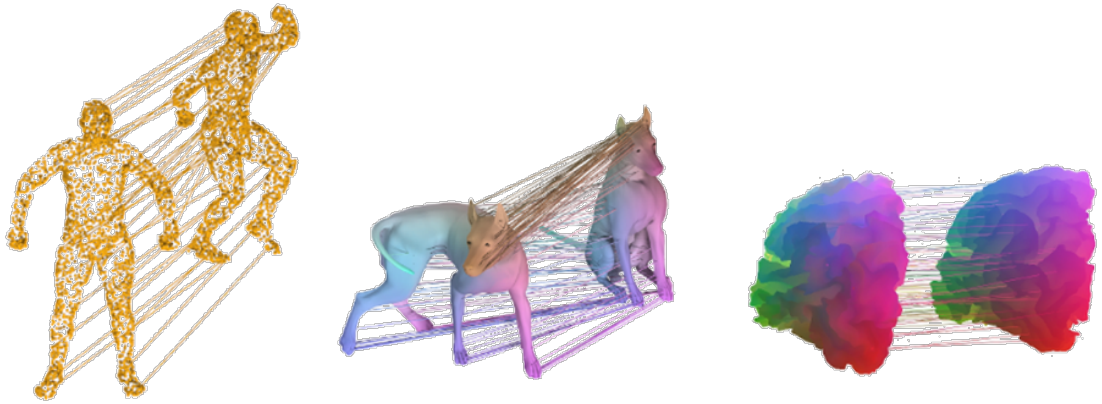
From Taubin, Siggraph 1995

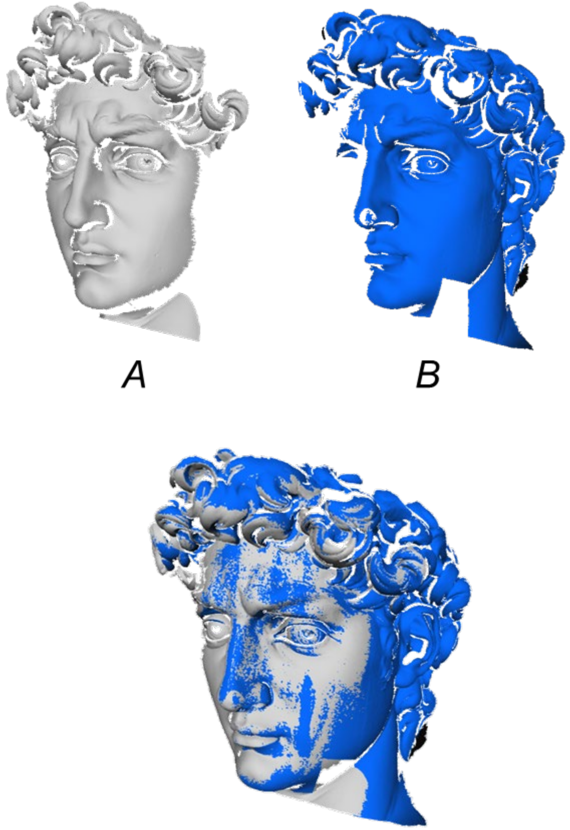
Conclusion

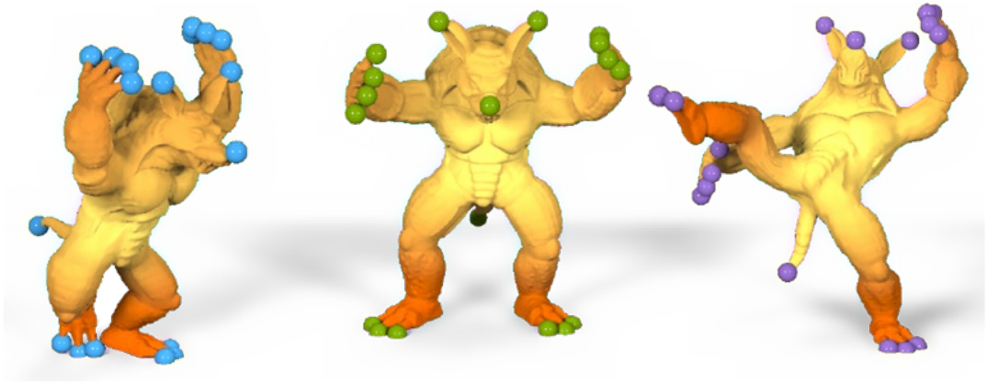
- Spectral Methods in Shape Analysis
  - Mesh Laplacians -- Laplace-Beltrami operator and its properties
  - Isometry invariance
  - Relatively easy to compute
  - Some applications
- Key message:
  - Laplacian matrices allow us to organize shape information in a multi-scale, easy to manipulate way.

