CS233, CME251: Geometric and Topological Data Analysis

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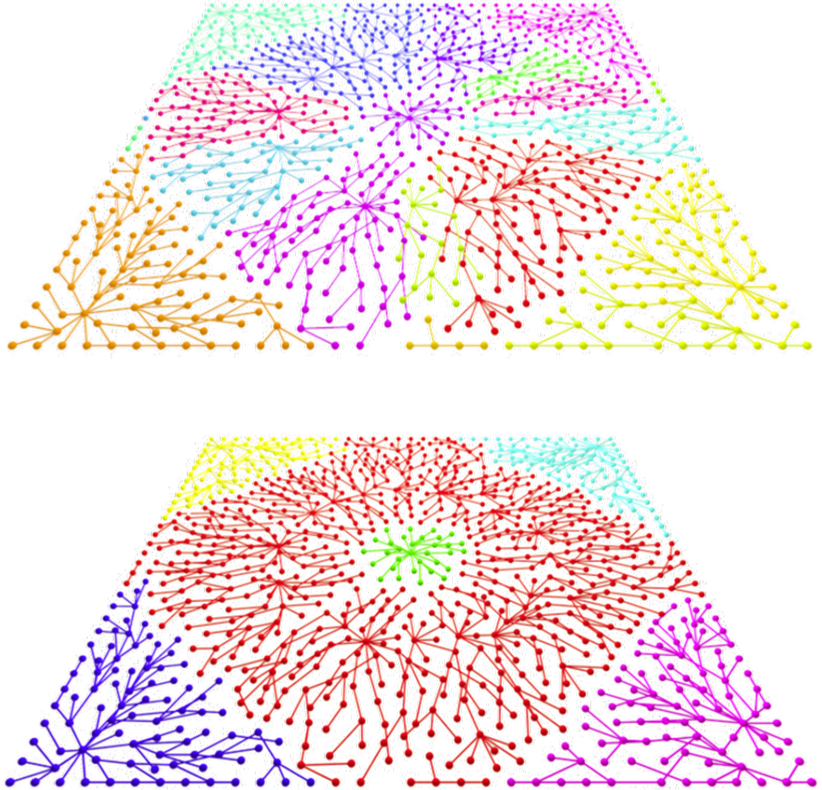
1891

Lecture 15
19 May 2021

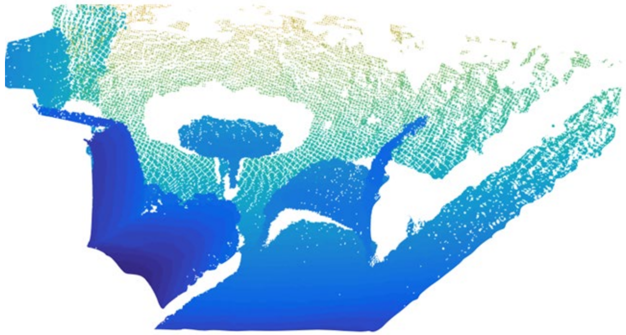


Leonidas Guibas Laboratory

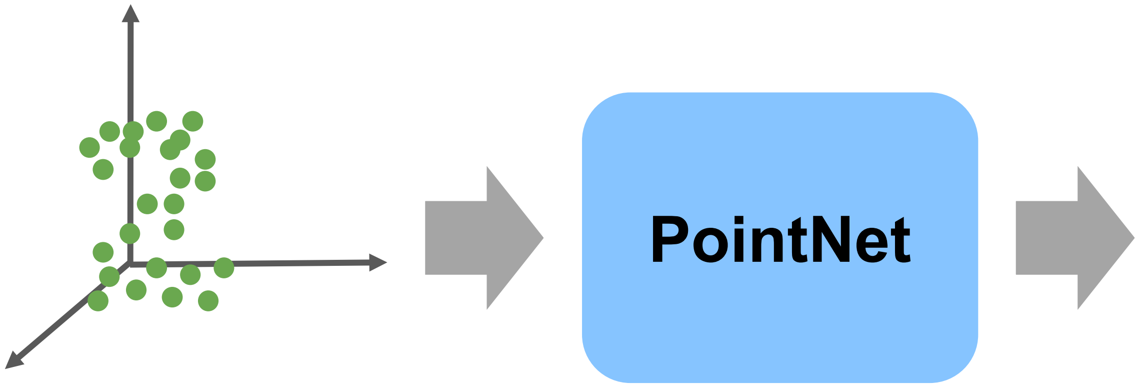
Geometric Computing

Last Time: Deep Learning on Point Cloud Data

Non-regular 3D data



Deep Nets for PCs: PointNet and PointNet++



PointNet

Object Classification

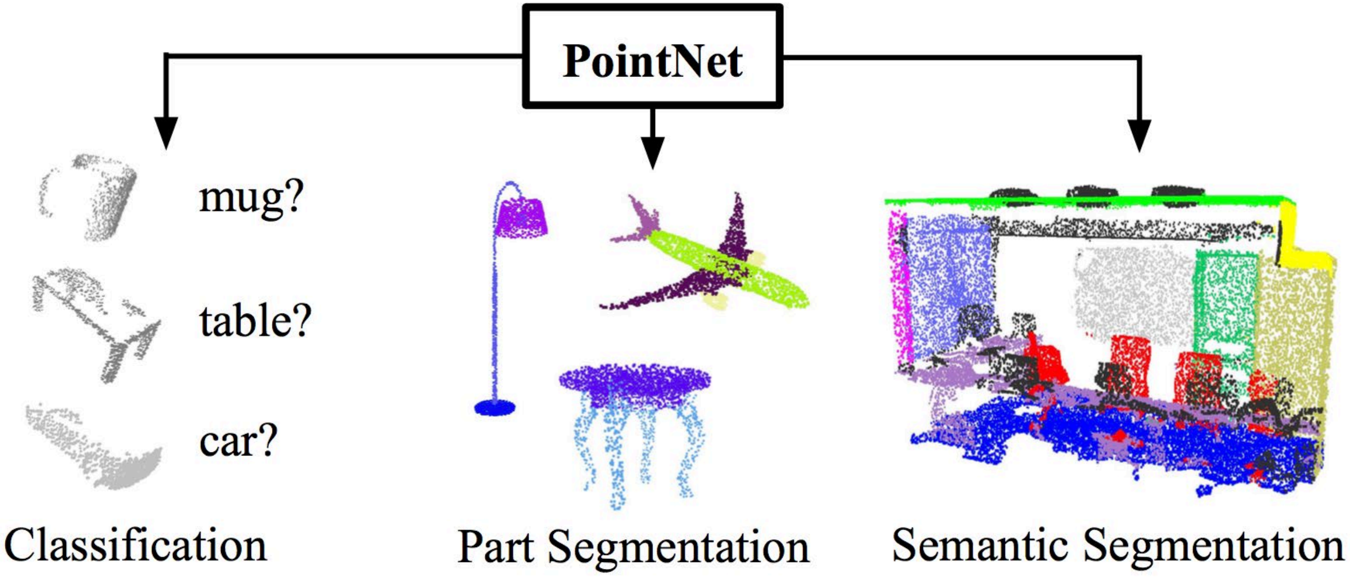
Object Part Segmentation

Semantic Scene Parsing

...

End-to-end learning for irregular point data

Unified framework for various tasks



Charles R. Qi, Hao Su, Kaichun Mo, Leonidas J. Guibas. PointNet: Deep Learning on Point Sets for 3D Classification and Segmentation. (CVPR'17)

Invariances

The model has to respect key desiderata for point clouds:

Point Permutation Invariance

Point cloud is a set of unordered points

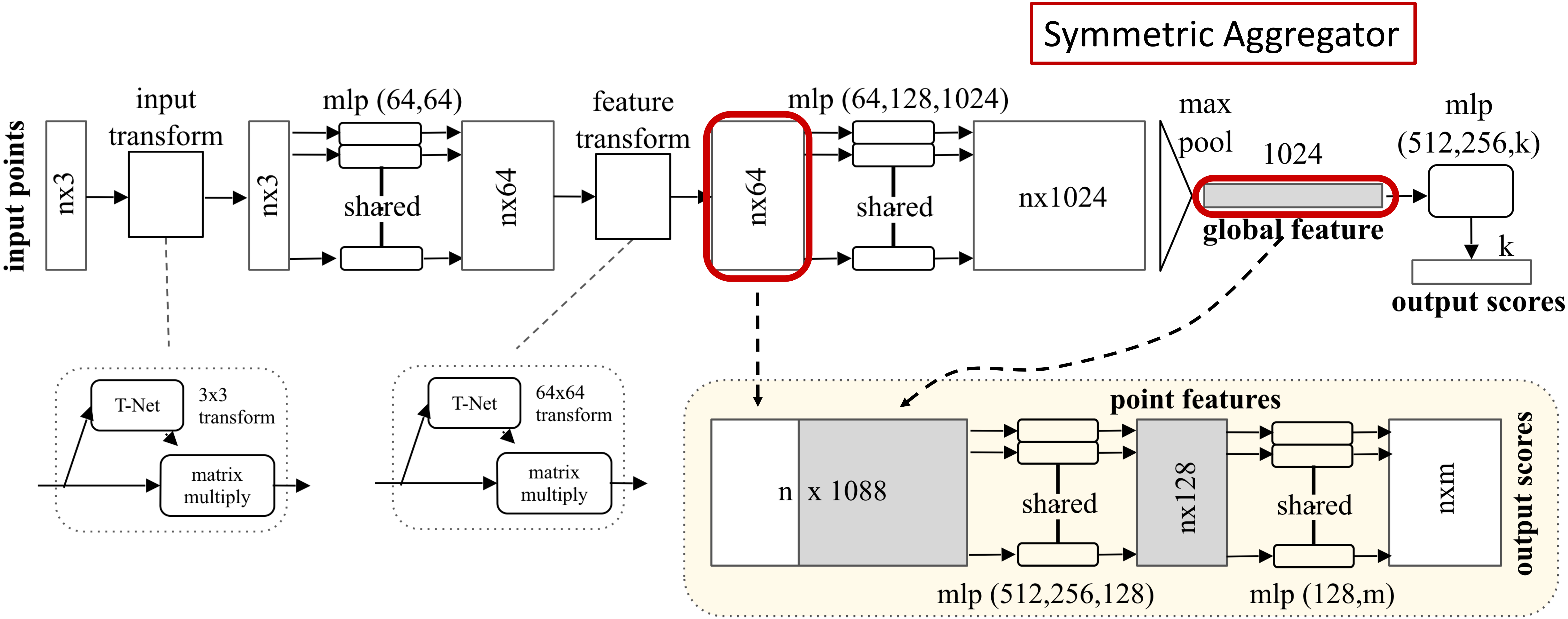
Spatial Transformation Invariance

Point cloud rigid motions should not alter classification results

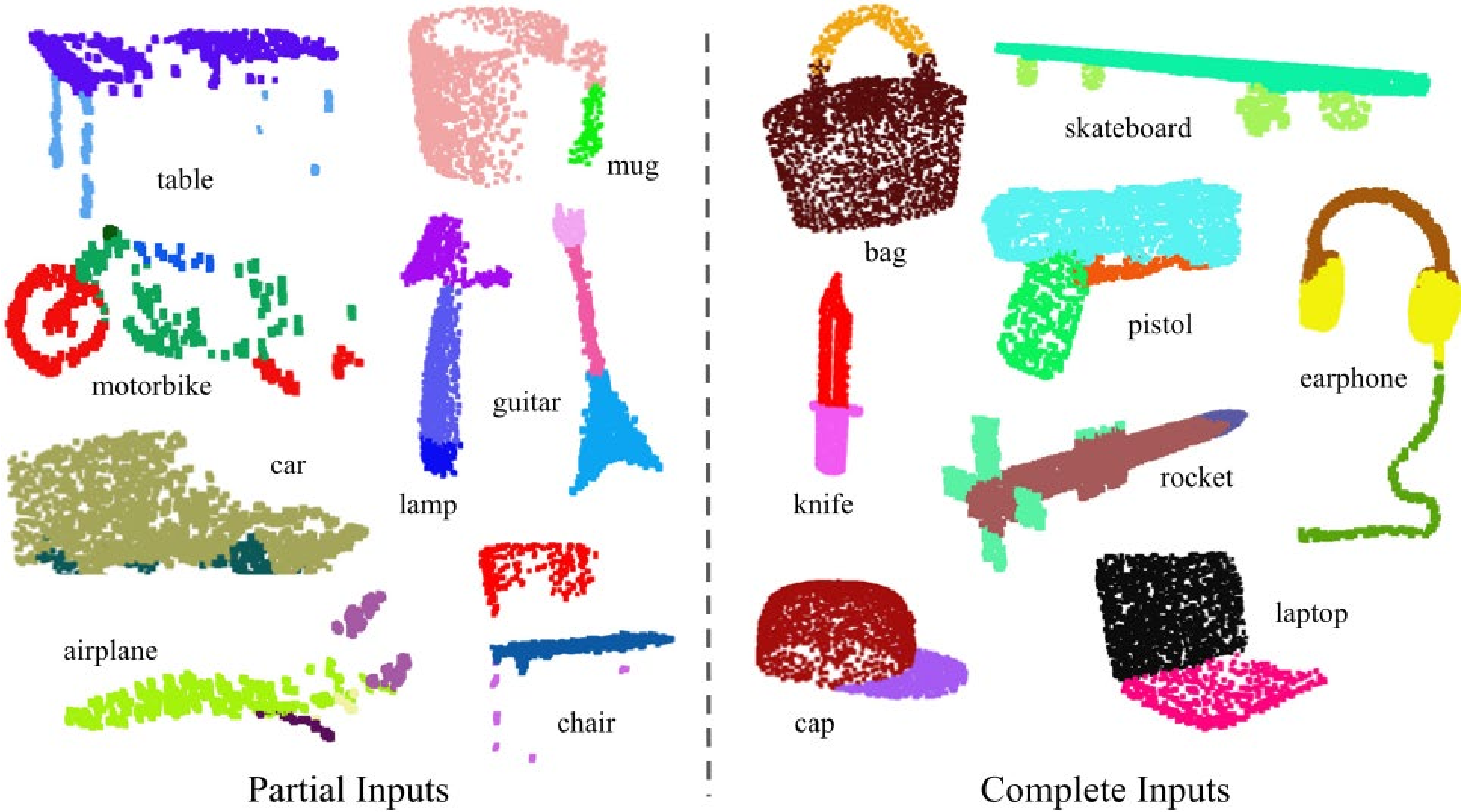
Sampling Invariance

Output a function of the underlying geometry and not the sampling

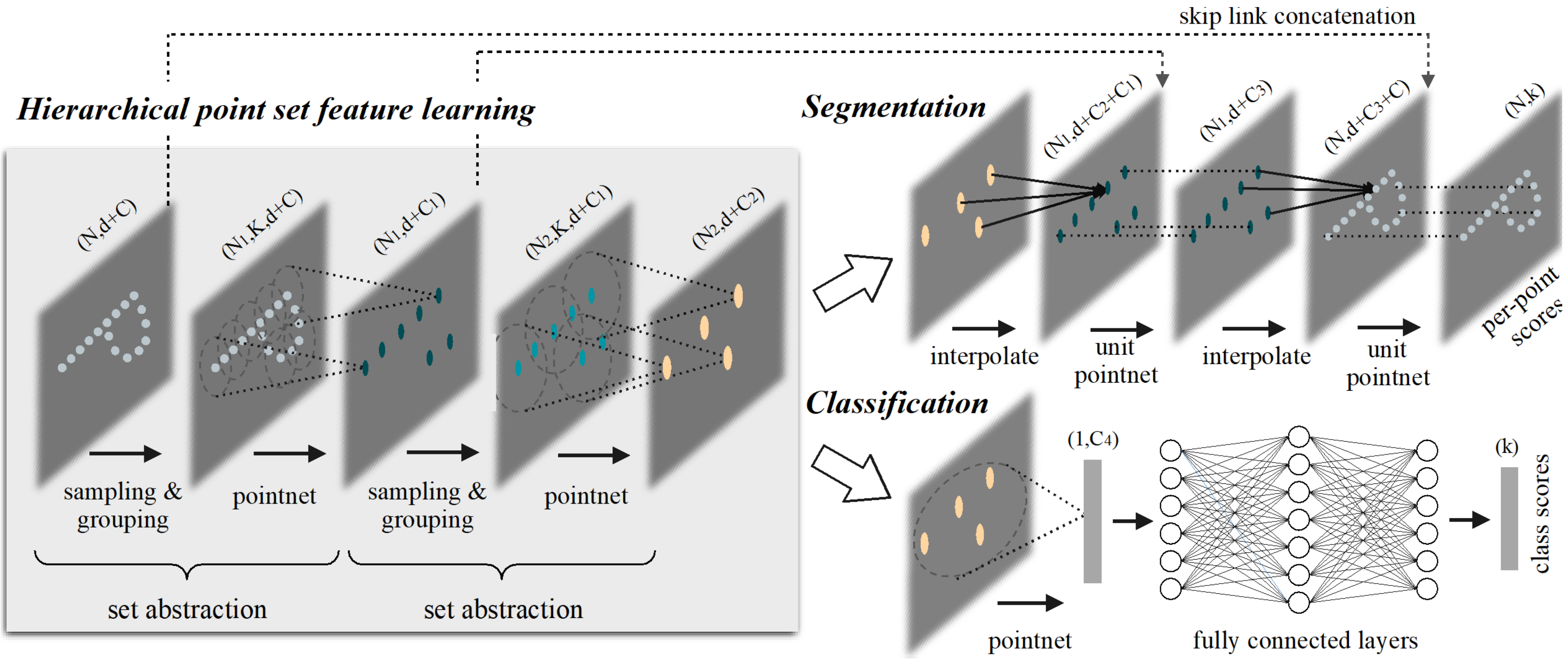
PointNet Classification and Segmentation



Results on Object Part Segmentation

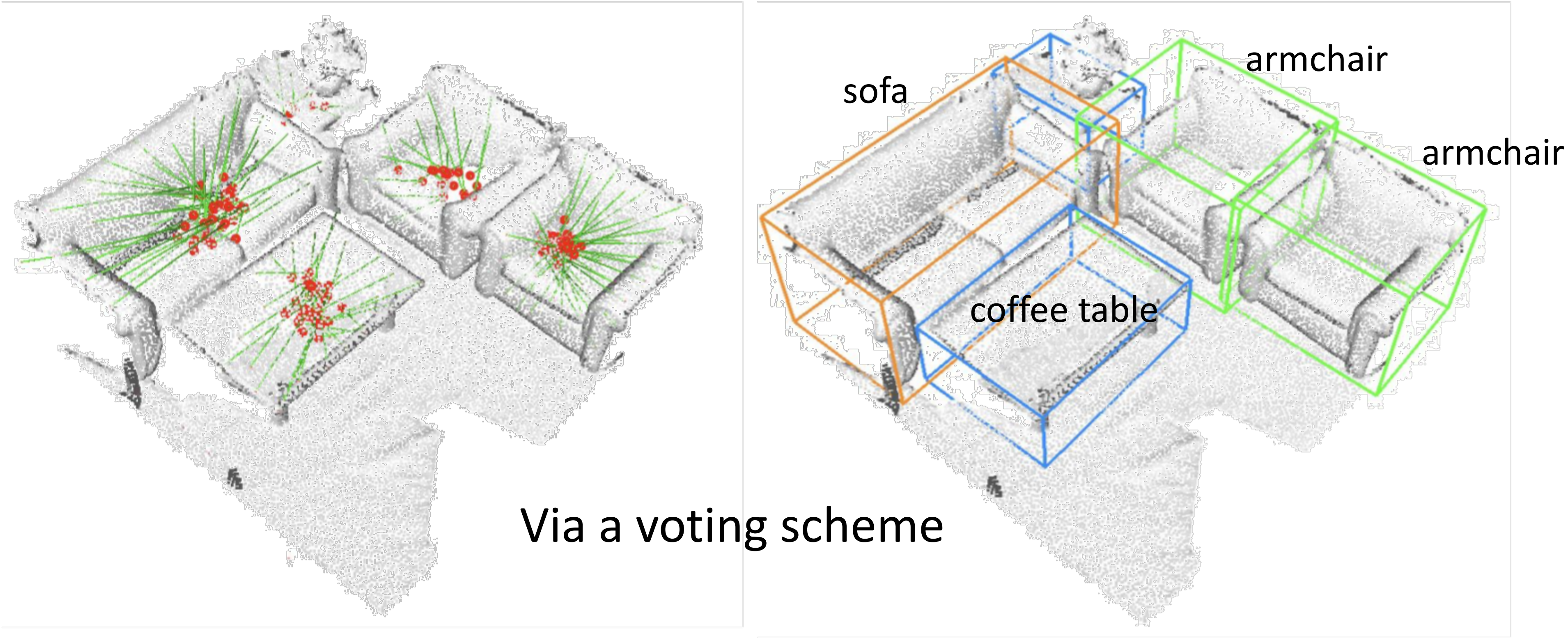


PointNet++ for Classification and Segmentation



Aggregation pattern is only a function of the spatial locations of the points

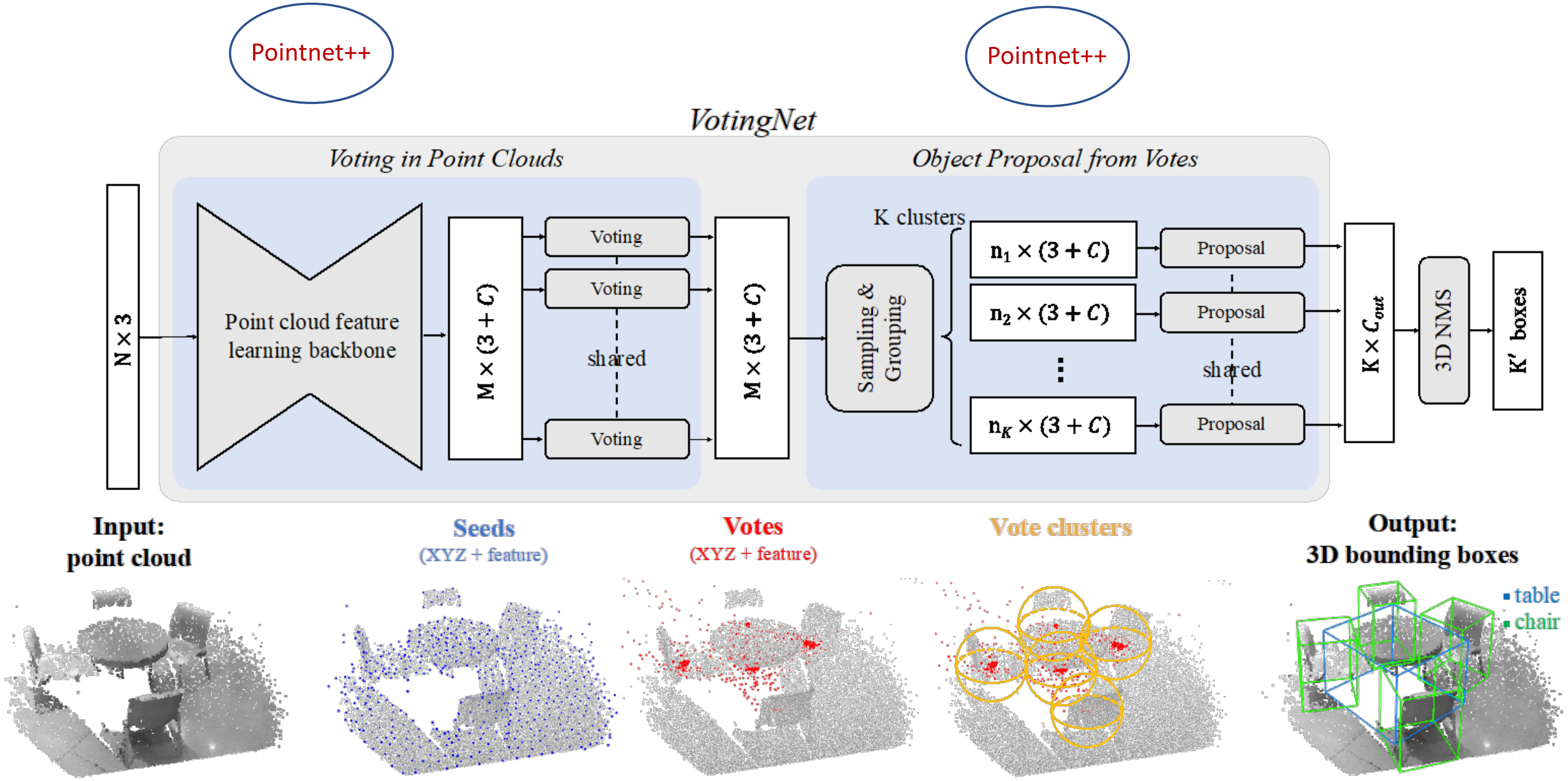
Point Cloud Object Amodal Bounding Box Detection



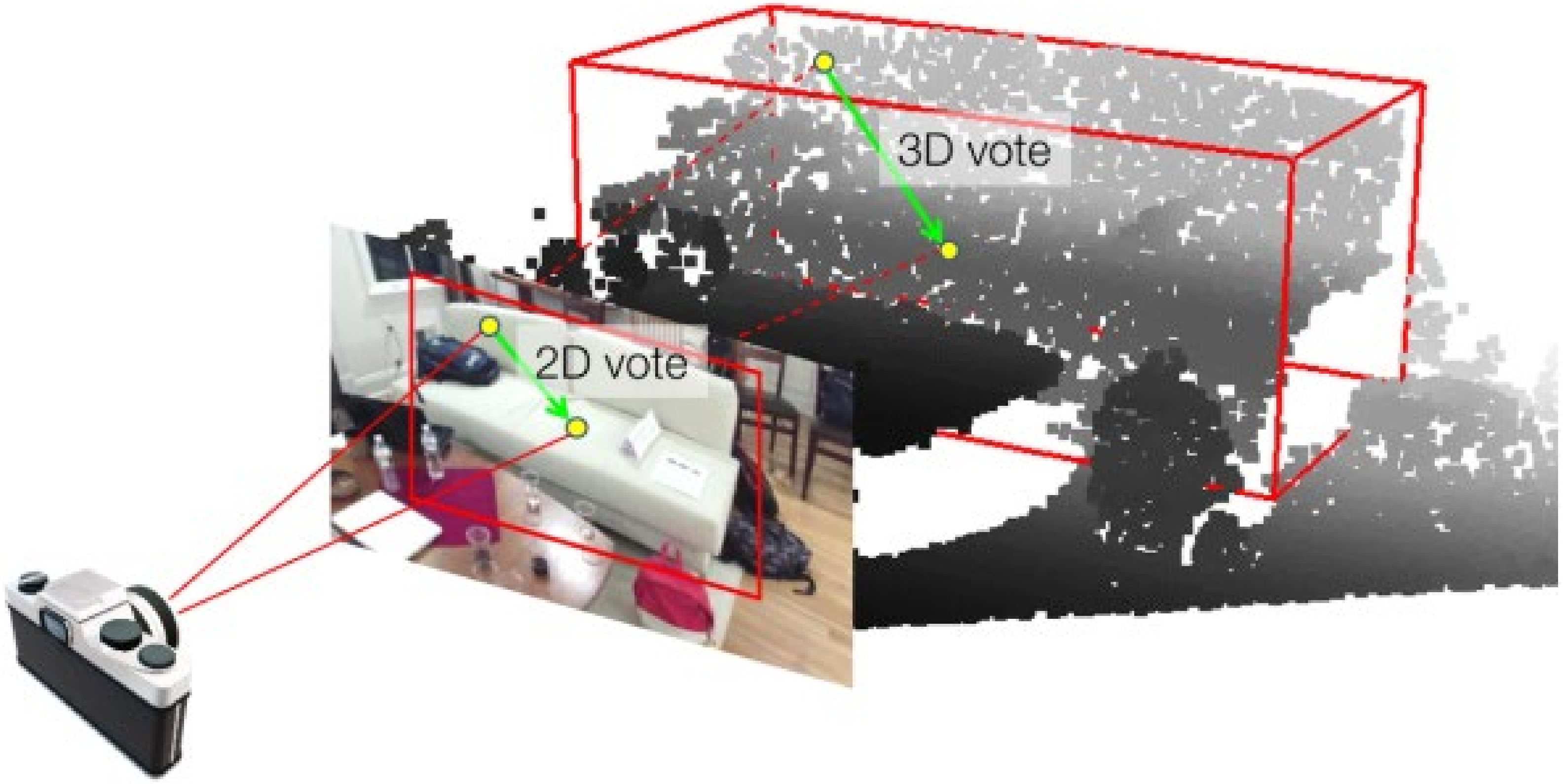
Via a voting scheme

- Charles R. Qi, Or Litany, Kaiming He, Leonidas J. Guibas. *Deep Hough Voting for 3D Object Detection in Point Clouds*. ICCV 2019.
- Charles R. Qi, Xinlei Chen, Or Litany, Leonidas J. Guibas. *ImVoteNet: Boosting 3D Object Detection in Point Clouds with Image Votes*. CVPR 2020.

VoteNet – A Two-Stage Approach

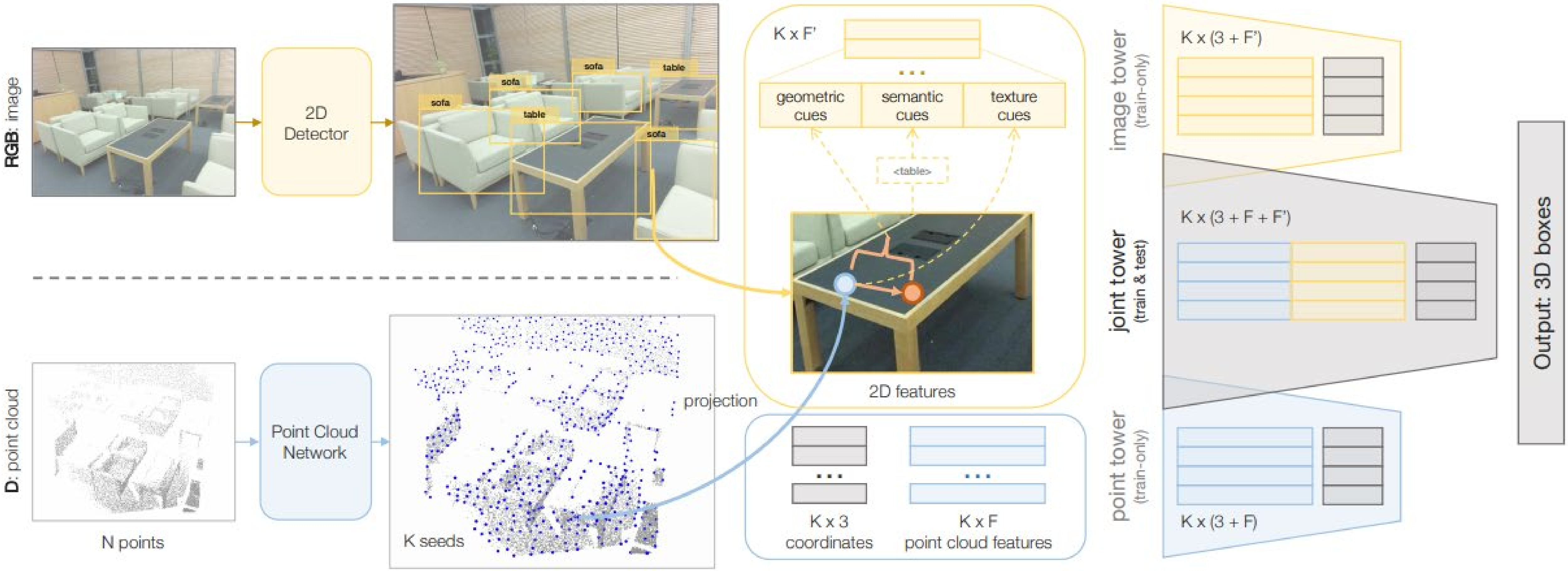


Basic idea: *ImVoteNet*

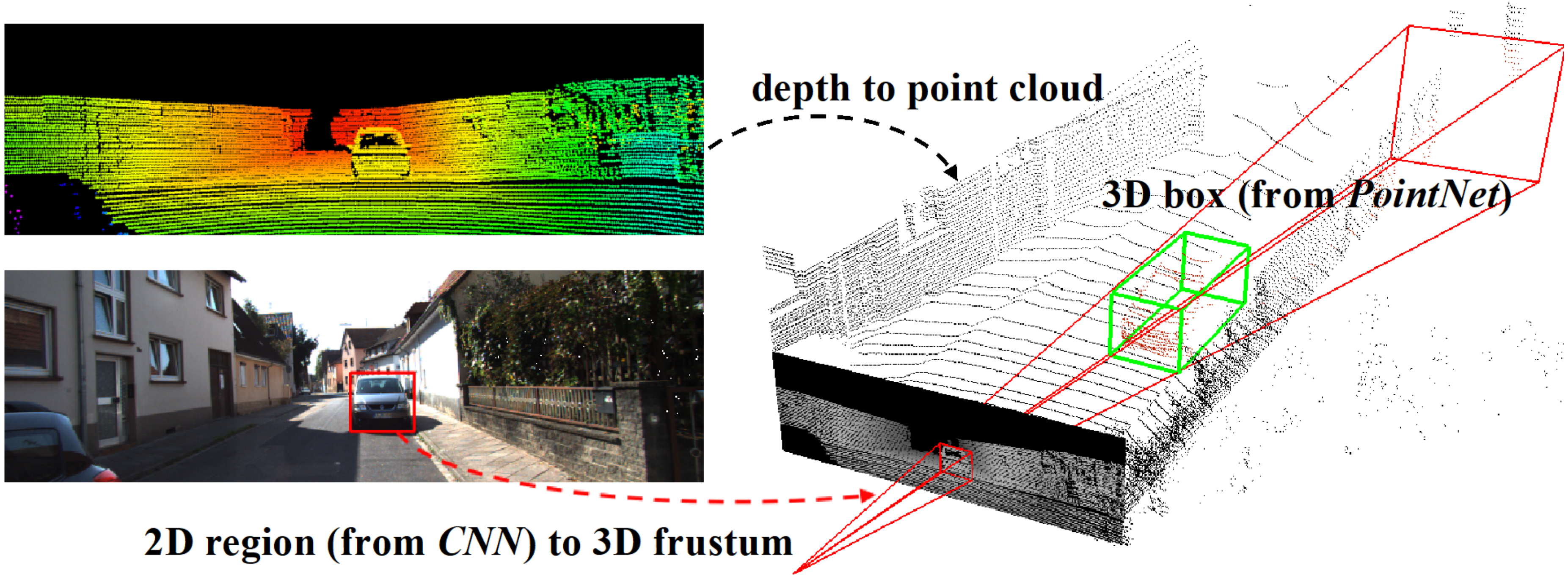


C. Qi, X. Chen, O. Litany, L.J. Guibas CVPR 2020

ImVoteNet Architecture



Frustum PointNets for 3D Object Detection



+ Leveraging mature 2D detectors for region proposal. greatly reducing 3D search space.

+ Solving 3D detection problem with 3D data and 3D deep learning.

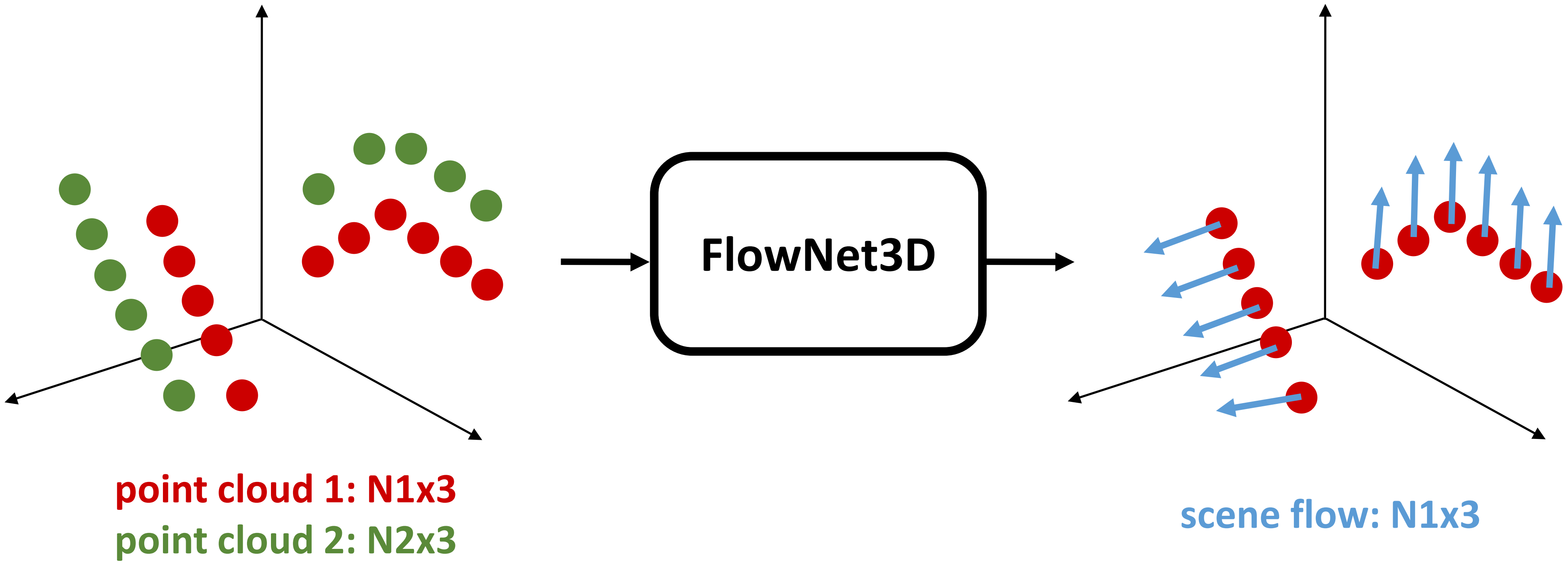
Charles R. Qi, Wei Liu, Chenxia Wu, Hao Su, Leonidas Guibas. Frustum PointNets for 3D Object Detection from RGB-D Data (CVPR 2018)

KITTI Results: Qualitative

Remarkable box estimation accuracy even with a dozen of points or with very partial point clouds.

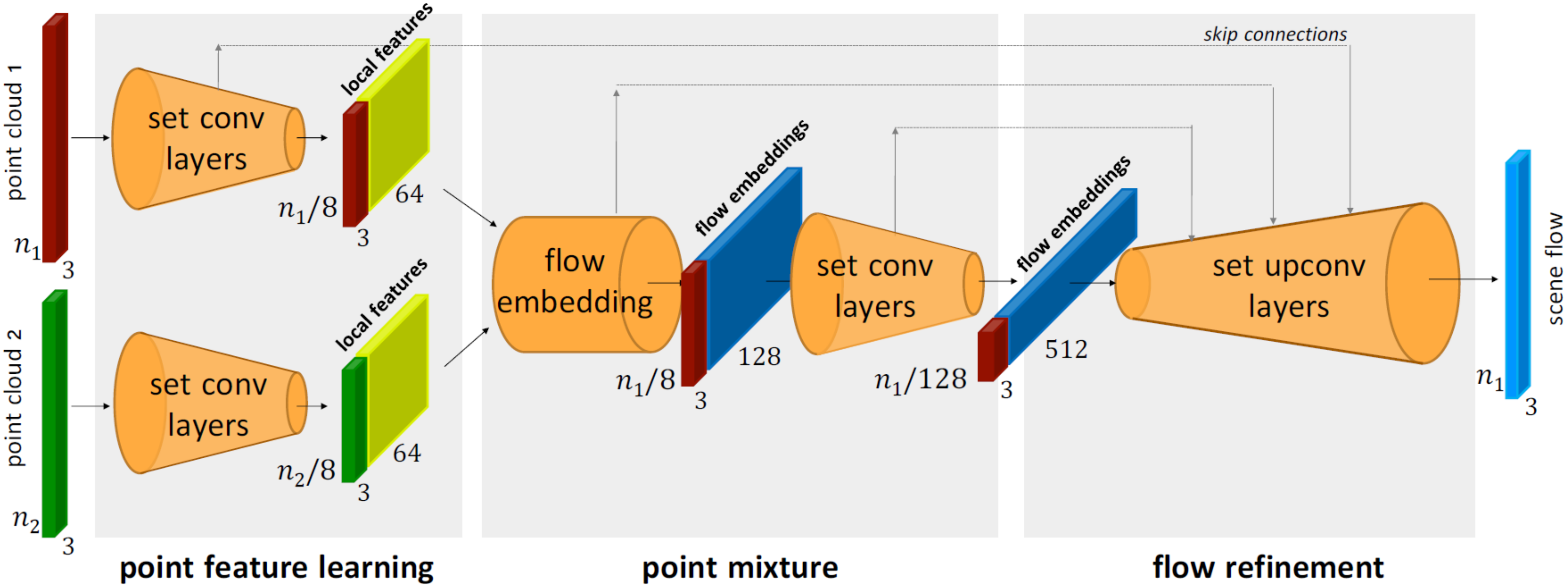
FlowNet3D – Point Clouds Over Time

- Directly learning scene flow in 3D point clouds, with 3D deep learning architectures.



Xingyu Liu, Charles R. Qi, Leonidas Guibas. Learning Scene Flow in 3D Point Clouds, (CVPR 2019).

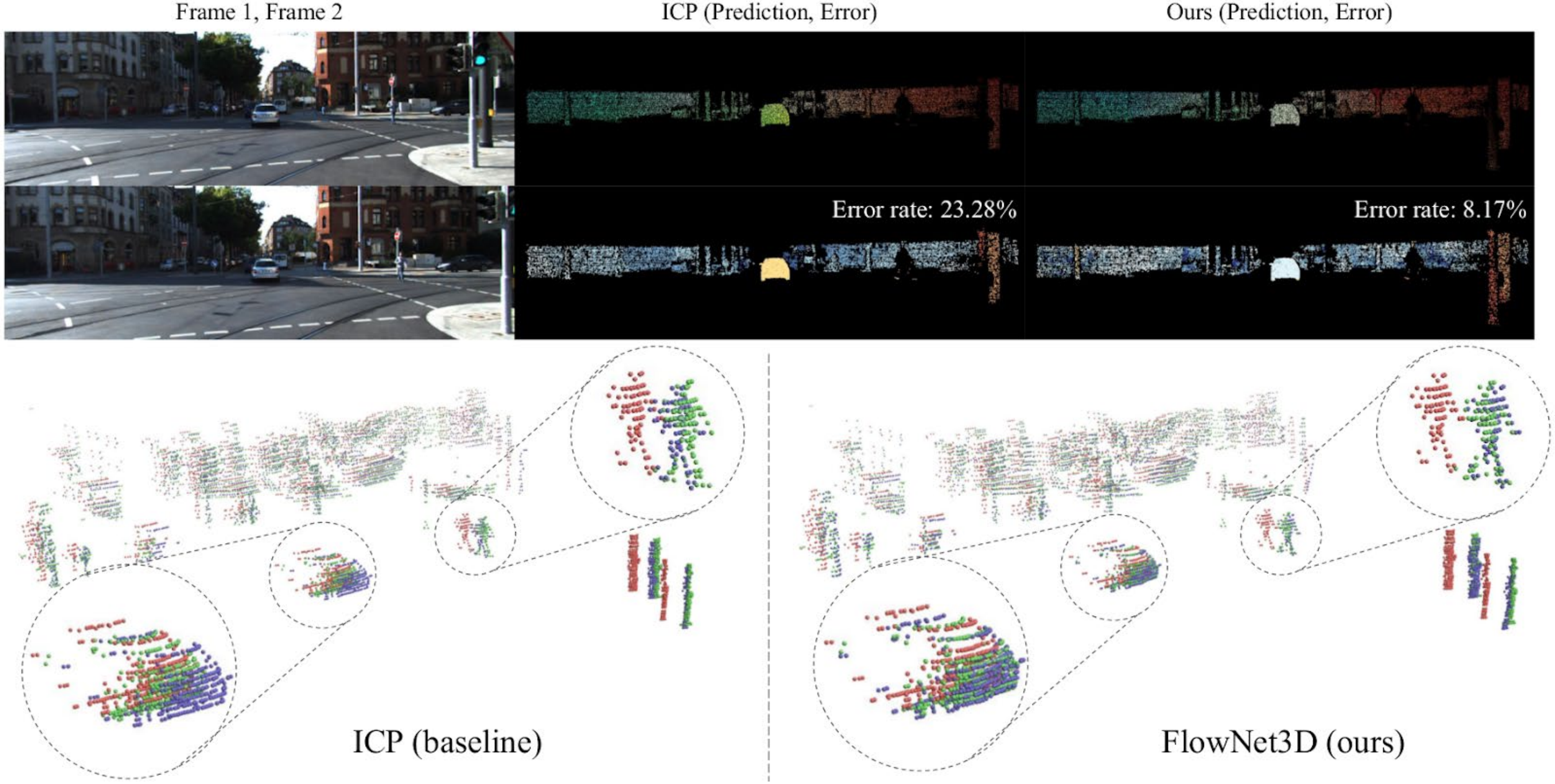
FlowNet3D – Point Clouds Over Time



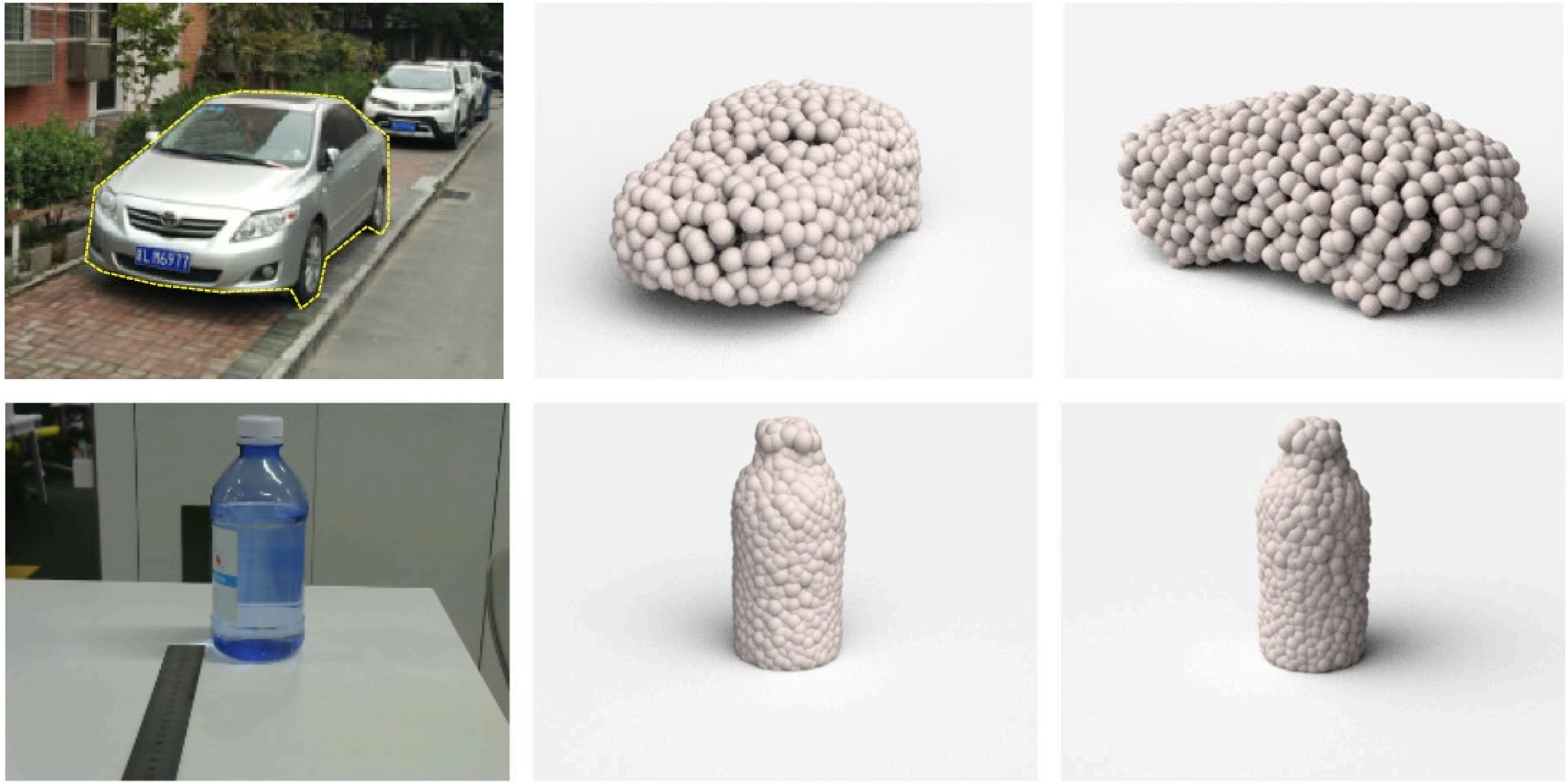
Composed of many many mini-pointnet++ modules …