Shape Alignments and Correspondences

Alignment and Correspondences, Shape Features

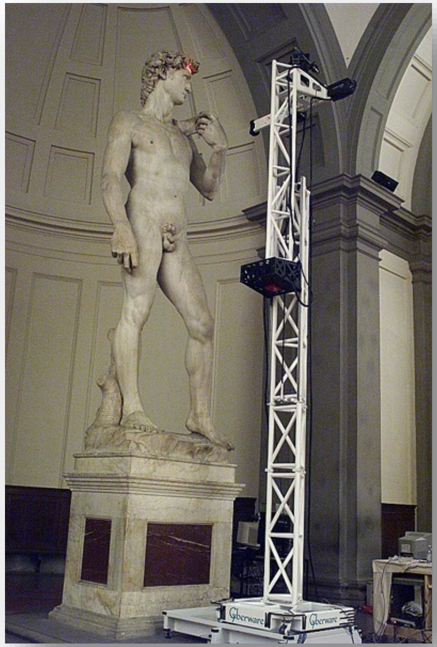


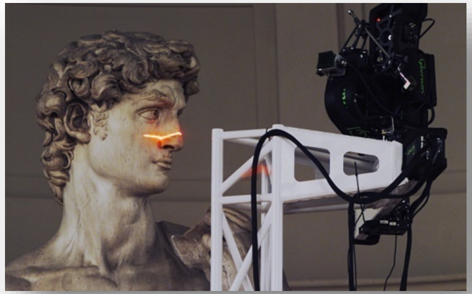




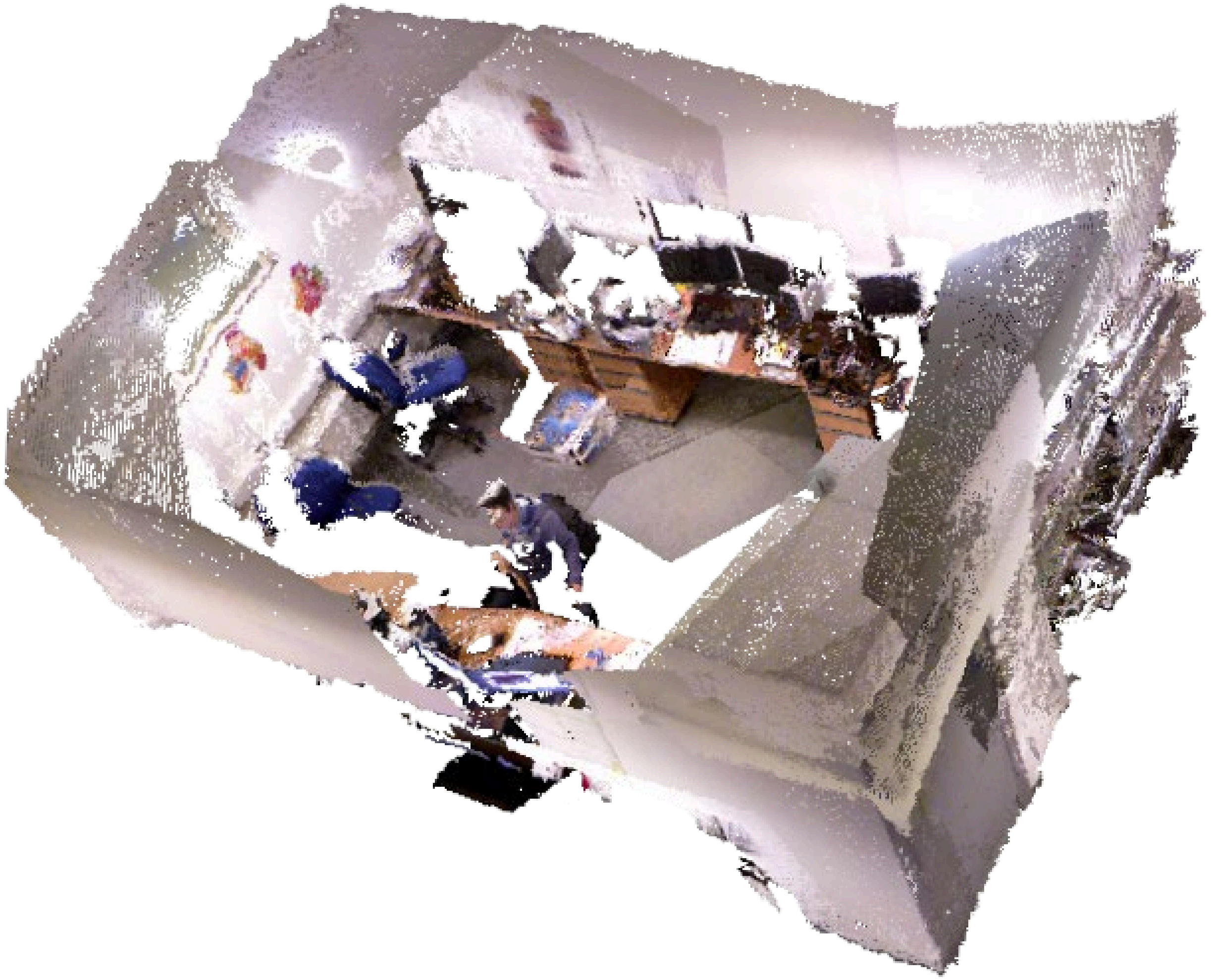
Aligning and Registering (Partial or Entire) Shapes