Pointnet++

KITTI Results



Point Cloud Synthesis from a Single Image

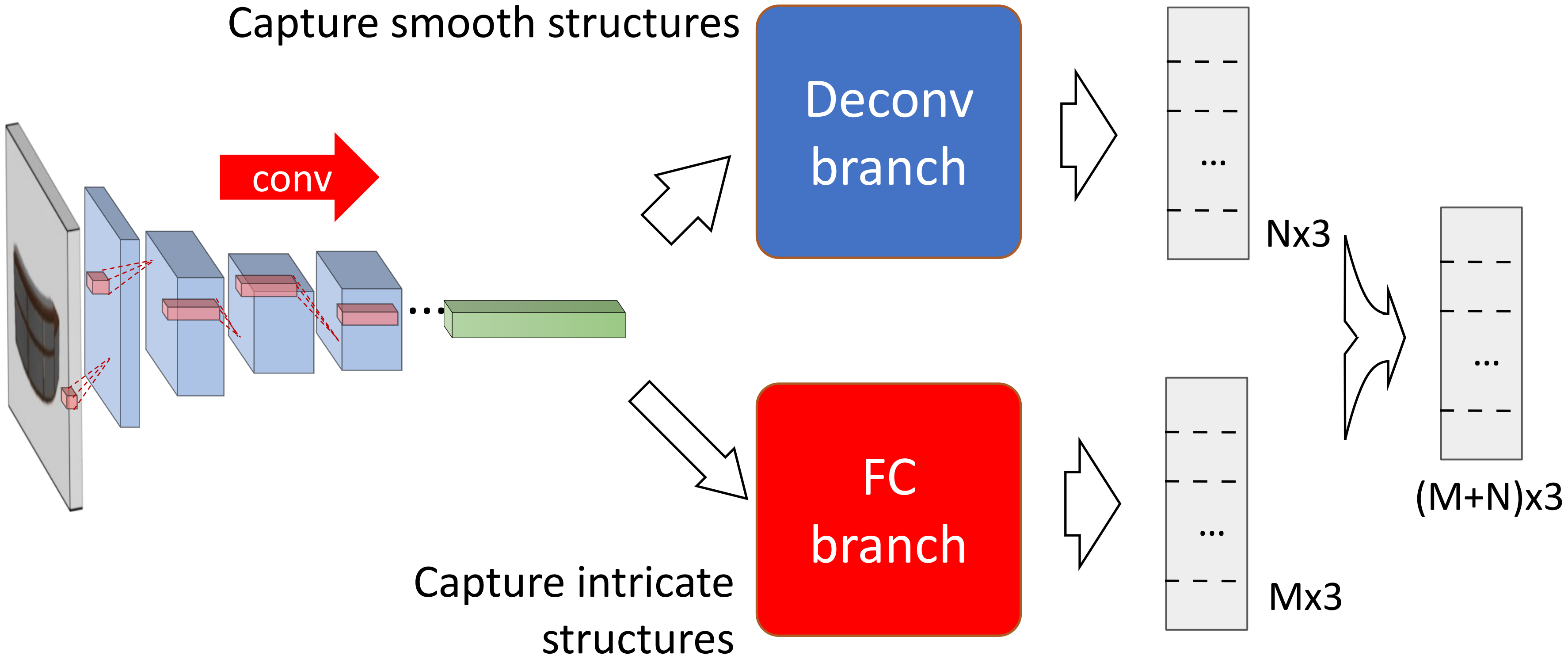


Input

Reconstructed 3D point cloud

[H. Su, H. Fan, LG, 2017]

Two-Branch Architecture



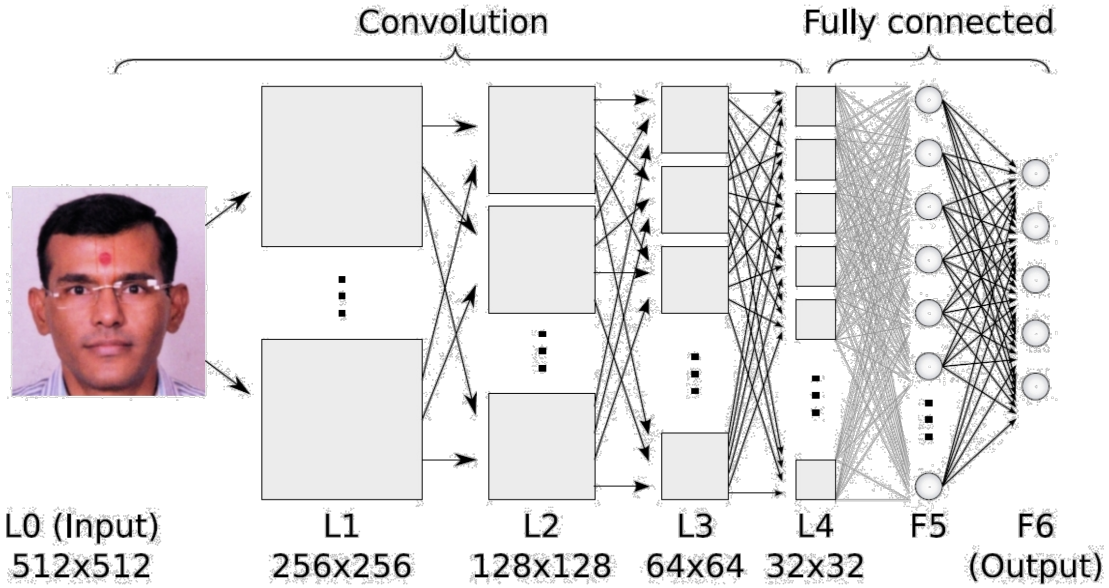
Capture smooth structures

Capture intricate structures

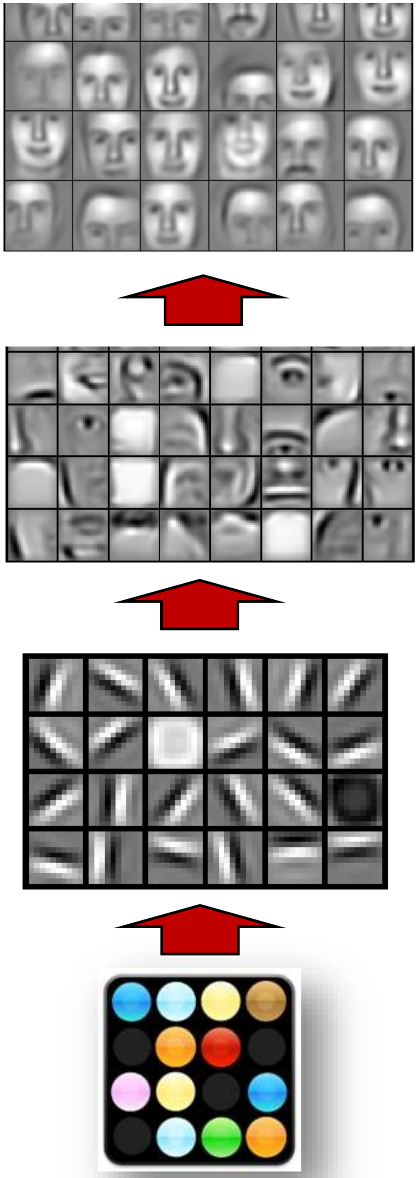
Set union by array concatenation

Today: Functional Maps
for
Joint Data Analysis

Vertical Learning Networks



[Makwana, 2016]



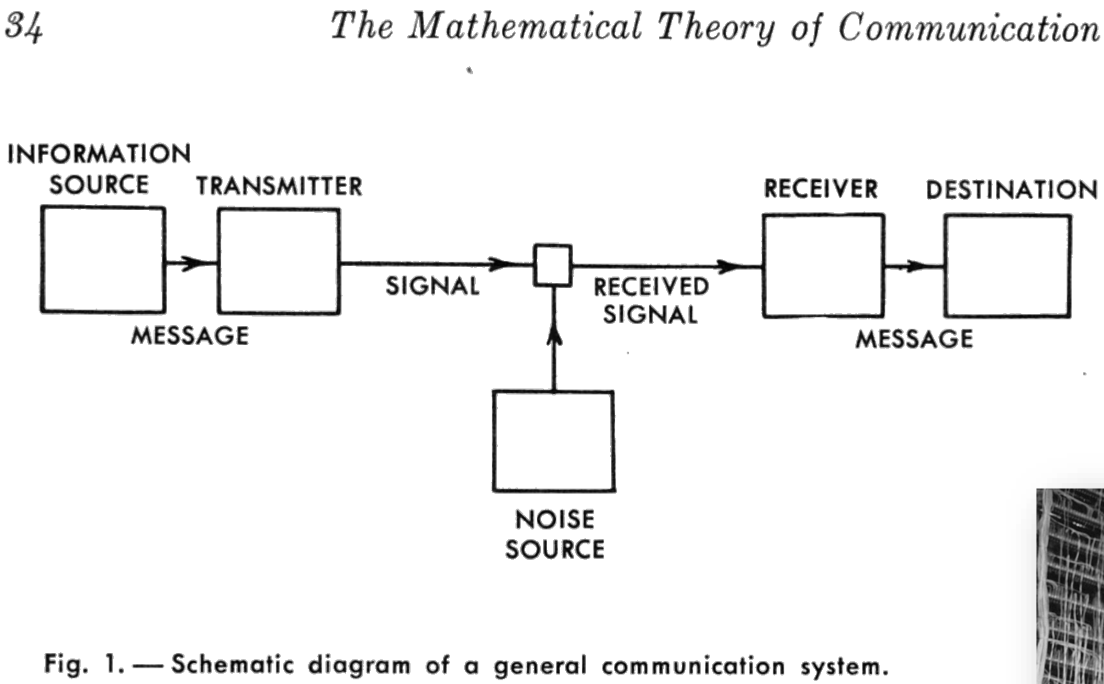
[Lee et al., 2009]

Data-driven feature learning at ascending abstraction layers

“Deep” nets

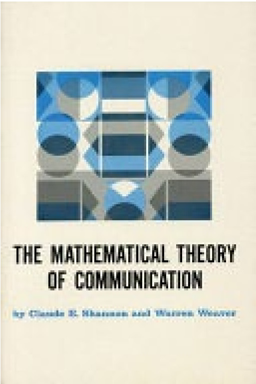
Horizontal Networks

Similarity as a communications channel

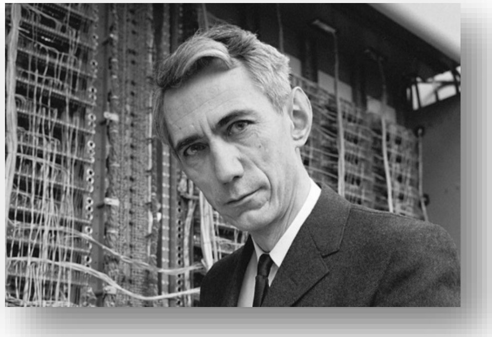


34 *The Mathematical Theory of Communication*

Fig. 1. — Schematic diagram of a general communication system.

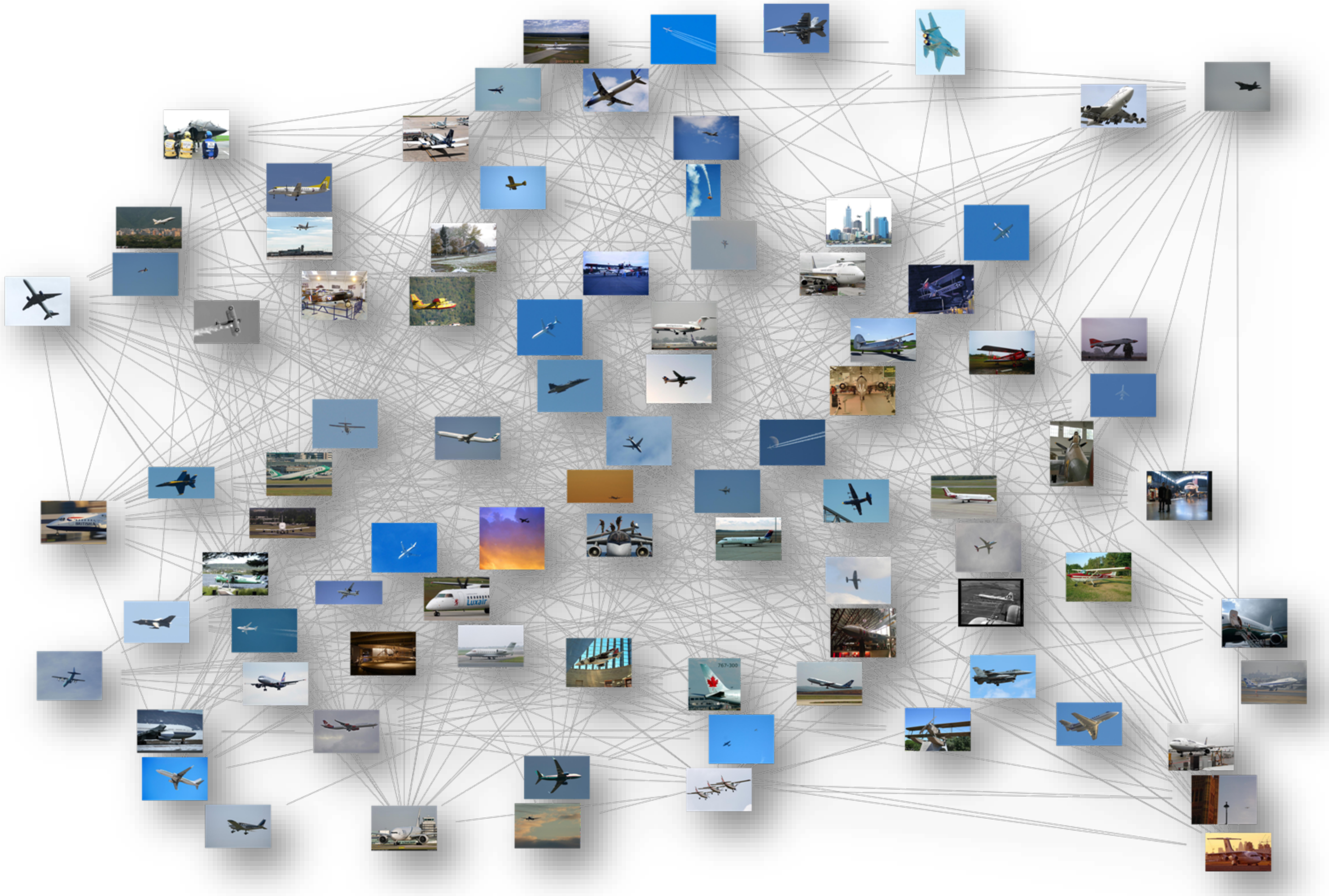


Claude Shannon

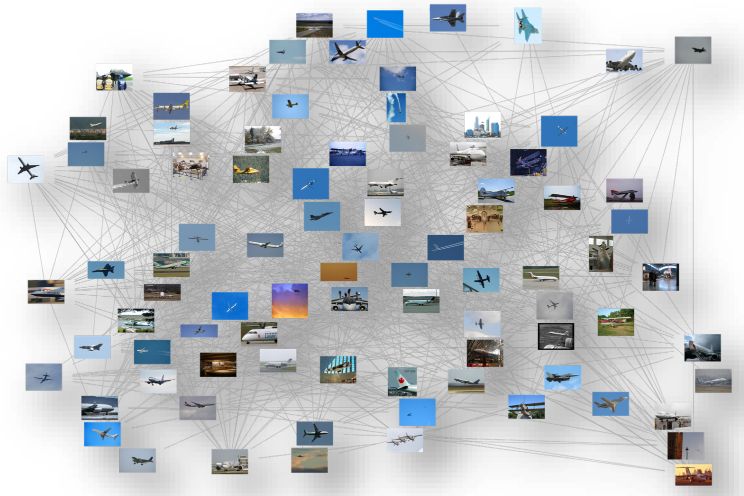


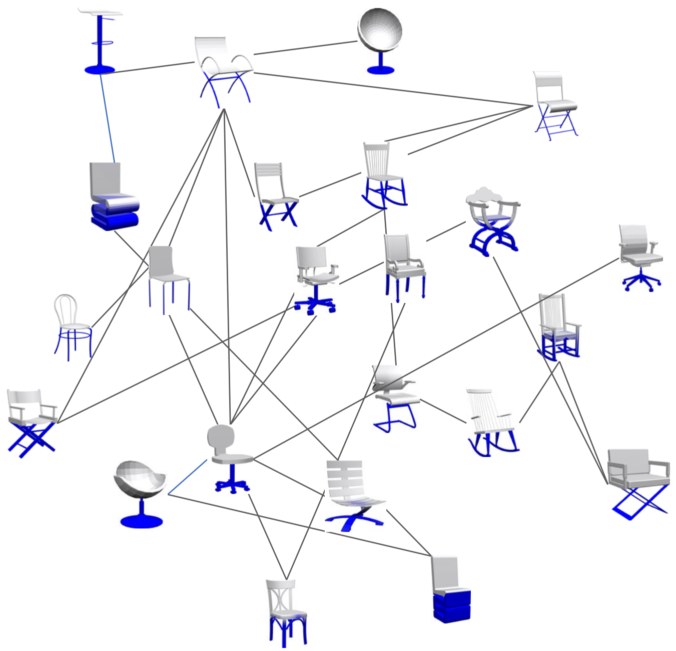
The Network View: Information Transport Between Visual Data

Networks of Images



Or of Shapes, Or of Both



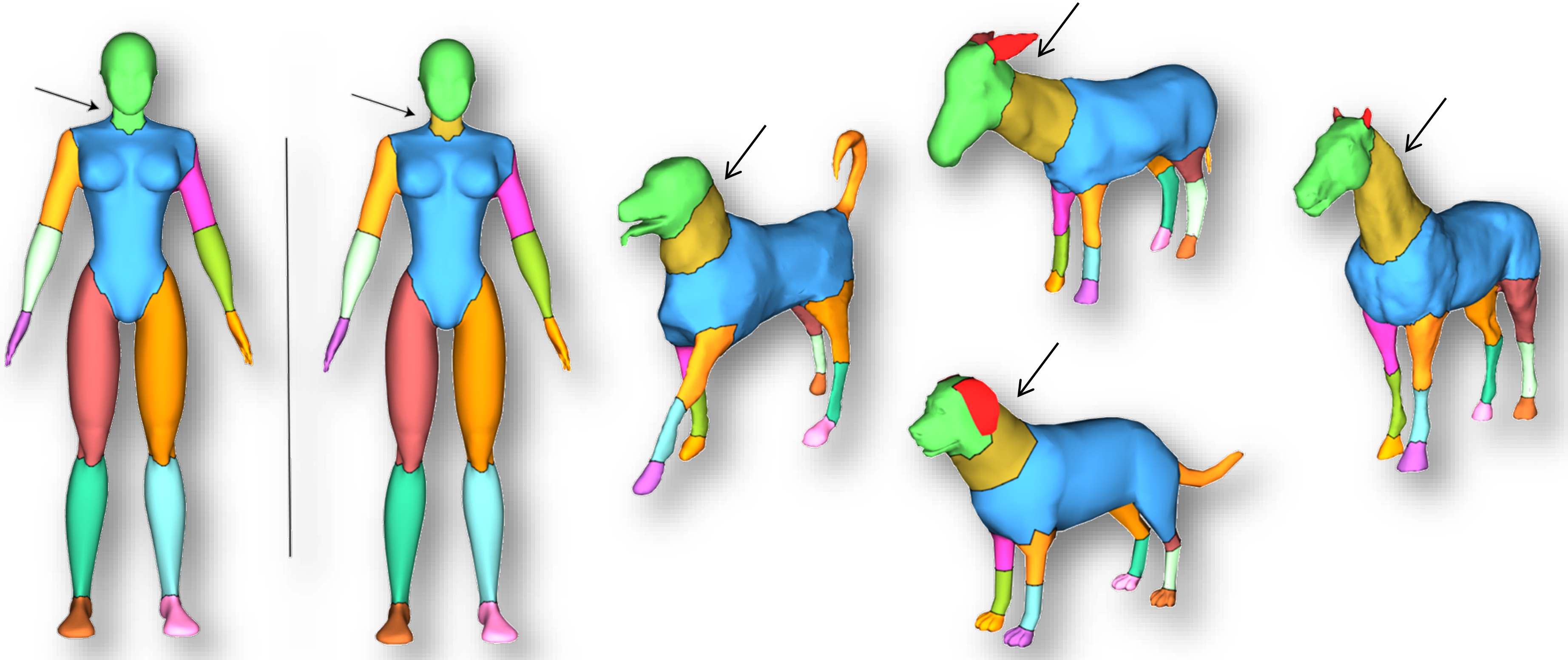


Relations Between Visual Data

Each Data Set Is Not Alone

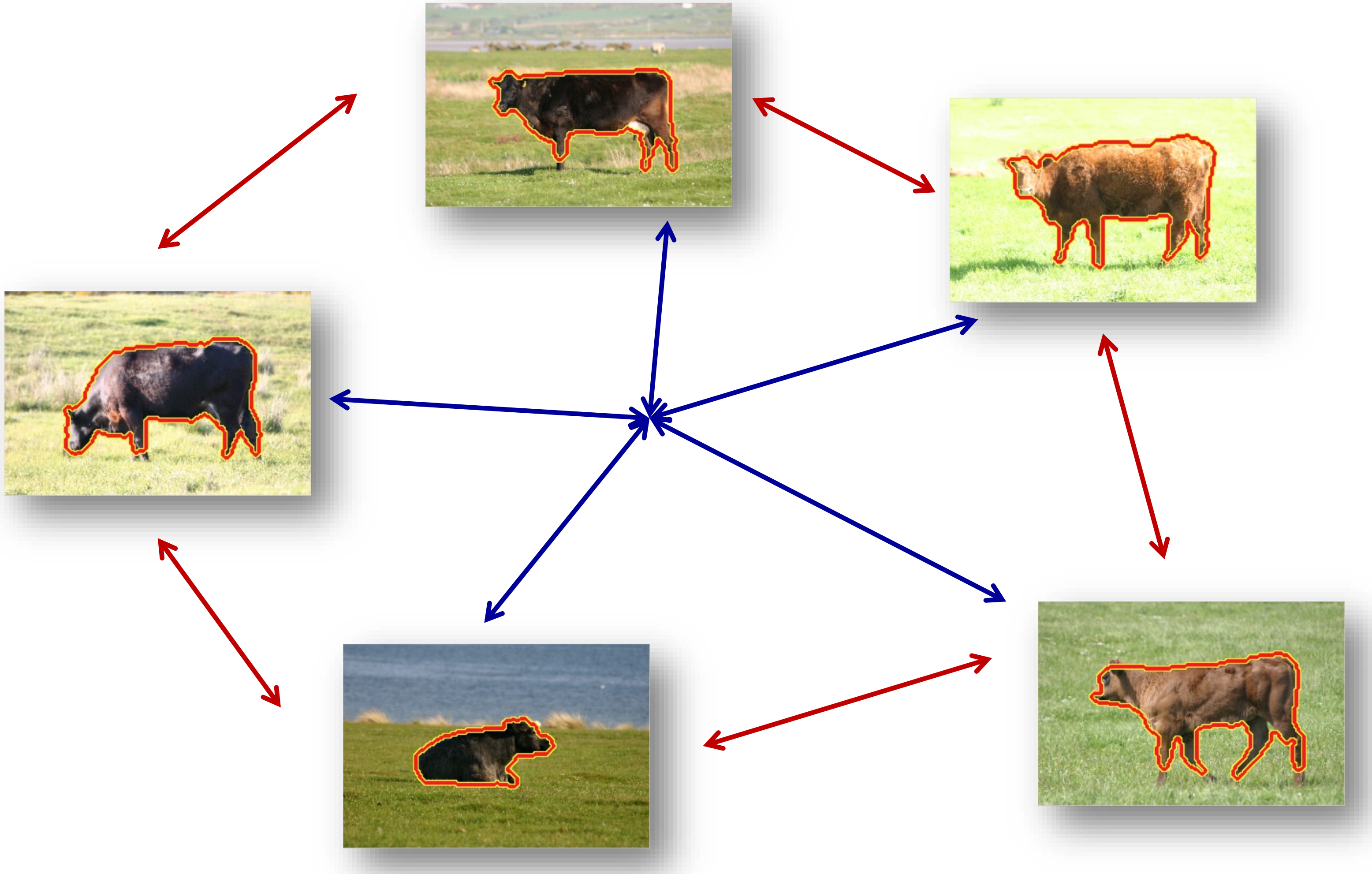
[Q. Huang, V. Koltun, L. Guibas, 2011]

- The interpretation of a particular piece of geometric data is deeply influenced by our interpretation of other related data

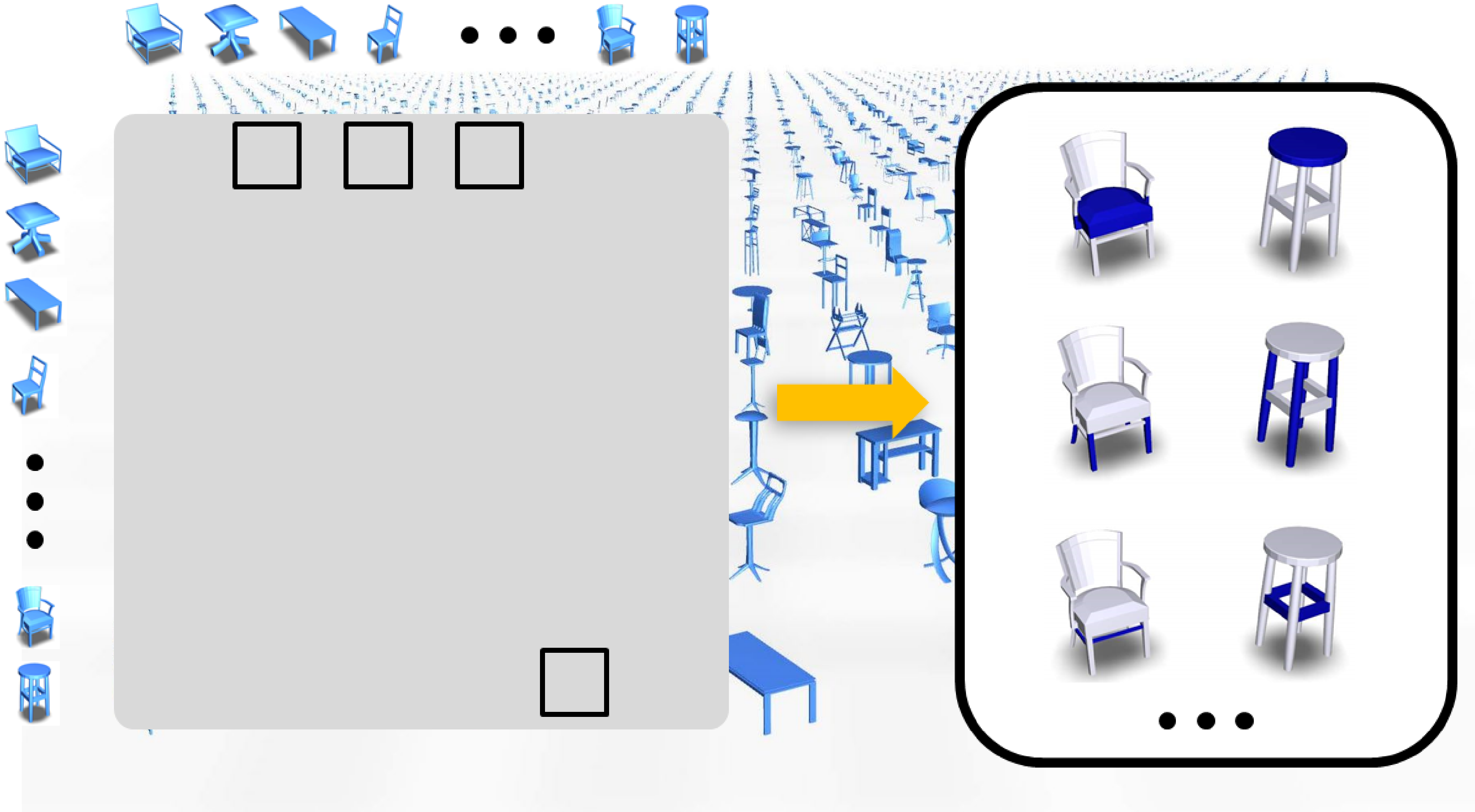


3D Segmentation

Co-Segmentation

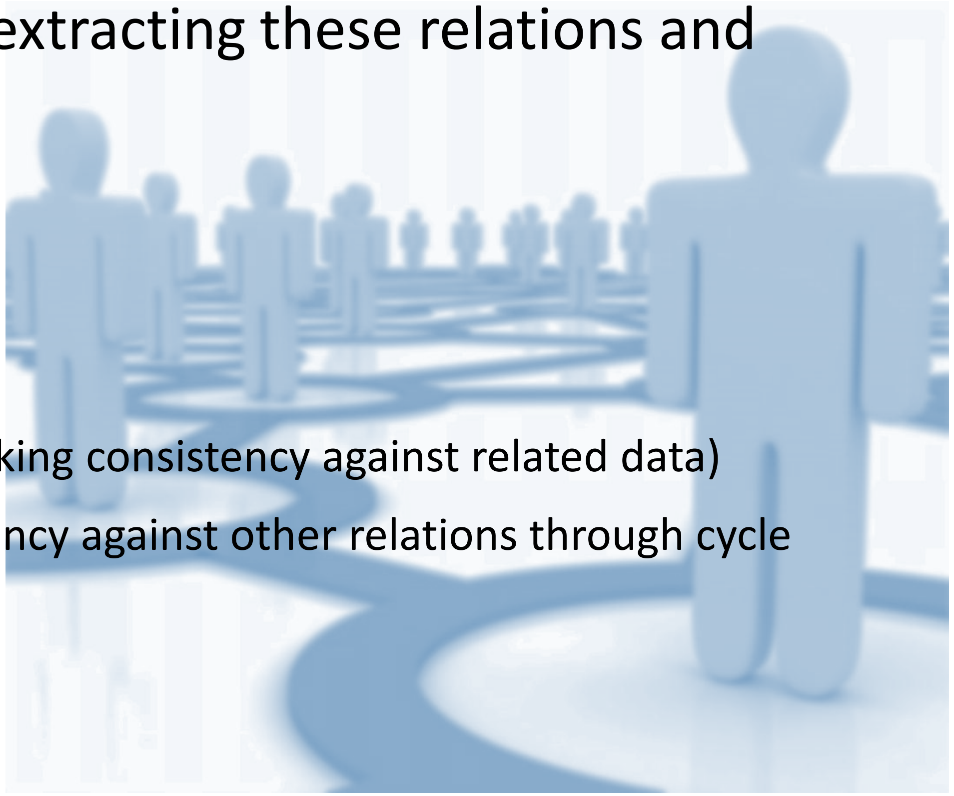


Semantic Structure Emerges from the Network



[Q. Huang, F. Wang, L. Guibas, ’14]

Societies, or Social Networks of Data Sets



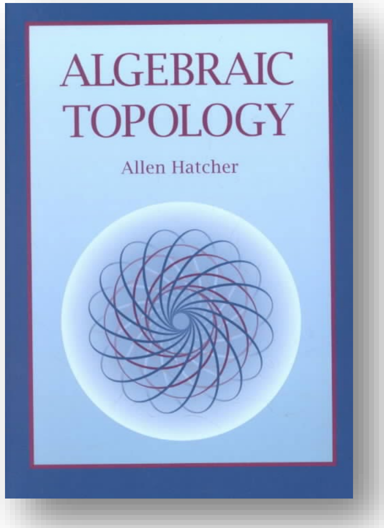
Our understanding of data can greatly benefit from extracting these relations and building relational networks.

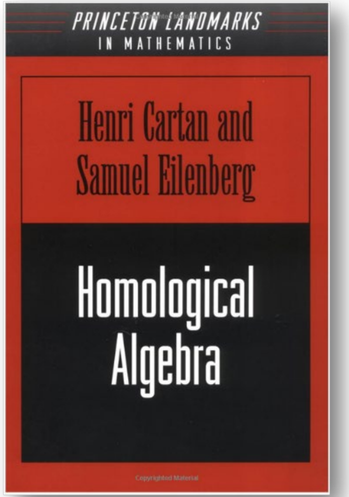
We can exploit the relational network to

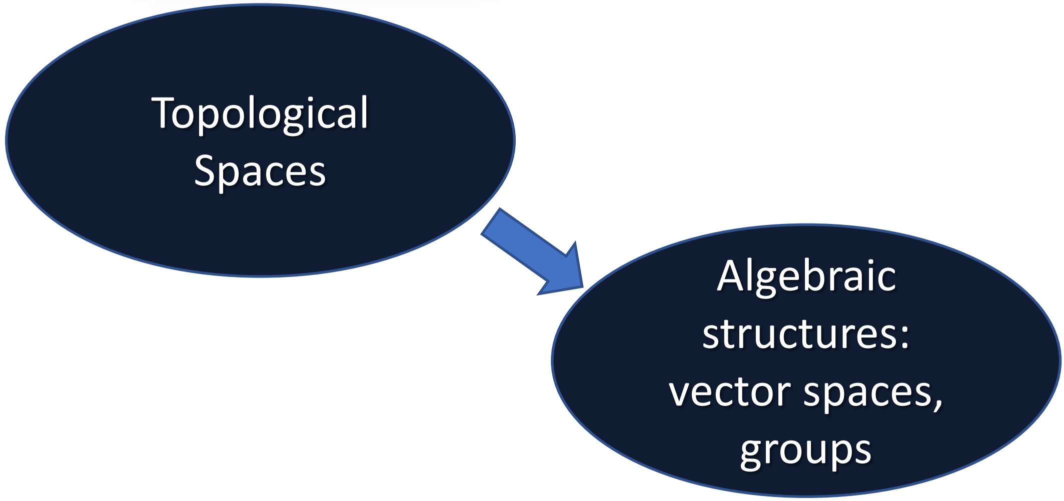
- transport information around the network
- assess the validity of operations or interpretations of data (by checking consistency against related data)
- assess the quality of the relations themselves (by checking consistency against other relations through cycle closure, etc.)
- extract shared structure among the data

Thus the network becomes the great regularizer in joint data analysis.

V+H: Functorial Data Analysis


ALGEBRAIC TOPOLOGY
Allen Hatcher


PRINCETON LANDMARKS IN MATHEMATICS
Henri Cartan and Samuel Eilenberg
Homological Algebra


Topological Spaces

Algebraic structures: vector spaces, groups

$$\begin{tikzcd}
H_*(X) \arrow[r, "\phi"] & H_*(Y) \\
X \arrow[u, "H_*"] \arrow[r, "f"'] & Y \arrow[u, "H_*"']
\end{tikzcd}$$

$$\begin{tikzcd}
L(X) \arrow[r, "\phi"] & L(Y) \\
X \arrow[u, "DN"] \arrow[r, "f"'] & Y \arrow[u, "DN"']
\end{tikzcd}$$

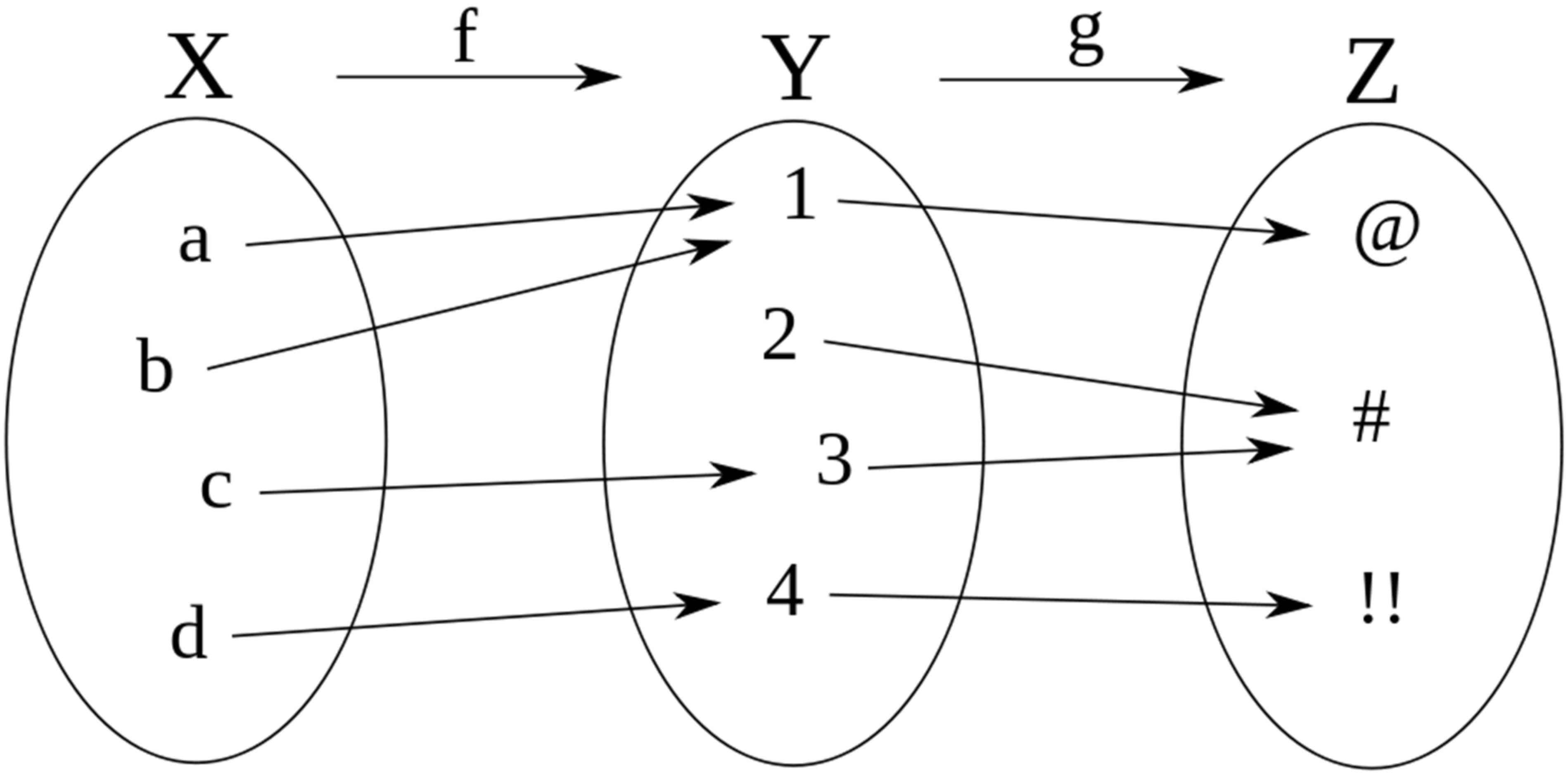
Maps as
First-Class Citizens

Maps

$$\phi : X \to Y$$

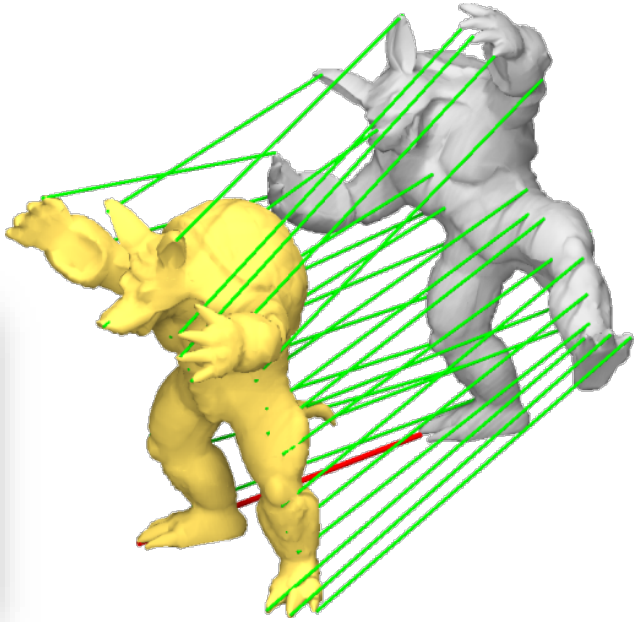
Map from *X* to *Y*

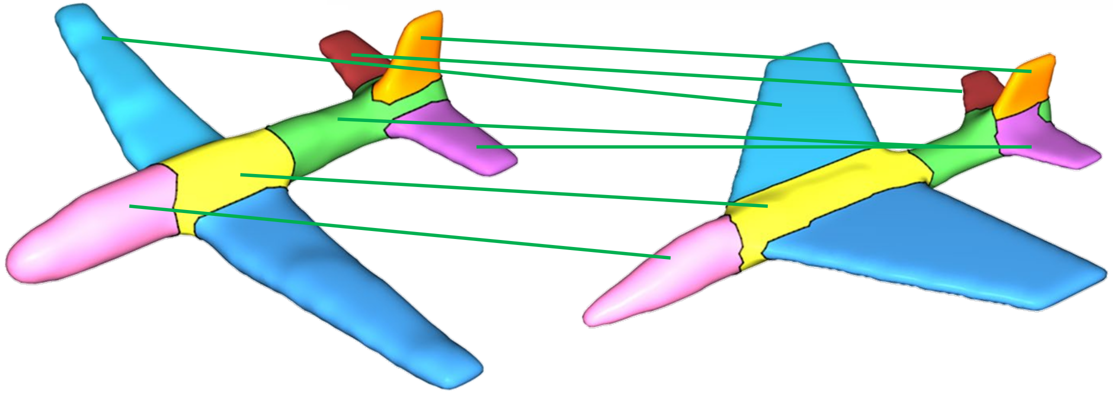
Algebraic Structure: Map Composition



Relationships as Correspondences or Maps

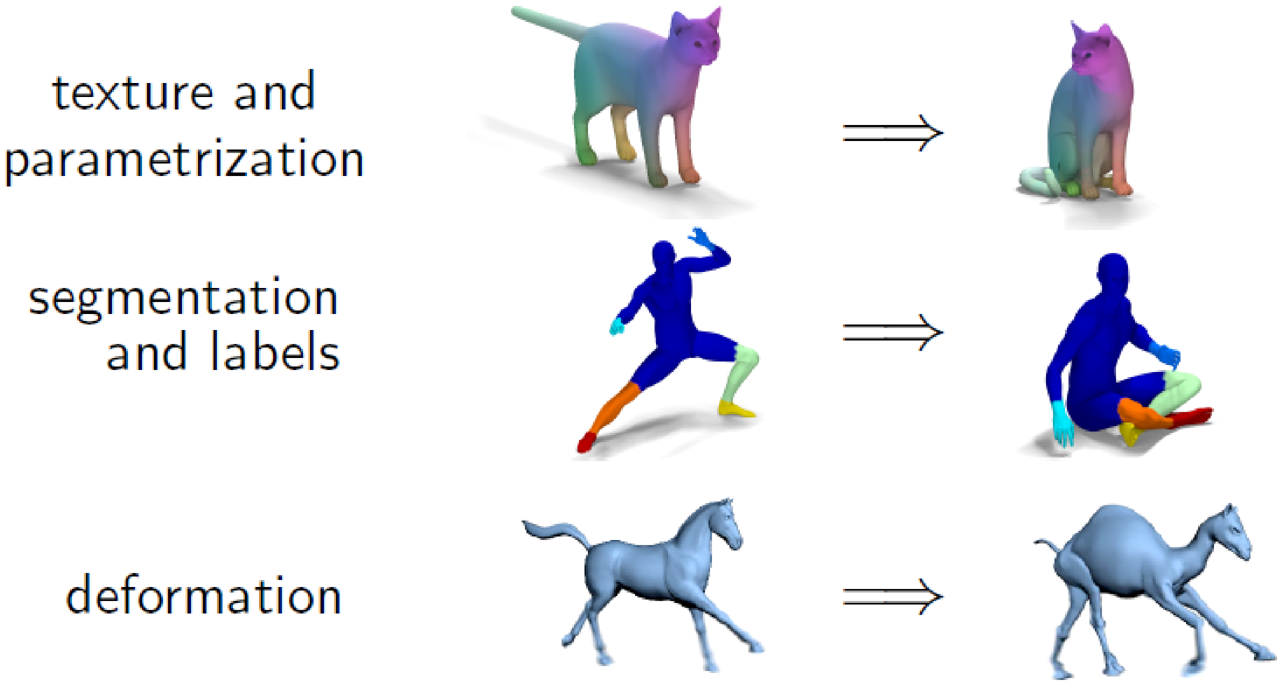
- Multiscale mappings
  - Point/pixel level
  - part level