- The need to align and match shapes arises in many domains
- A classic example is shape acquisition through a 3D scanner

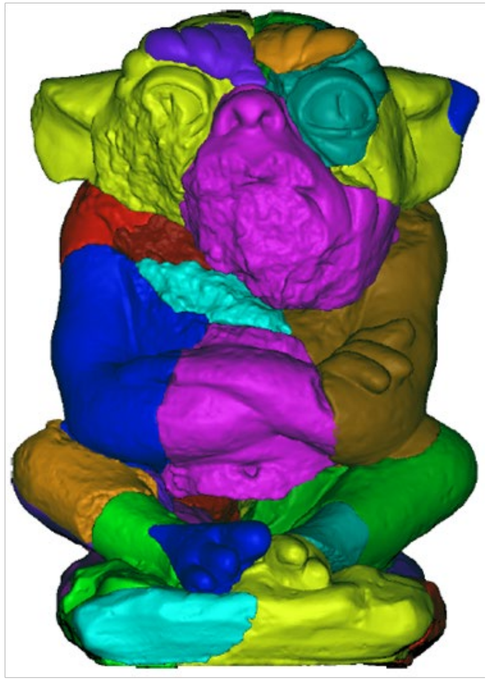




Simultaneous Localization and Mapping (SLAM)



Cultural Artifact Reconstruction



Protein Structure Alignment

Human (red) and fly (yellow) thierodoxins, compared

Measuring Success: Shape Distances

Given two shapes *A* and *B*, we are interested in defining a distance or (dis-)similarity measure