Maps capture what is the same or similar across two data sets

Correspondences or Maps are Information Transporters



Matching Enables Information Transport

IEEE TRANSACTIONS ON PATTERN ANALYSIS AND MACHINE INTELLIGENCE, VOL. 24, NO. 24, APRIL 2002

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Shape Matching and Object Recognition Using Shape Contexts

Serge Belongie, *Member, IEEE*, Jitendra Malik, *Member, IEEE*, and Jan Puzicha

Image Matching via Saliency Region Correspondences

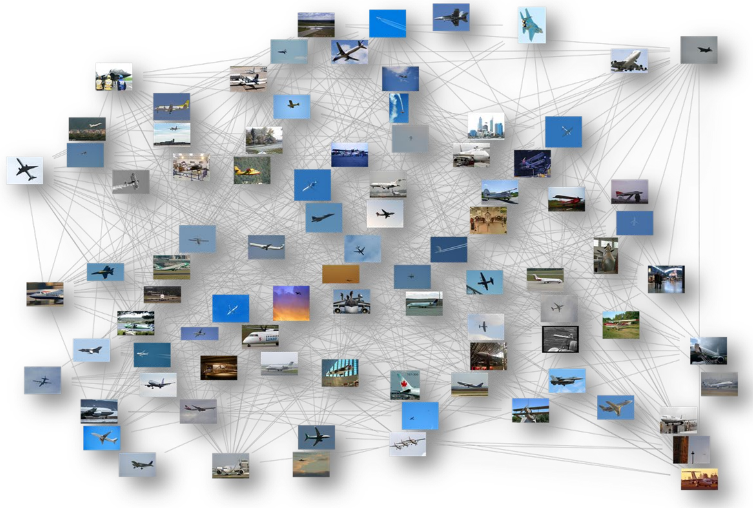
Alexander Toshev, Jianbo Shi, and Kostas Daniilidis
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA 19104, USA

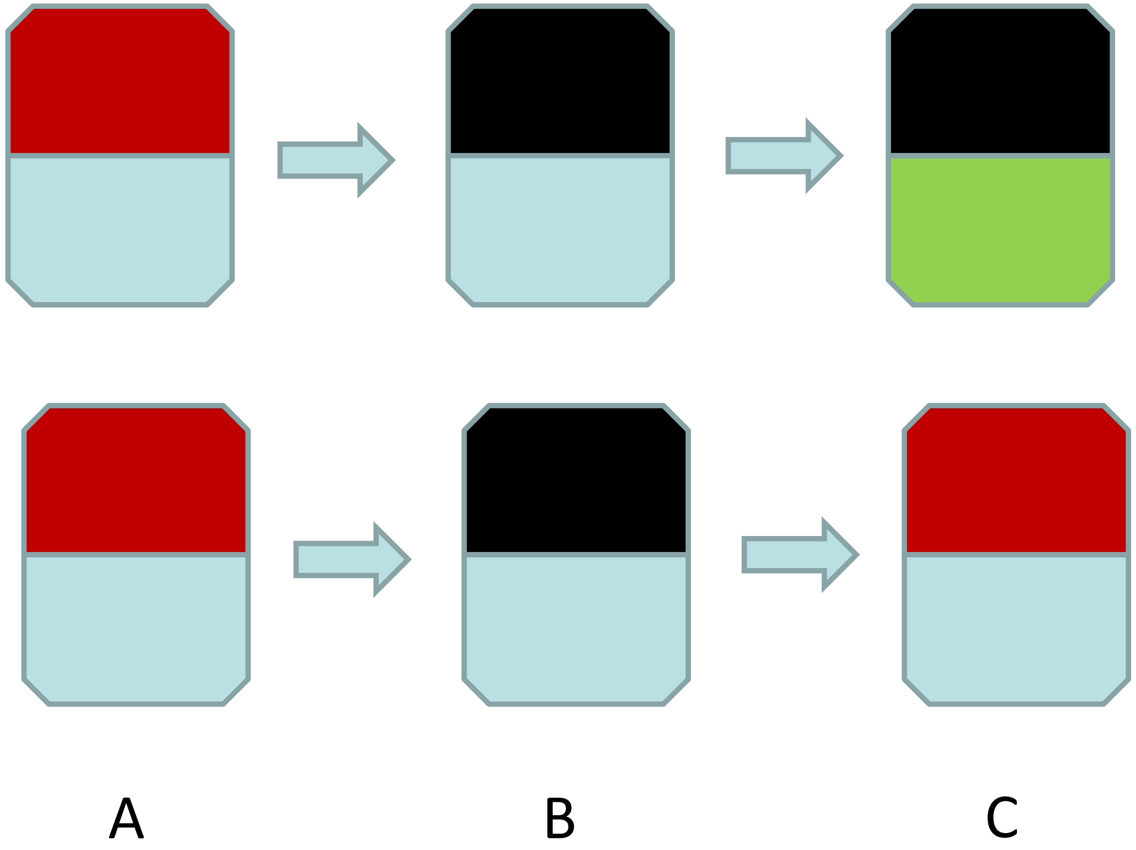
toshev@seas.upenn.edu, jshi@cis.upenn.edu, kostas@cis.upenn.edu

Learning Graph Matching

Tibério S. Caetano, Li Cheng, Quoc V. Le and Alex J. Smola
Statistical Machine Learning Program, NICTA and ANU
Canberra ACT 0200, Australia

Maps vs. Distances/Similarities Networks vs. Graphs

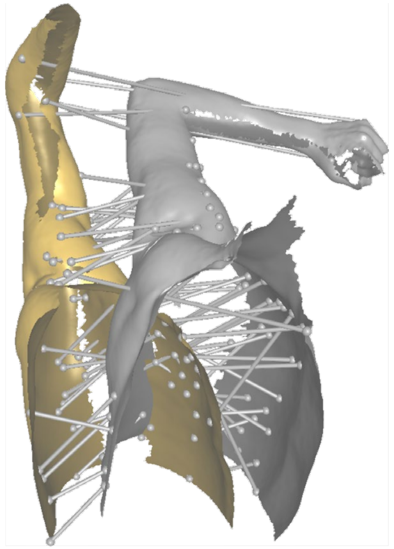




Persistence of correspondences

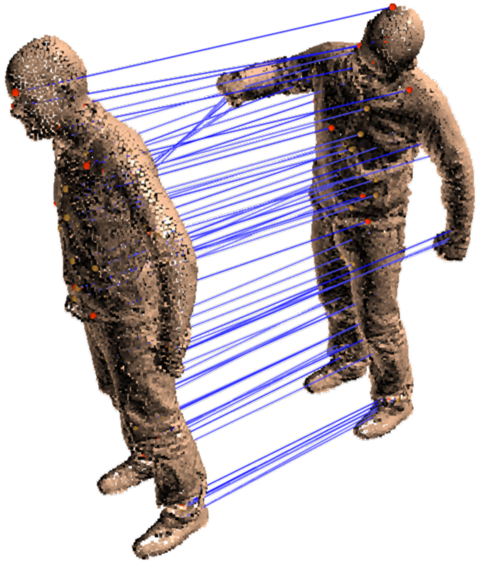
What Makes a Map Good?

- 1st order descriptor or feature preservation (e.g., corners go to corners)
- 2nd order attribute preservation (e.g., Euclidean or geodesic distances)



- Smoothness or continuity
- Respect for “internal structure” (symmetries, etc)

- Semantic correctness may still be elusive

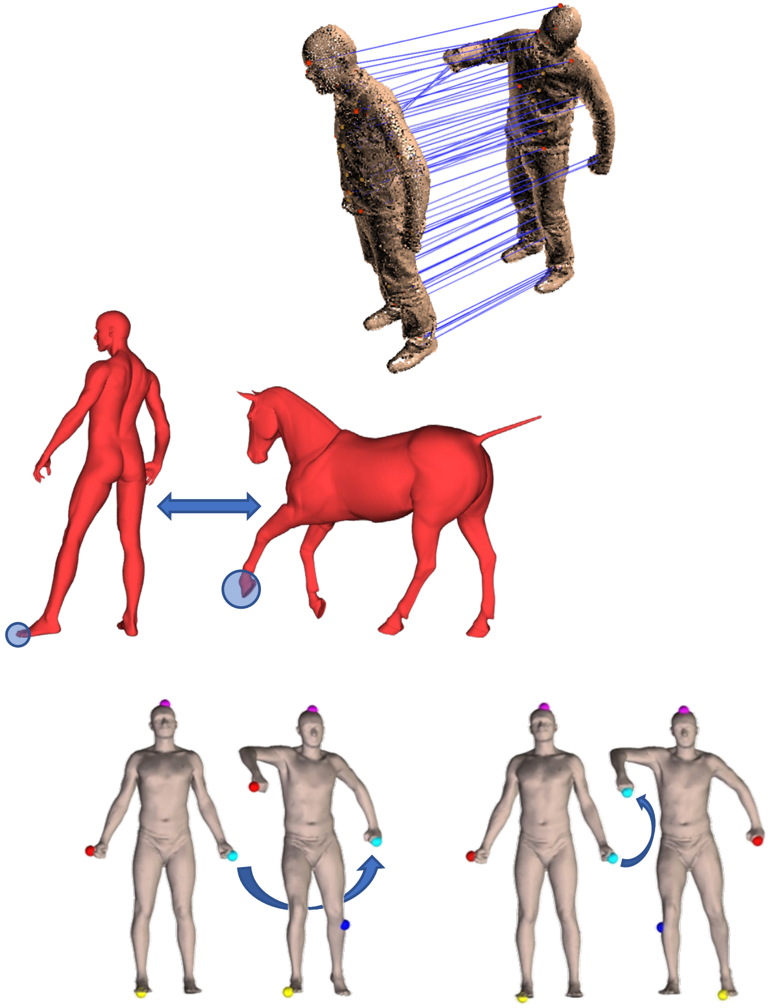


Problems and Issues

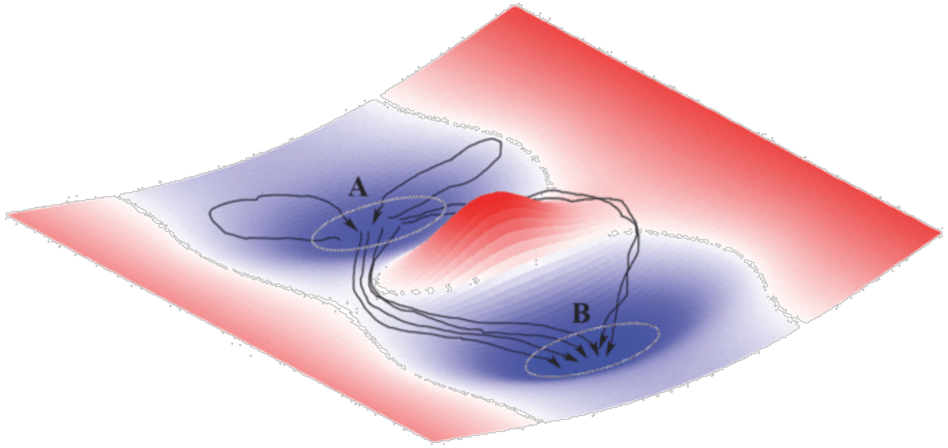
Symmetry, ambiguity, scale, bad data

Maps Challenges: Representation and Computation

- Map representation as a data structuring problem
  - hard to encode compactly
  - hard to select correct scale
  - hard to enforce consistency across abstraction levels
- Also, difficult to compute
  - typically based on matching features/descriptors
  - symmetries, discrete and continuous, lead to ambiguities
  - 2nd order attribute preservation leads to NP-hard quadratic assignment problems

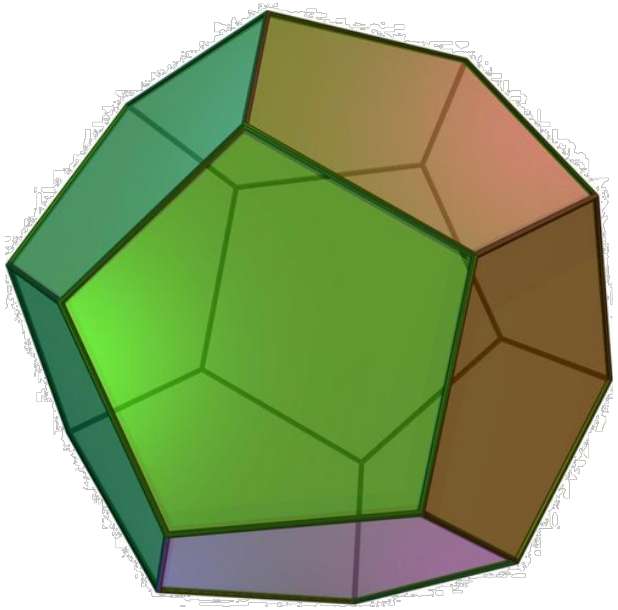


Non-Convex, Combinatorial Optimization



multiple minima

NP-hard quadratic assignments

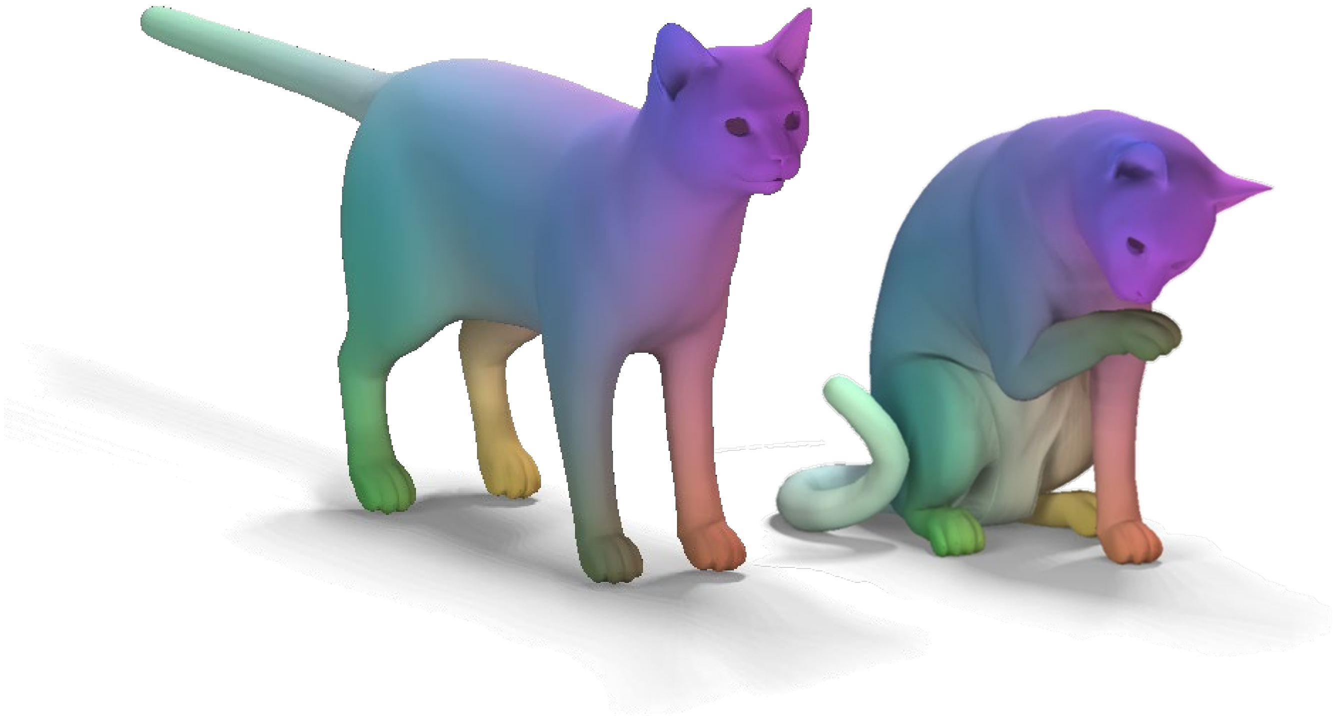


n! permutations

Symmetry, ambiguity, scale, bad data

A Potential Way Out

Find alternative representation more amenable to optimization



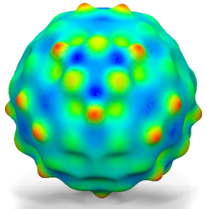
Redefine the notion of map

Technical Approach: Function Spaces and Functional Maps

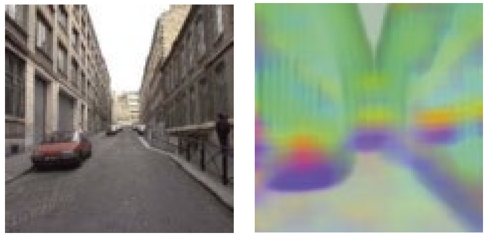
Shapes: From a Particle View to a Wave View …

A Dual View: Functions and Operators

- Functions on data
  - Properties, attributes, descriptors, part indicators, etc.
  - But also beliefs, opinions, etc
- Operators on functions
  - Maps of functions to functions
    - Laplace-Beltrami operator on a manifold

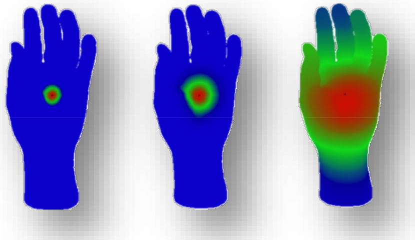

Curvature


Parts


SIFT flow, C. Liu 2011

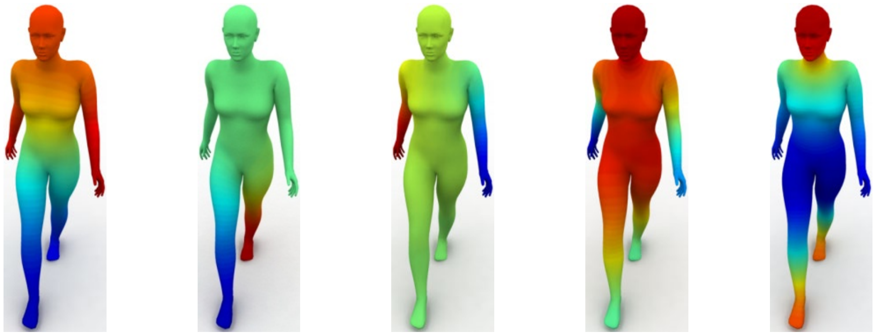
$$M$$

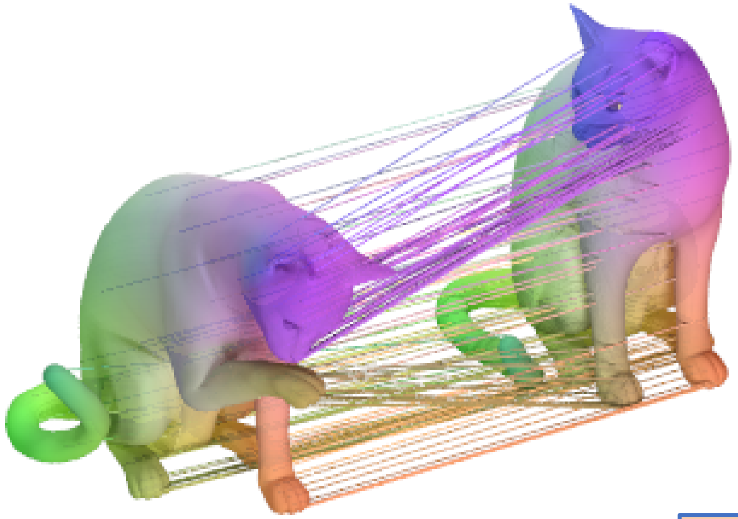
$$\Delta : C^{\infty}(M) \to C^{\infty}(M), \Delta f = \text{div } \nabla f$$



$$\frac{\partial u}{\partial t} = -\Delta u$$

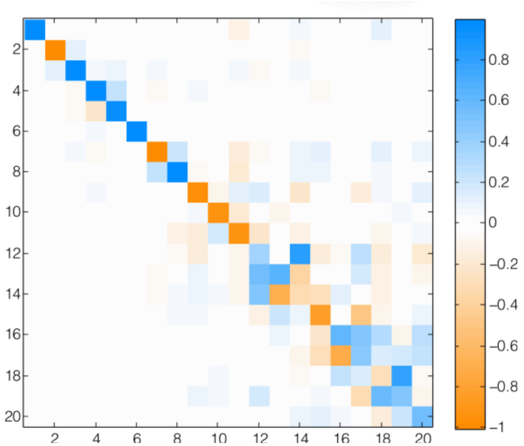
heat diffusion


Laplace Beltrami eigenfunctions

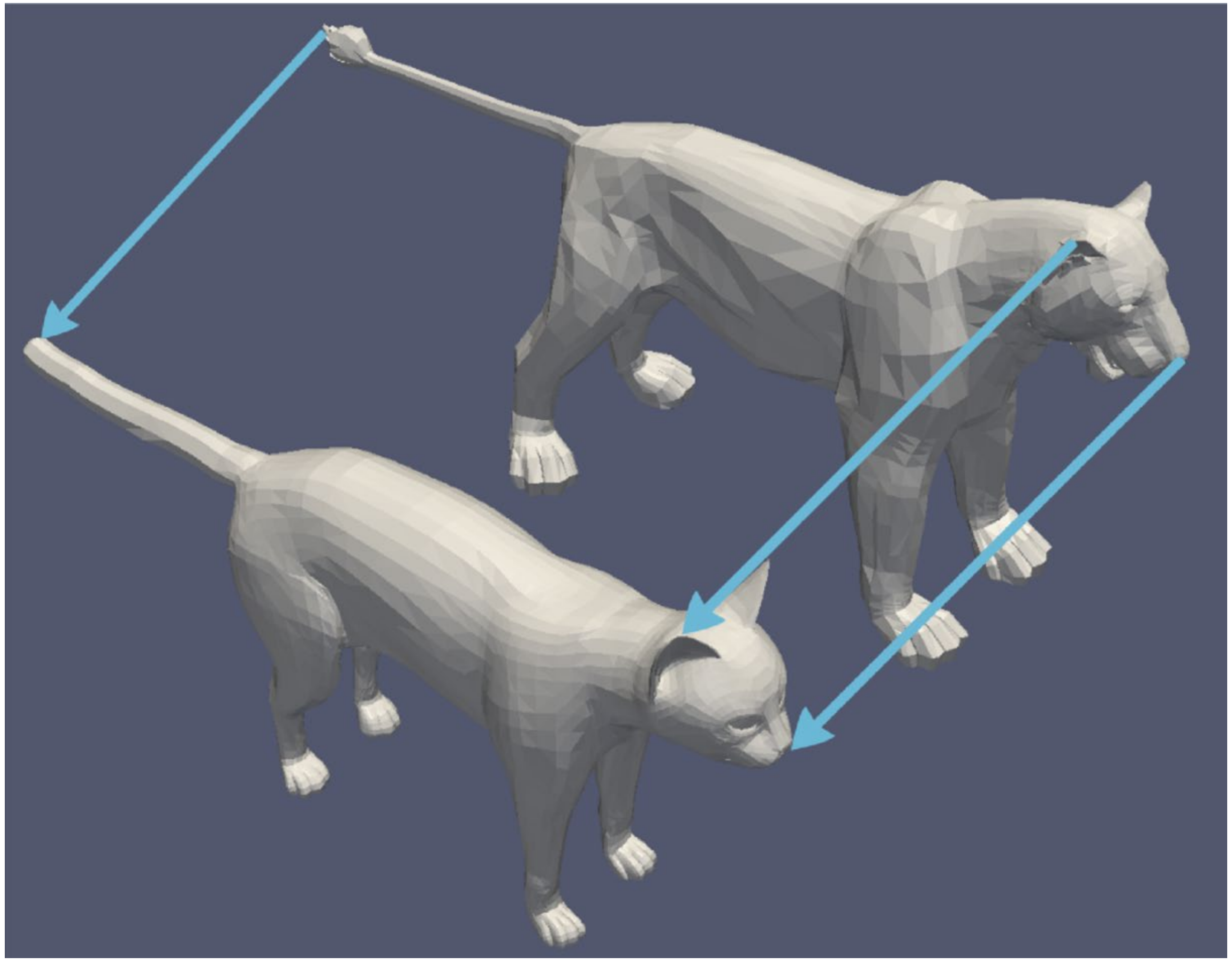


Functional Maps
(a.k.a. Operators)

[M. Ovsjanikov, M. Ben-Chen, J. Solomon, A. Butscher, L. G., Siggraph ’12]

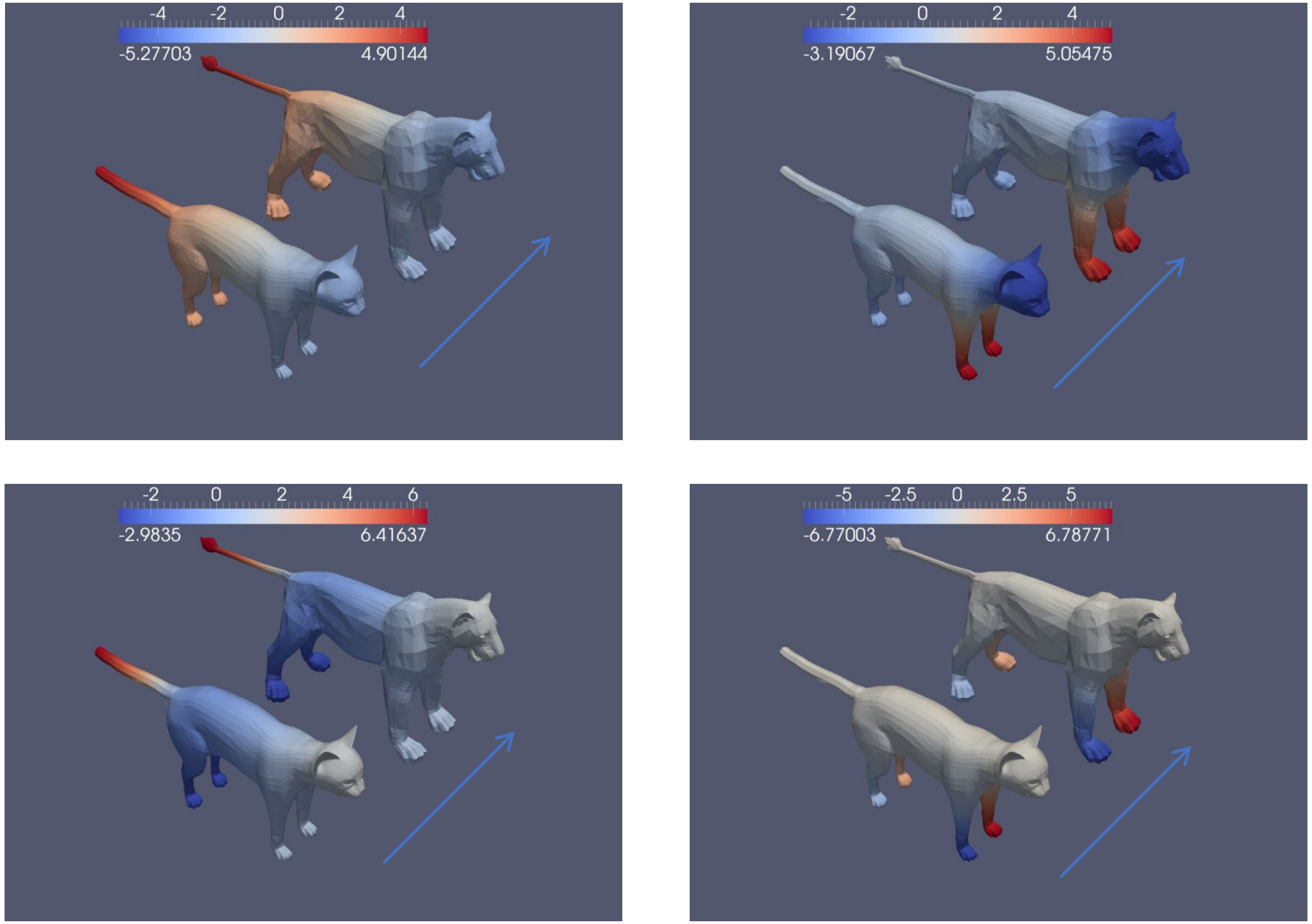


Starting from a Regular Map ϕ



ϕ: lion → cat

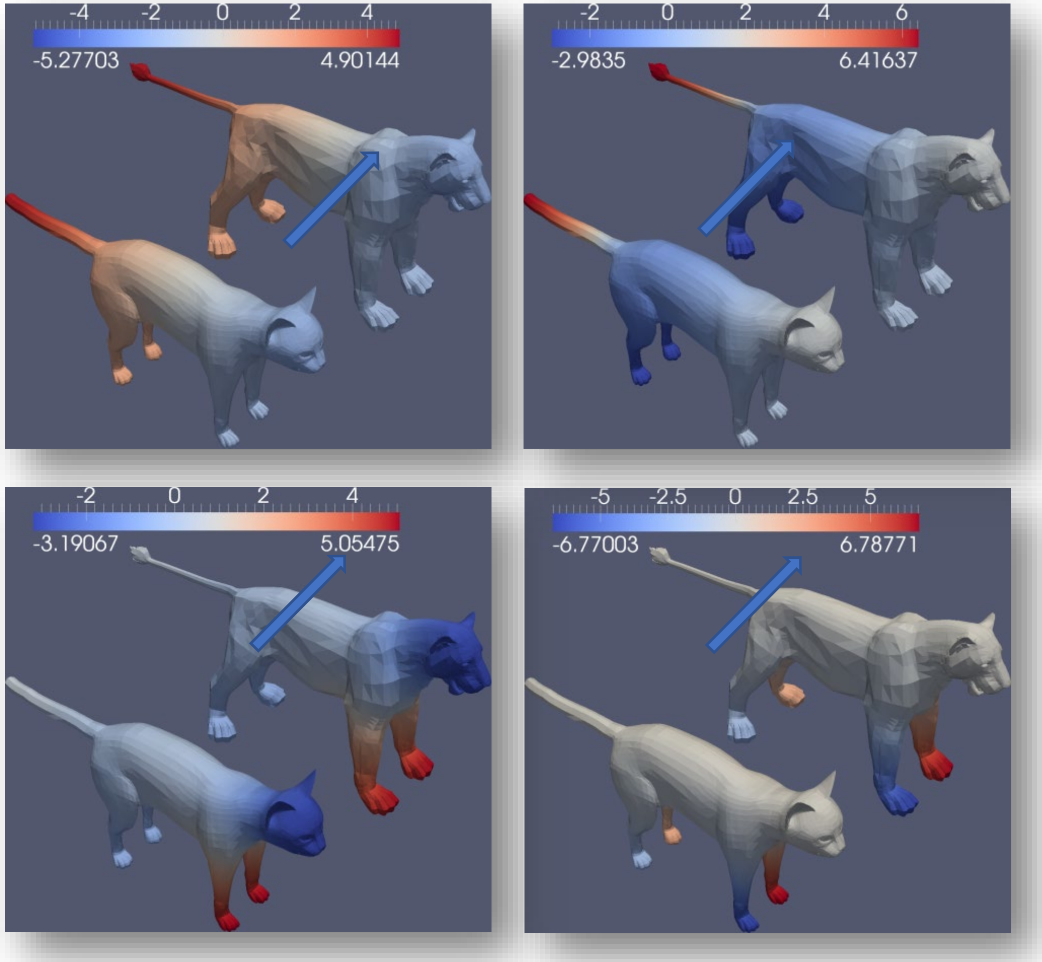
Attribute Transfer via Pull-Back



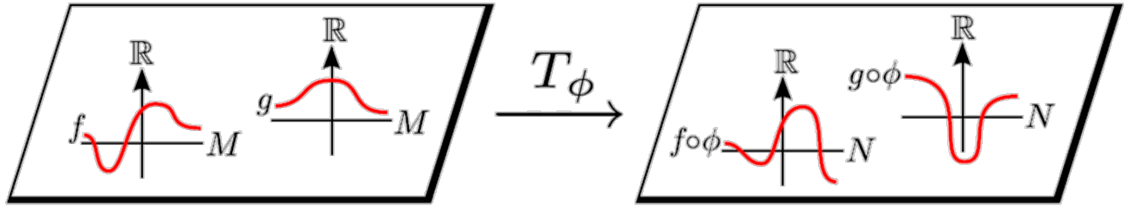
T_ϕ: cat → lion

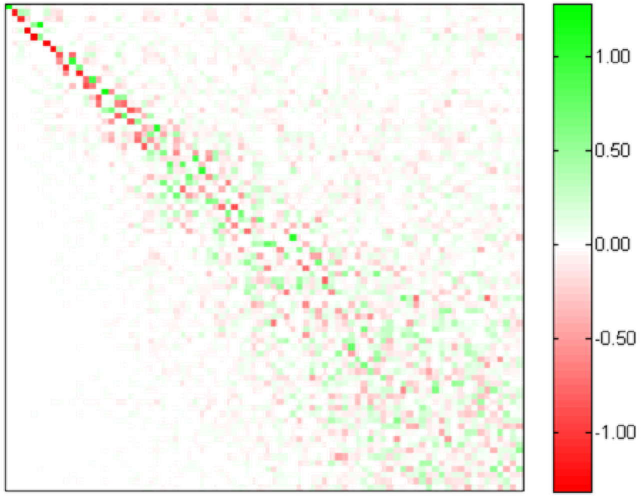
A Contravariant Functor

from cat to lion



Functions on cat are transferred to lion using T_ϕ





T_ϕ is a linear operator (matrix)

$$T_\phi : L^2(cat) \to L^2(lion)$$

Functional Map

$$\boxed{\phi : M \to N}$$

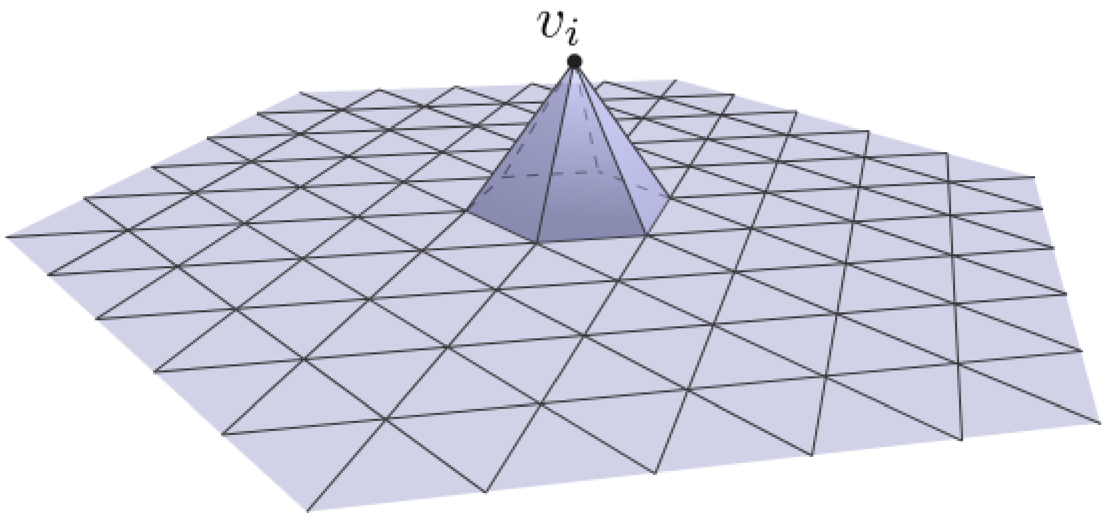
$$\boxed{T_\phi : L^2(N) \to L^2(M)}$$

Dual of a point-to-point map

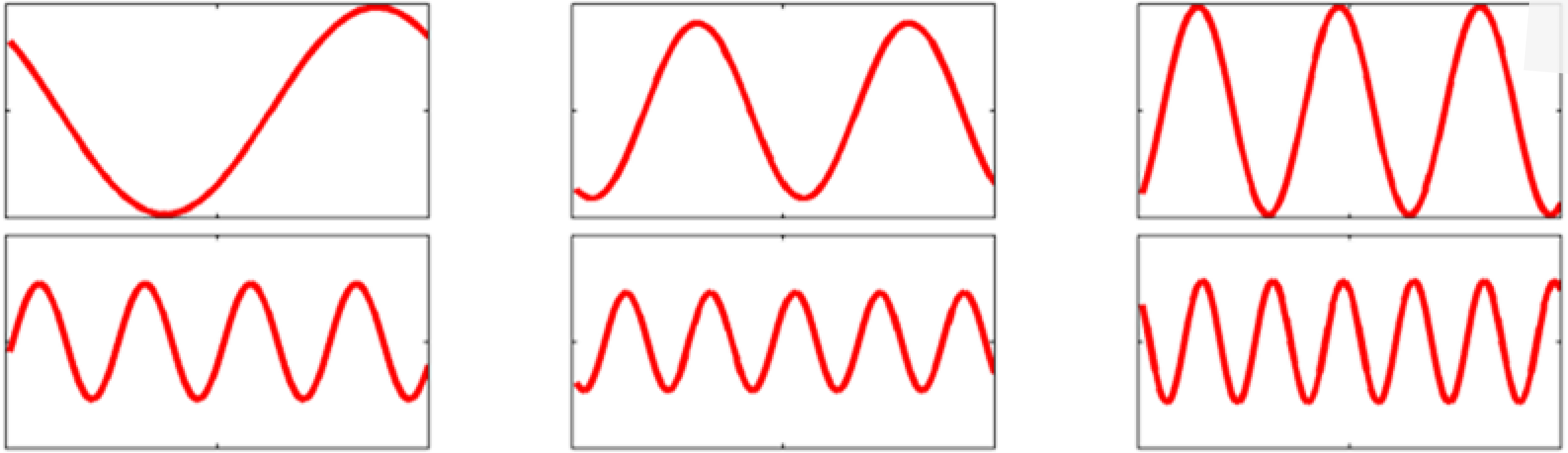
Encoding Functions: Bases for a Function Space

Point basis
Finite-element basis

Local bases



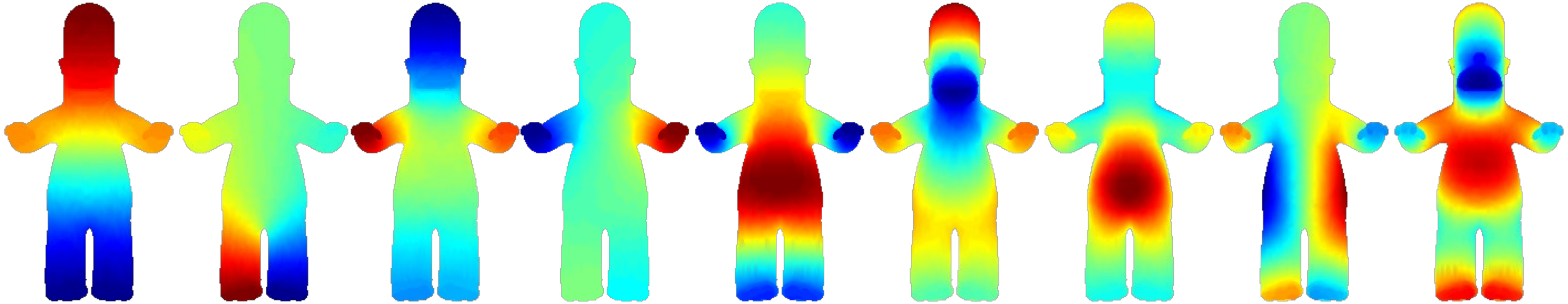
Hierarchical Bases for a Function Space



Fourier

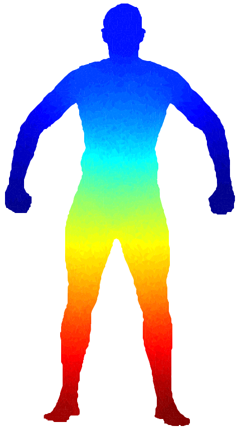
Laplace-Beltrami

global support



Application of Basis

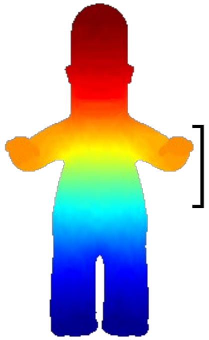
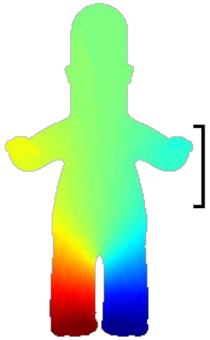
$f(x) = a_1 \cdot$  $+ a_2 \cdot$  $+ a_3 \cdot$  $+ \cdots$

$\phi :$  $\longrightarrow$ 

Homer Simpson

Application of Basis

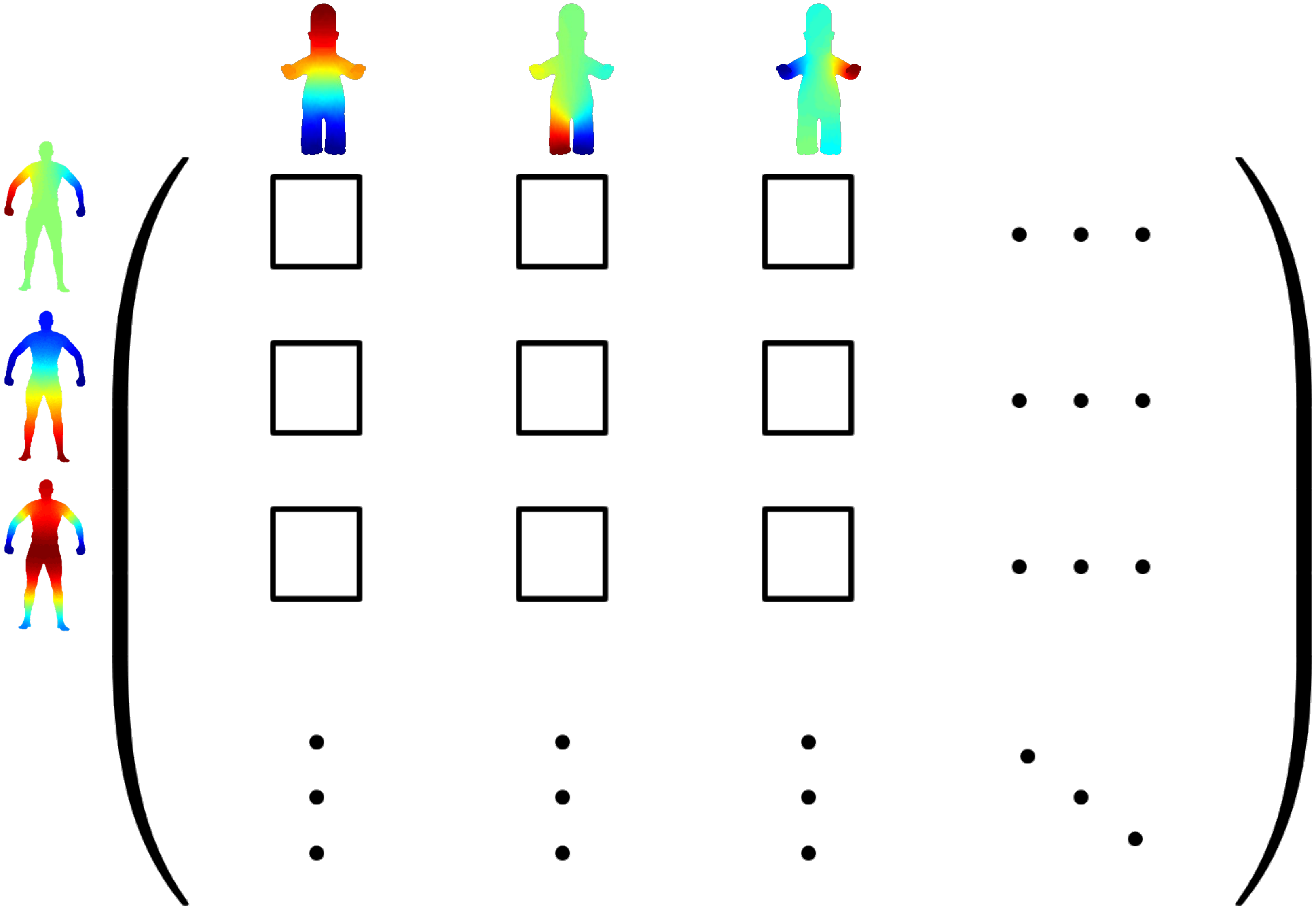
$T_\phi[f](x) = T_\phi[a_1 \cdot$  $+ a_2 \cdot$  $+ a_3 \cdot$  $+ \cdots]$

$= a_1 T_\phi[$  $] + a_2 T_\phi[$  $] + a_3 T_\phi[$  $] + \cdots$

$\underbrace{\qquad\qquad} \qquad \underbrace{\qquad\qquad} \qquad \underbrace{\qquad\qquad}$

Enough to know these

Functional Map Matrix



Functional Map Representation

Definition

For a fixed choice of basis functions $\{\phi^M\}$ and $\{\phi^N\}$, and a bijection $T : M \to N$, define its **functional representation** as a matrix C, s.t. for all $f = \sum_i a_i \phi_i^M$, if $T_F(f) = \sum_i b_i \phi_i^N$ then:

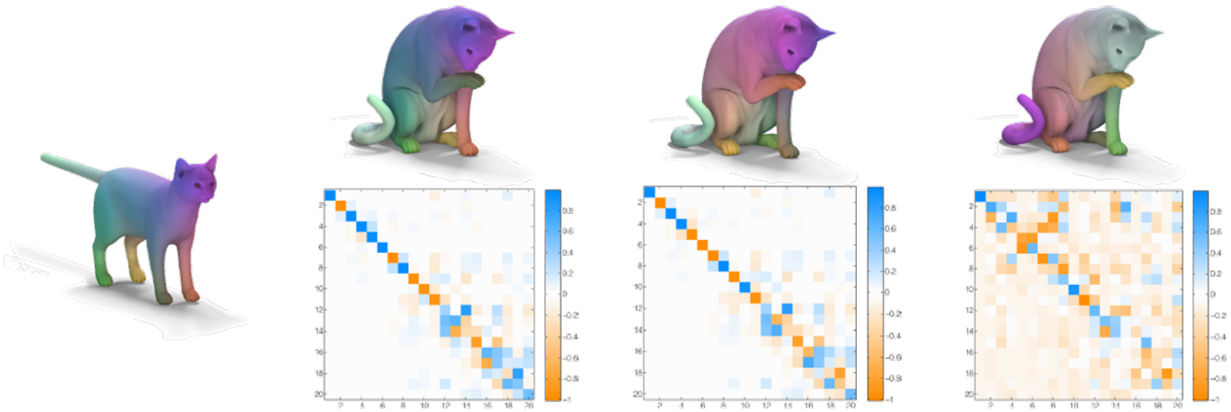
$$\mathbf{b} = C\mathbf{a}$$

If $\{\phi^M\}$ and $\{\phi^N\}$ are both orthonormal w.r.t. some inner product, then

$$C_{ij} = \left\langle T_F(\phi_i^M), \phi_j^N \right\rangle .$$

Maps as Linear Operators

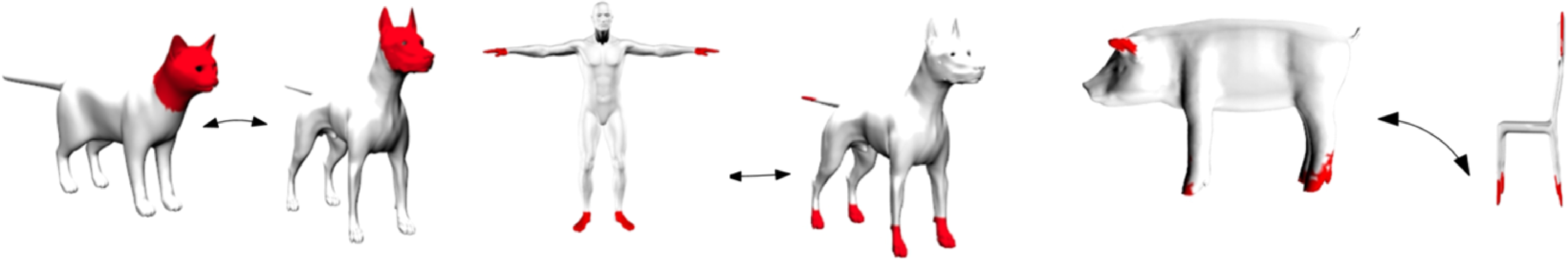
- An ordinary shape map lifts to a linear operator mapping the function spaces
- With a truncated hierarchical basis, compact representations of functional maps are possible as ordinary matrices
- Map composition becomes ordinary matrix multiplication
- Functional maps can express many-to-many associations, generalizing classical 1-1 maps



source direct symmetric head to tail

Using truncated Laplace-Beltrami basis

FMaps Can Represent Broader Classes of Correspondences



FMaps include point-to-point maps, but are much more general

Estimating the Mapping Matrix Directly

Suppose we don't know *C*. However, we expect a pair of functions $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ to correspond. Then, *C* must satisfy:

$$C\mathbf{a} \approx \mathbf{b}$$

where $f = \sum_i \mathbf{a_i} \phi_i^M, \quad g = \sum_i \mathbf{b}_i \phi_i^N$

Functional Correspondences

Given enough $\{\mathbf{a_i}, \mathbf{b}_i\}$ pairs in correspondence, we can recover C through a linear least squares system.