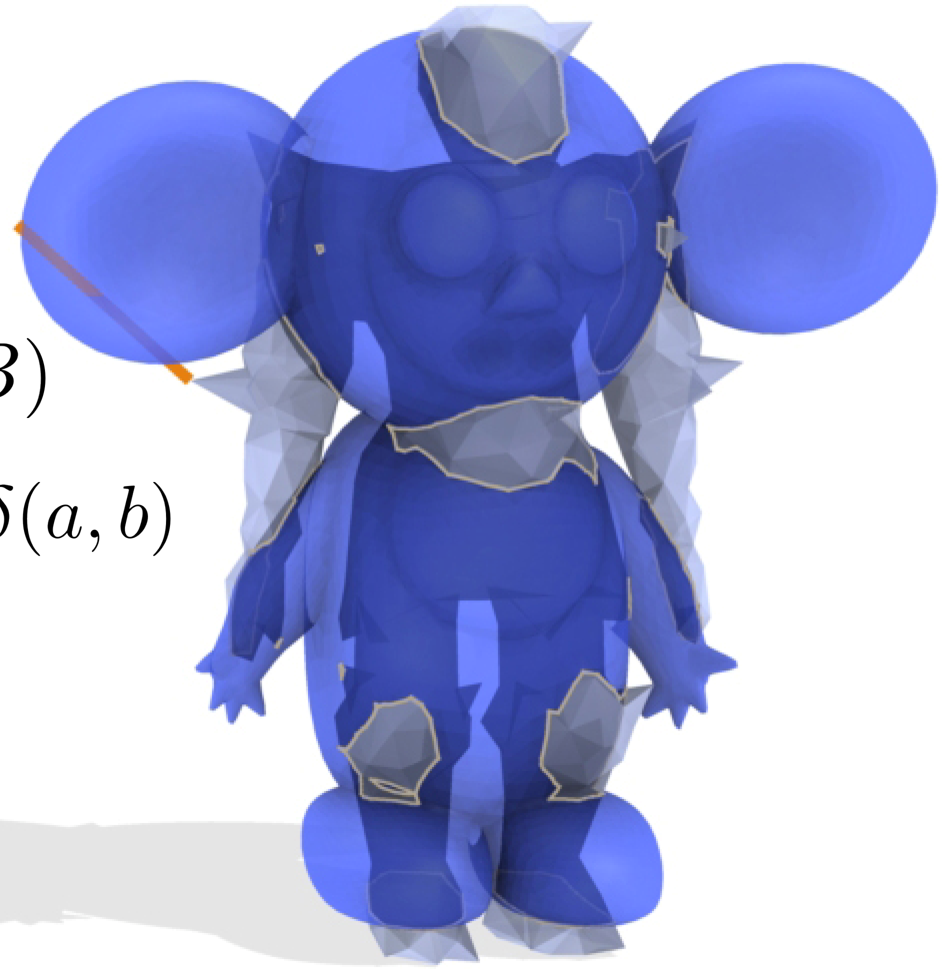
$$\min_{T} \delta(A, T(B))$$ [extrinsic]

Such measures are crucial in shape similarity search, shape classification, and in general for defining ML loss functions.

Hausdorff Distance

$$max_{a \in A} \; \delta(a, B)$$

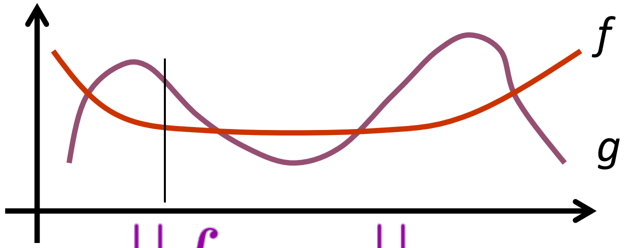
$$\delta(a, B) = \min_{b \in B} \; \delta(a, b)$$



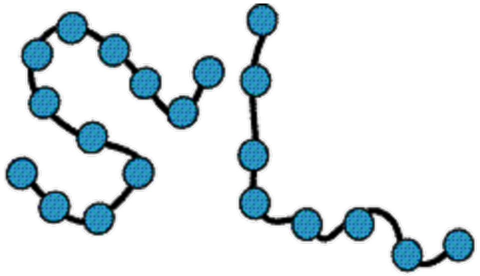
Hausdorff is a worst-case error – in some cases we prefer MSE

Issues about Distance Metrics

- We are all familiar with function norms (L_2, etc.). The common parametrization establishes correspondences. We don't have that for structures or shapes.
- Partial matches need to be considered -- notion of support σ for the match.
- What group of aligning transforms is to be considered?
- Is the resulting distance a metric?