Function Preservation Constraints

Suppose we don’t know *C*. However, we expect a pair of functions $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ to correspond. Then, *C* must be s.t.

$$C\mathbf{a} \approx \mathbf{b}$$

Function preservation constraint is quite general and includes:

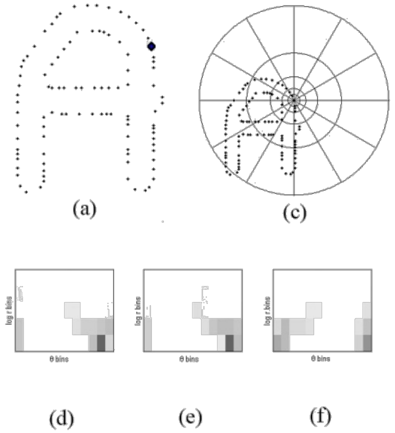
- Descriptor preservation (e.g. Gaussian curvature, spin images, HKS, WKS).
- Landmark correspondences (e.g. distance to the point).
- Part correspondences (e.g. indicator function).
- Texture preservation

“Probe functions”

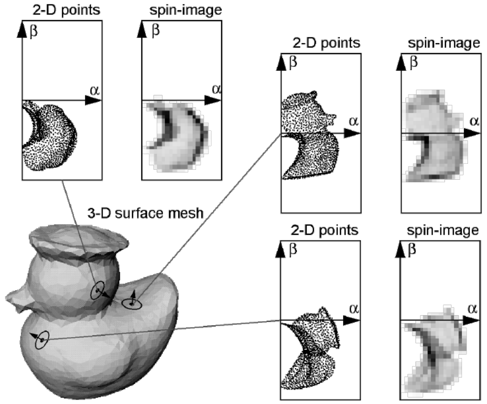
injection of low-level knowledge or supervision

Plenty of Functions: Descriptors for Points and Parts

- For shapes, there are many descriptors with various types of invariances

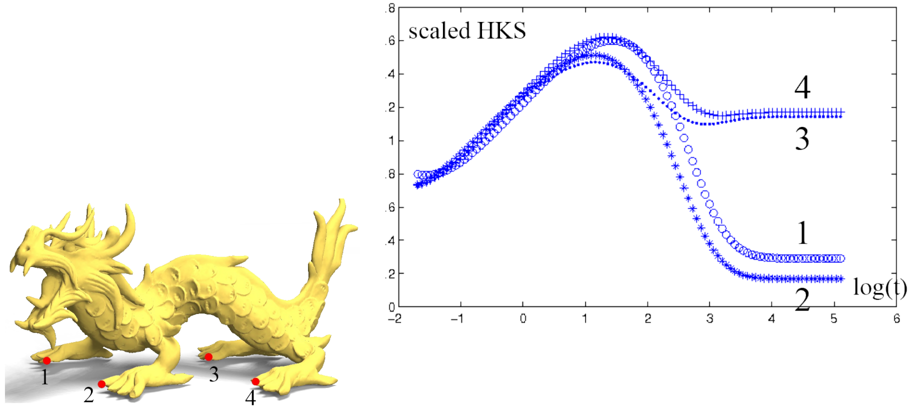


Shape Contexts:
[Belongie et al. ’00, Frome et al. ’04]



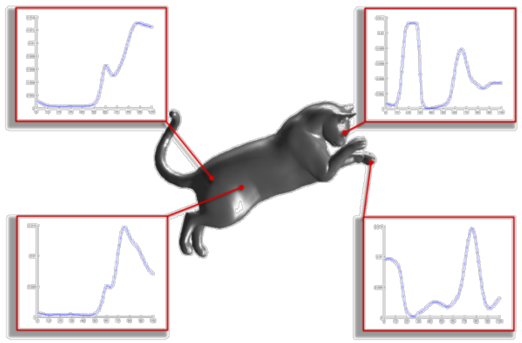
Spin Images:
[Johnson, Hebert ’99]

Rigid invariance (extrinsic)

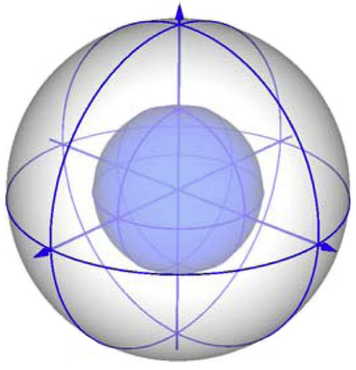


Heat Kernel Signatures (HKS):
[Sun, Ovsjanikov, G. ’08]]

Isometric invariance (intrinsic)



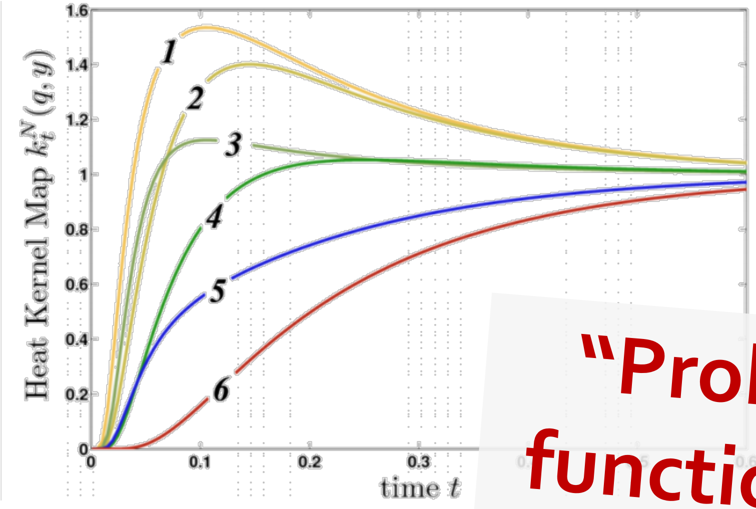
Wave Kernel Signatures (WKS):
[Aubry et. al. ‘11]



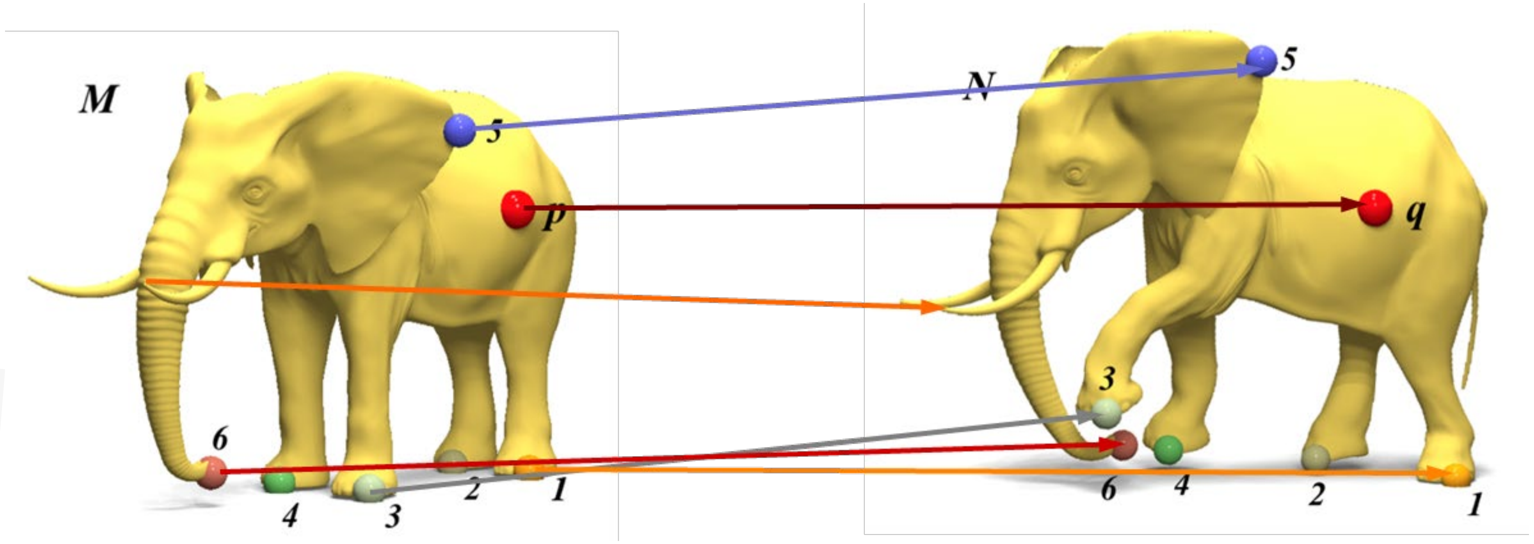
SHOT

Map Estimation

$$CD_1 = D_2 \implies C = D_2 D_1^{-1}$$



“Probe function”



Map from a linear solve

Commutativity Symmetry Regularization

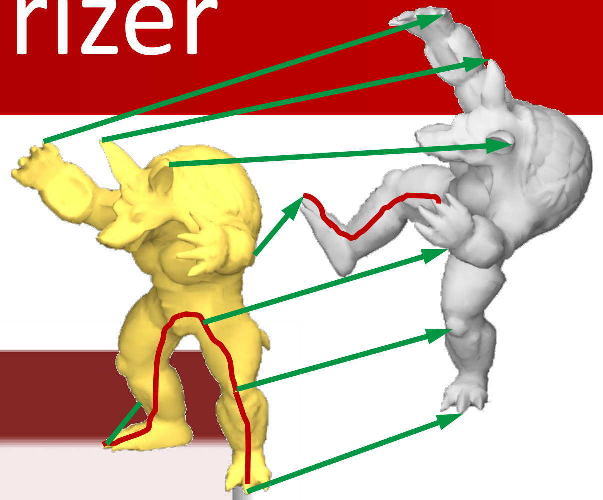
In addition, we can phrase an operator commutativity constraint: given two operators $S_1 : \mathcal{F}(M, \mathbb{R}) \to \mathcal{F}(M, \mathbb{R})$ and $S_2 : \mathcal{F}(N, \mathbb{R}) \to \mathcal{F}(N, \mathbb{R})$

$$\begin{tikzcd}
\mathcal{F}(M, \mathbb{R}) \arrow[r, "C"] \arrow[d, "S_1"'] & \mathcal{F}(N, \mathbb{R}) \arrow[d, "S_2"] \\
\mathcal{F}(M, \mathbb{R}) \arrow[r, "C"] & \mathcal{F}(N, \mathbb{R})
\end{tikzcd}$$

Thus: $CS_1 = S_2 C$ or $\|CS_1 - S_2 C\|$ should be minimized

Note: this is a linear constraint on C. S_1 and S_2 could be symmetry operators or e.g. Laplace-Beltrami or heat operators.

Isometry (Length Preservation) Regularizer



Lemma 1:

The mapping is *isometric*, if and only if the functional map matrix commutes with the Laplacian:

$$C\Delta_1 = \Delta_2 C$$

Differentiate and then transport

Transport and then differentiate

Δ_1 Laplacian on Shape 1
Δ_2 Laplacian on Shape 2

Conformal (Angle Preservation) Regularization

Lemma 3:

If the mapping is *conformal* if and only if:

$$C^T \Delta_1 C = \Delta_2$$

Using these regularizations, we get a very efficient shape matching method.

Volume Preservation Regularizer

Lemma 2:

The mapping is *locally volume preserving*, if and only if the functional map matrix is *orthonormal*:

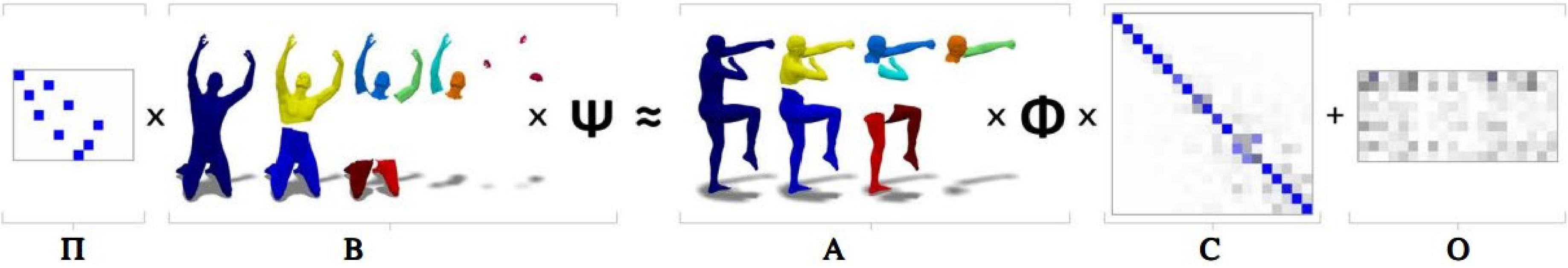
$$C^T C = \mathrm{I}$$

Rotations/reflections in functions space

Sparcity in a Localized Basis

$$\min \|C\|_{2,1}$$

Sum of Euclidean norms of cols



Sparse Modeling of Intrinsic Correspondences (Pokrass, Bronstein², Sprechmann, Sapiro)

Basic FMaps Pipeline

Given a pair of shapes: $\mathcal{M}, \mathcal{N}$

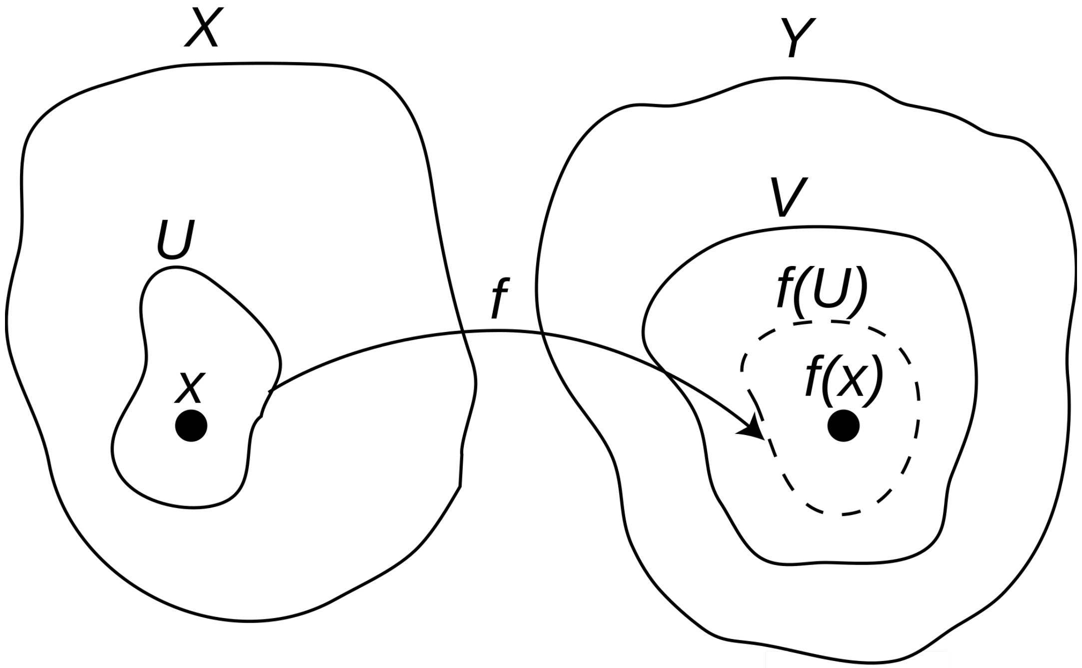
1. Compute the first *k* (~80-100) eigenfunctions of the Laplace-Beltrami operator. Store them in matrices: $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$
2. Compute descriptor functions (e.g., Heat Kernel Signatures) on $\mathcal{M}, \mathcal{N}$. Express them in $\mathbf{A}, \mathbf{B}$, as columns of $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$
3. Solve $C_{\text{opt}} = \underset{C}{\arg\min} \|C\mathbf{A} - \mathbf{B}\|^2 + \|C\Delta_{\mathcal{M}} - \Delta_{\mathcal{N}}C\|^2 + \ \ldots$

   $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$: diagonal matrices of eigenvalues of LB operator
4. Convert the functional map C_{opt} to a point to point map *T*.

Side Comment: Map Continuity

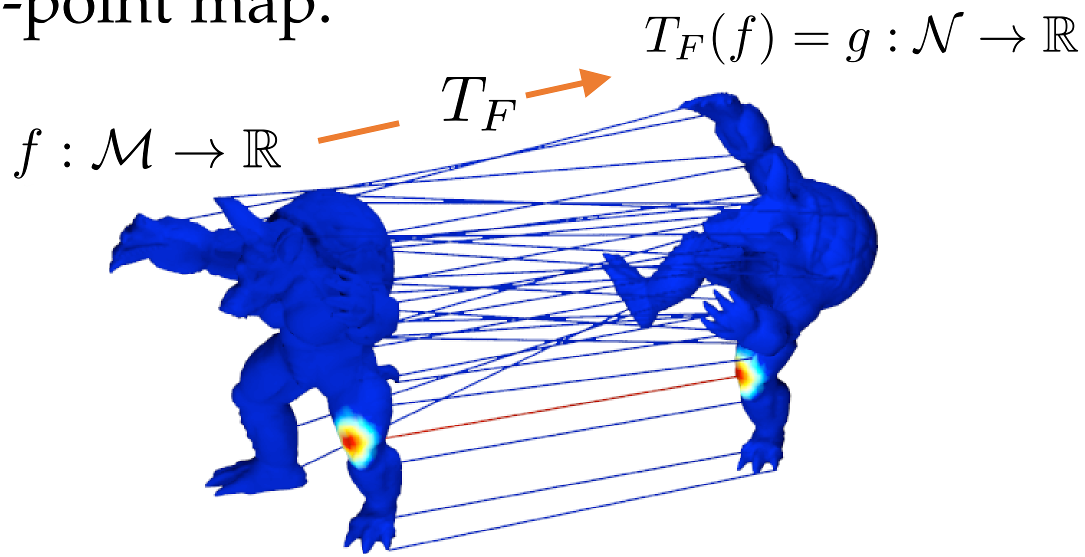
- Not explicitly enforced

- Implicit in the choice of basis



Conversion to Point-to-Point

Given a functional map C, we would like to convert to to a point-to-point map.



$T_F(f) = g : \mathcal{N} \to \mathbb{R}$

T_F

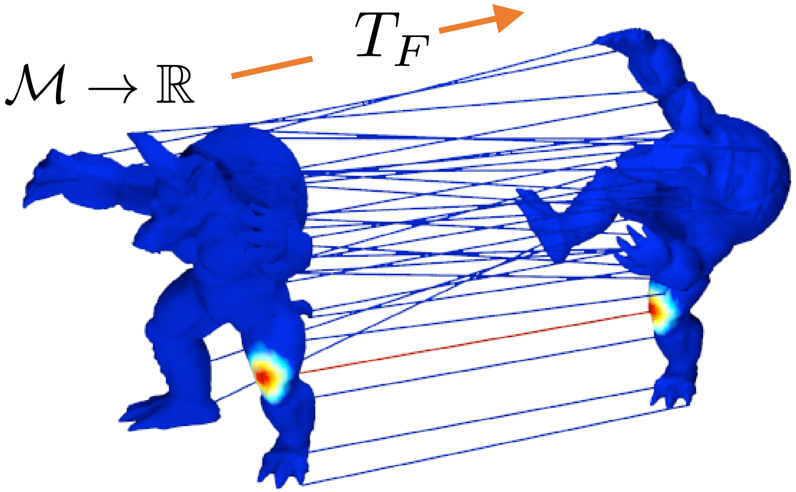
$f : \mathcal{M} \to \mathbb{R}$

Option 1: declare $T(x) = \arg\max_{y} \Phi_{\mathcal{N}} C \delta_x$

Problems: high computational complexity $O(n_{\mathcal{M}} n_{\mathcal{N}})$, low accuracy.

Conversion to Point-to-Point

Given a functional map C, we would like to convert to to a point-to-point map.



$f : \mathcal{M} \to \mathbb{R}$

T_F

$T_F(f) = g : \mathcal{N} \to \mathbb{R}$

Option 2: declare $T(x) = \arg\min_{y} \|\delta_y - C\delta_x\|_2$

Advantages: computational complexity $O(n_{\mathcal{M}} \log n_{\mathcal{N}})$,
higher accuracy (e.g., works with the identity map).