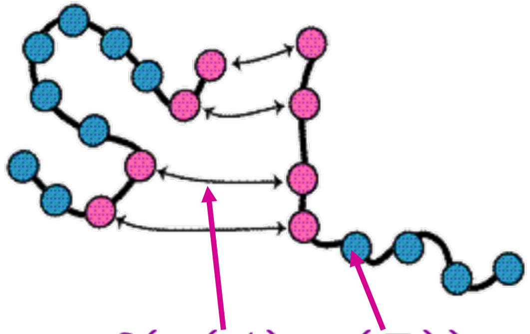


$$||f - g||_2$$



$$\delta(A, C) \leq \delta(A, B) + \delta(B, C)$$

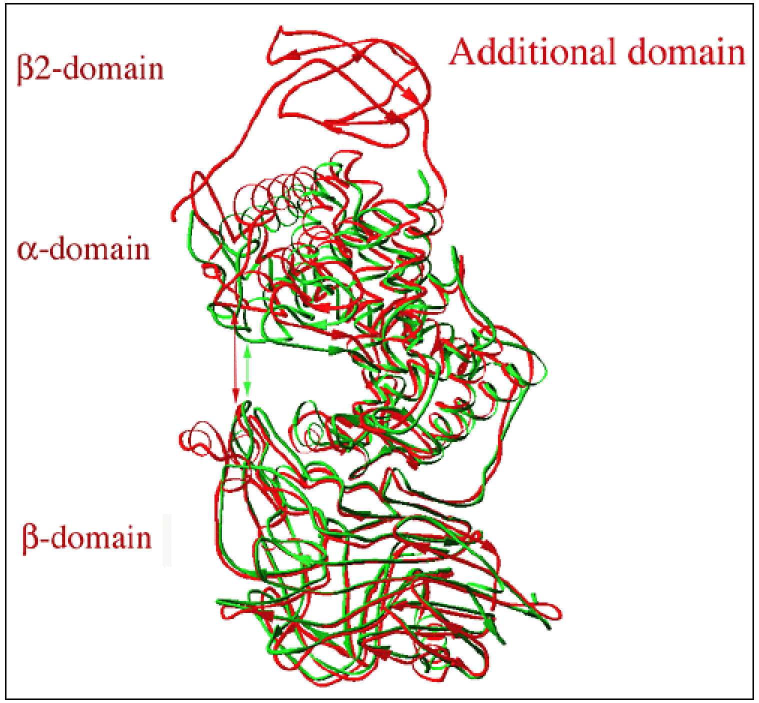
Not for partial matches

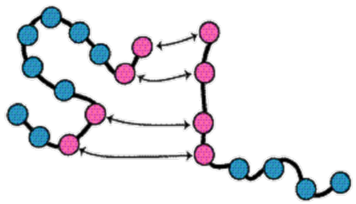


$$\delta(\sigma(A), \sigma(B))$$

Simultaneous Estimation

- We are given two shapes *A* and *B*, each in its own coordinate system
- We must establish correspondences between certain parts (the alignment supports) of *A* and *B*
- We must find an optimal transform that best aligns the supports of *A* and *B*
- We must score this choice of supports and transform to produce a distance measure δ



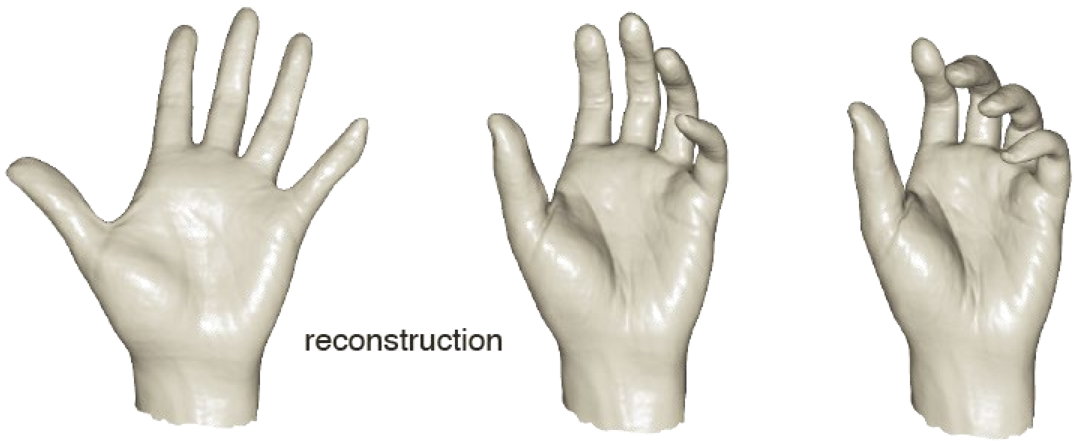


In computing the score,

1. what distance metric do we use?
2. how do we aggregate distances?
3. how do we trade-off larger supports for larger aggregate distance?

Degrees of Freedom

- Transform estimation
  - A rigid motion has 6 degrees of freedom (3 for translation and 3 for rotation)
  - We typically estimate the motion using many more pairs of corresponding points, so the problem is overdetermined (which is good, given noise, outliers, etc – use least squares approaches)
  - More general transforms require more degrees of freedom. When shape deformations are allowed, the degrees of freedom can grow very rapidly



A Double Whammy

- Estimate correspondences
- Estimate the aligning transform

- Gives rise to combinatorial searches
- Transforms can be non-linear

Hard optimization problems!

Good features help

Low-dimensionality of some transforms helps

That’s All