From Functional to Point-to-Point Maps

- Can try transporting delta functions individually -- expensive

Primal

$$p \mapsto q$$

Dual

$$\begin{array}{r} \\ \\ \\ \\ p \\ \\ \\ \\ \end{array}\begin{pmatrix}0\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\\\vdots\\0\\0\\0\\\vdots\\0\end{pmatrix}\begin{array}{l} \\ q \\ \\ \\ \\ \\ \\ \\ \end{array}$$

$$\delta_x = (\phi_1^M(x), \phi_2^M(x), \phi_3^M(x), \ldots)$$

From Functional to Point-to-Point Maps

$$T(x) = \arg\min_{y} \|\delta_y - C\delta_x\|_2$$

$$C\Phi_M^\top \leftrightarrow \Phi_N$$

Image of each point on surface M

Each point on surface N in LB basis

So transport, and then use nearest neighbor search

Incorporating Orthonormality

In many practical situations we would expect a volume-preserving map, which implies:

$$C^T C = Id$$

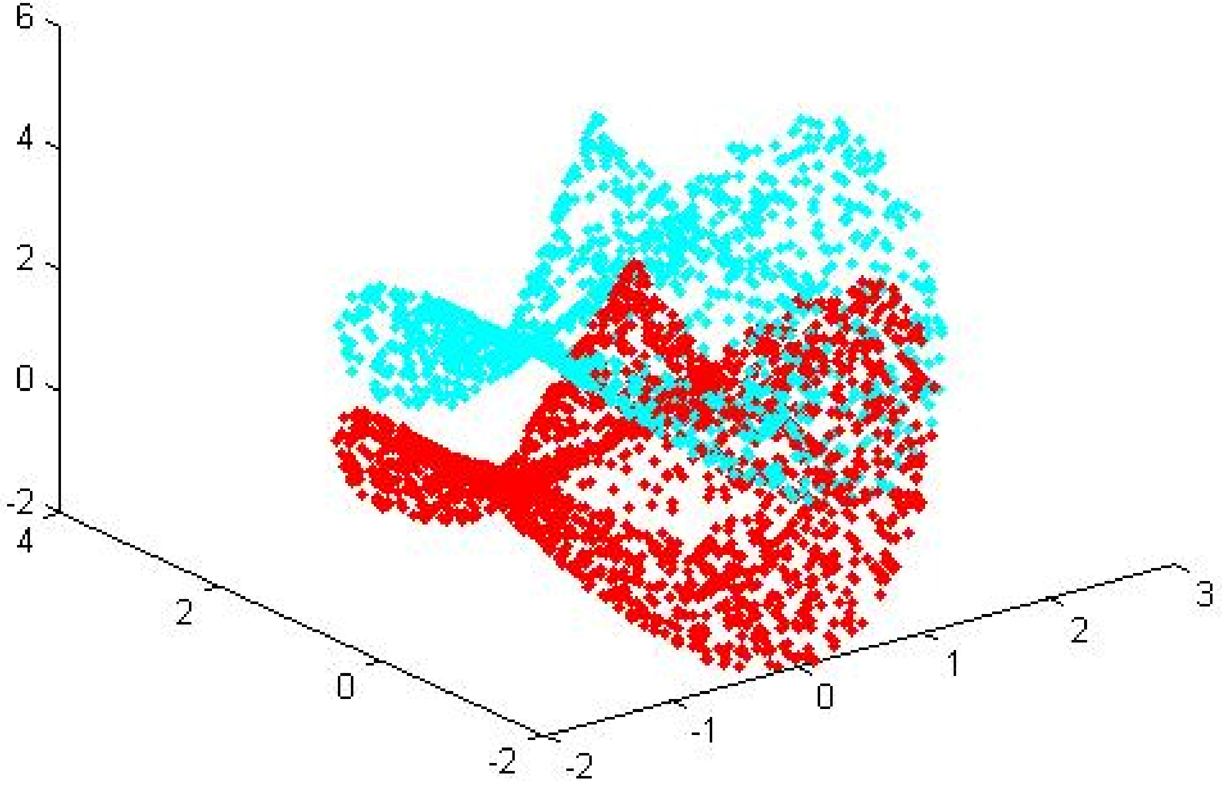
Option: use post-processing to enforce this constraint.

Iterate:

1. Compute the point-to-point map T.
2. Solve for the functional map: $\underset{C,\ \text{s.t. } C^T C = Id}{\arg\min} \sum_{x \in \mathcal{M}} \|C\delta_x - \delta_{T(x)}\|_2^2$

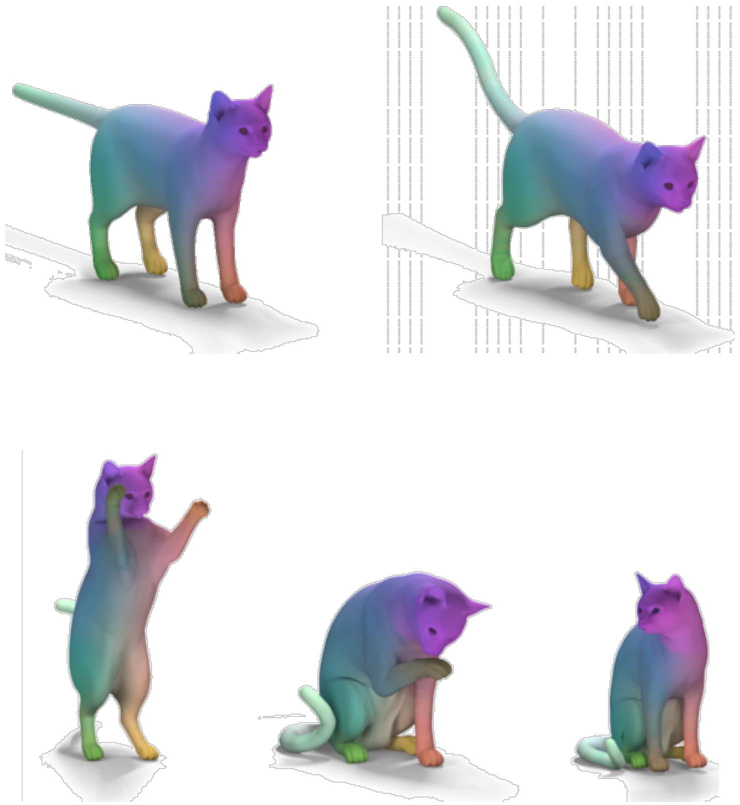
Exactly the same objective as ICP, but in higher dimension. Can use the same method!

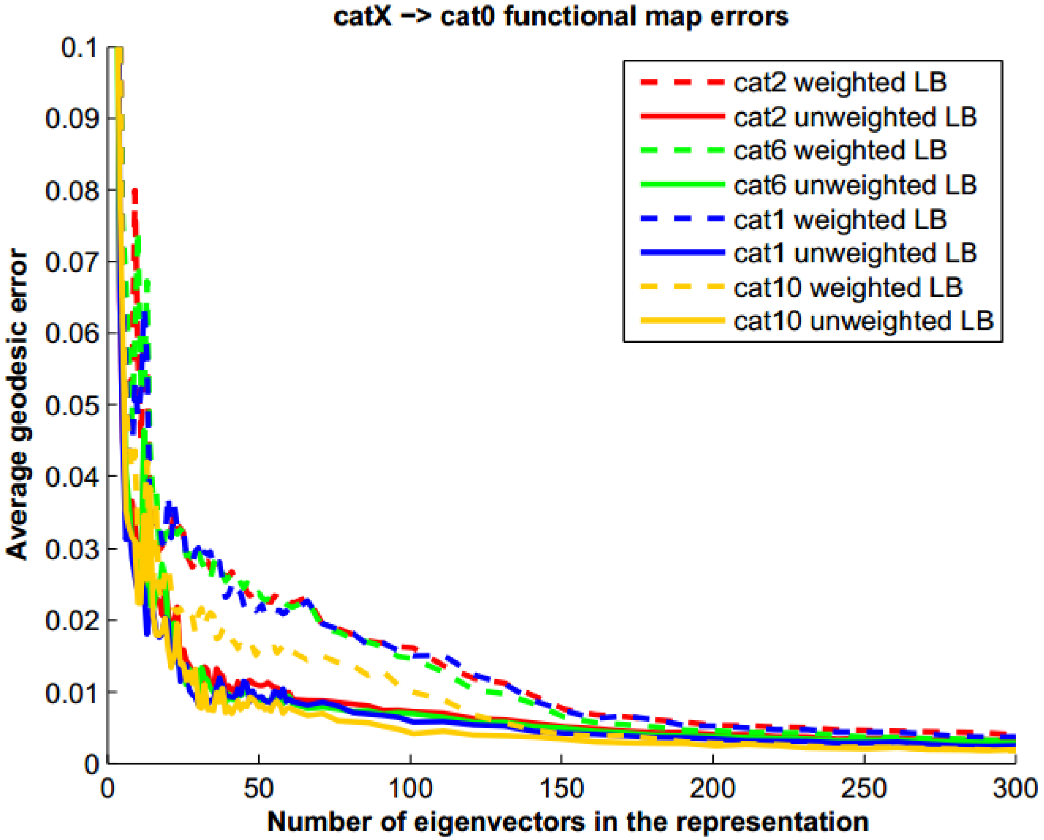
From Functional to Point-to-Point Maps



ICP in Function Space!

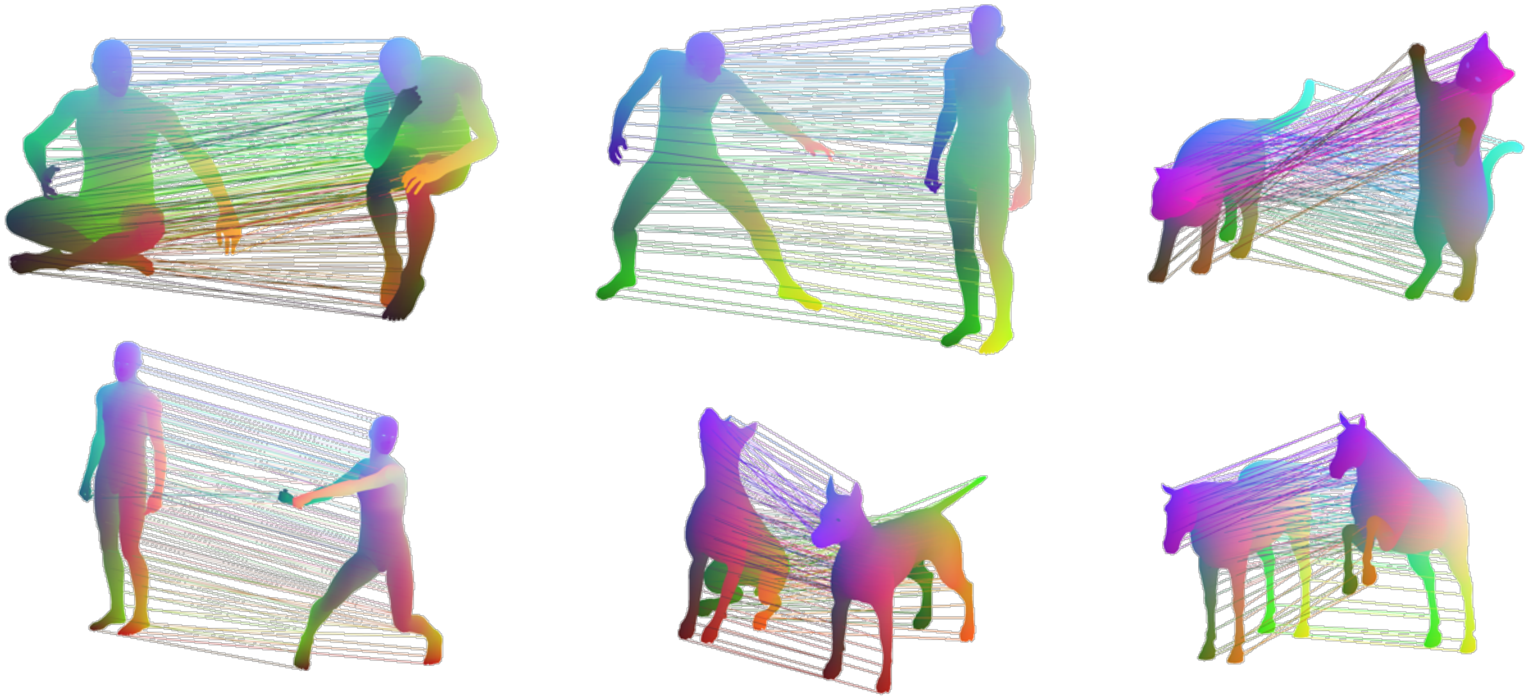
Ground Truth Comparison





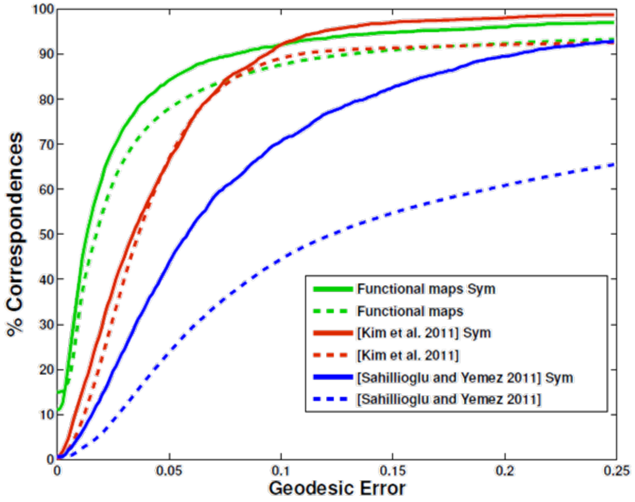
Results

A very simple method that puts together many constraints and uses 100 basis functions gives reasonable results:

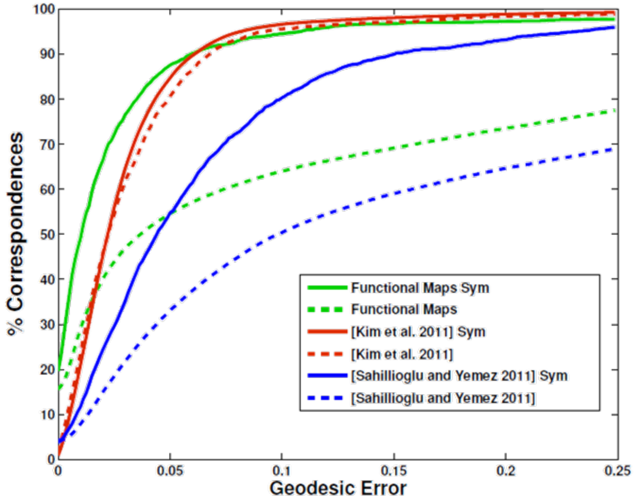


Map Estimation Quality

A very simple method that puts together a modest set of constraints and uses 100 basis functions outperforms state-of-the-art:



SCAPE



TOSCA

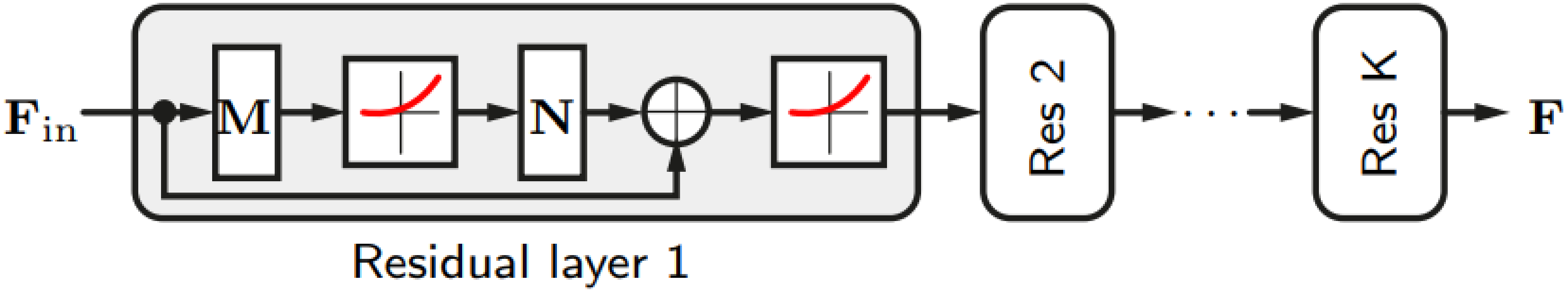
Roughly 10 probe functions + 1 part correspondence

Much Follow-On Work on FMaps

Deep Functional Maps

Deep Functional Maps

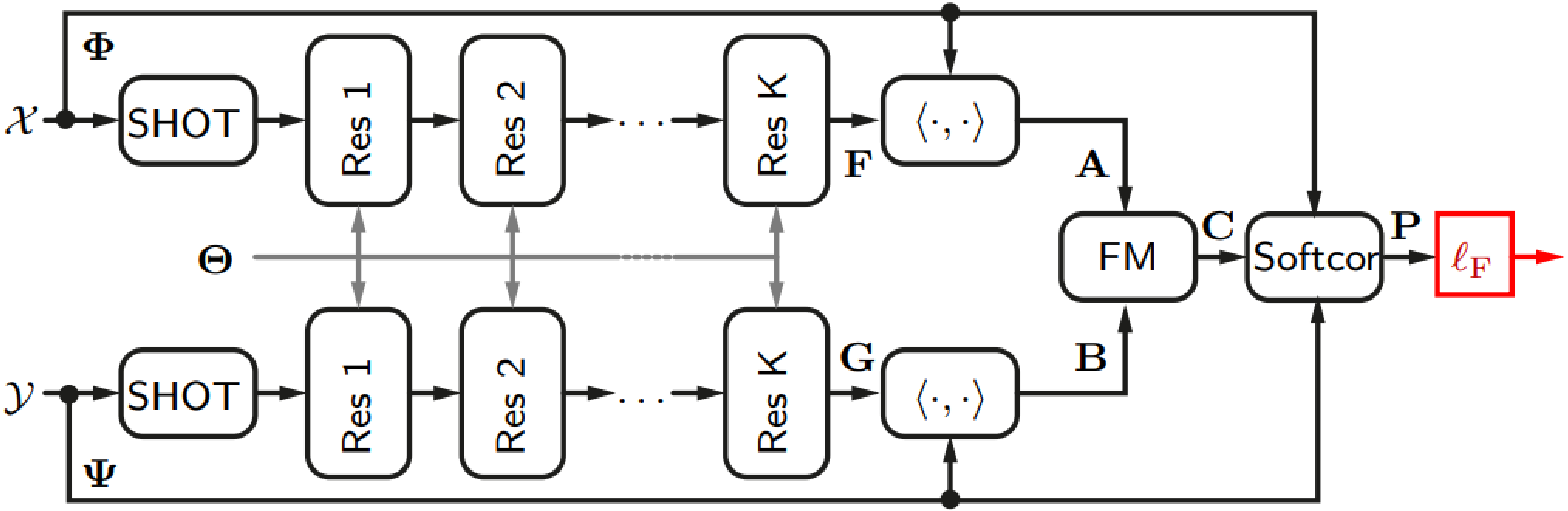
- Main idea: Learn pointwise descriptors resulting in *best functional maps*
- Residual network (ResNet) for descriptor learning:



- **Input:** pointwise descriptor $\mathbf{F}_{\text{in}}$ (e.g., 352-dimensional SHOT)
- N fully-connected residual layers with ELU activation parameterized by $\mathbf{\Theta} = \{\mathbf{M}_1, \mathbf{N}_1, \dots, \mathbf{M}_K, \mathbf{N}_K\}$
- **Output:** transformed pointwise descriptor $\mathbf{F}$

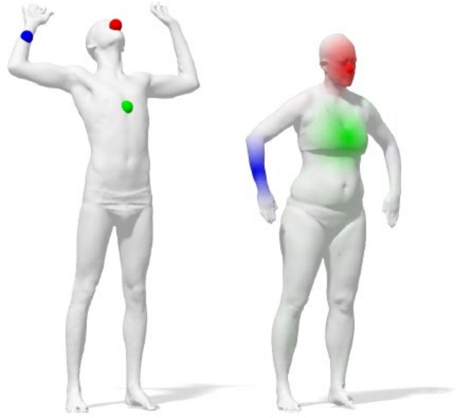
Litany, Remez, Rodola, Bronstein (ICCV’17)

FMNet



- **Functional map layer:** $\mathbf{C} = \arg\min \|\mathbf{CA} - \mathbf{B}\|_F^2$
- **Soft correspondence layer:** $\mathbf{P} = |\mathbf{\Psi} \mathbf{C} \mathbf{\Phi}^\top|_{\|\cdot\|}$
- Functional map loss:

$$\ell_F = \sum_{(x,y)\in(\mathcal{X},\mathcal{Y})} P(x,y)\, d_{\mathcal{Y}}(y, \pi^*(x)) = \|\mathbf{P} \odot \mathbf{D}_{\mathcal{Y}}\|_F$$



Making Functional Maps More Point-to-Point

Making Functional Maps Point-to-Point

Question:

When does a linear functional mapping correspond to a pull-back by a point-to-point map?

(Known) Theoretical result:

A functional map is point-to-point iff it preserves pointwise products of functions:

$$C(fh) = C(f)C(h) \; \forall \; f, h \quad (fh)(x) = f(x)h(x)$$

J. von Neumann, *Zur operatoren methode in der klassichen Mechanik*, Ann. of Math.(2) 33 (1932)

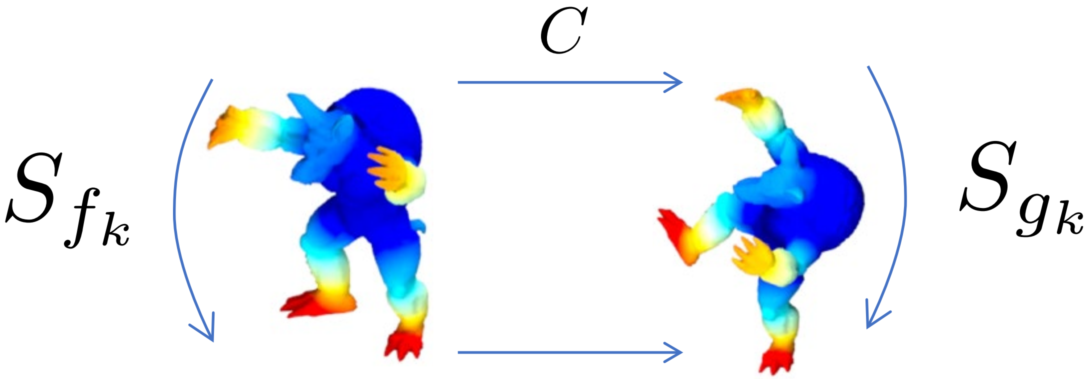
Making Functional Maps Point-to-Point

Approach

Represent descriptor functions via their action on functions through multiplication:

Functions as operators

$$S_{f_k} = \Phi_{\mathcal{M}}^{+} Diag(f_k)\Phi_{\mathcal{M}}$$
$$S_{g_k} = \Phi_{\mathcal{N}}^{+} Diag(g_k)\Phi_{\mathcal{N}}$$



$$CS_{f_k} = S_{g_k}C$$

Informative Descriptor Preservation via Commutativity for Shape Matching, Nogneng, O., Eurographics 2017

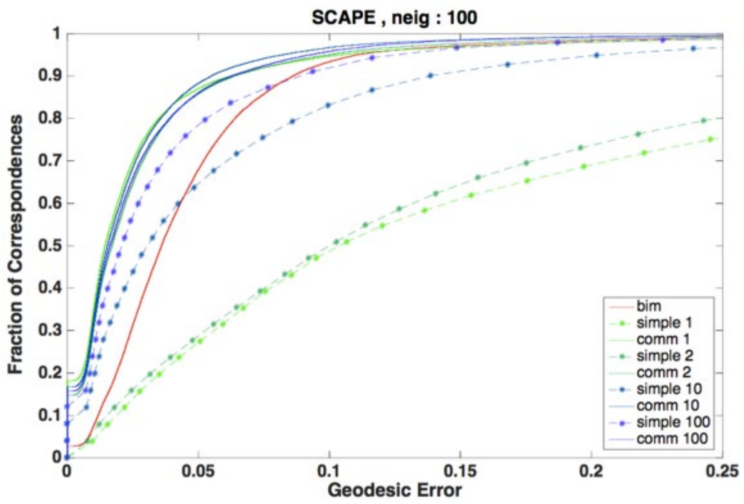
Extended Basic Pipeline

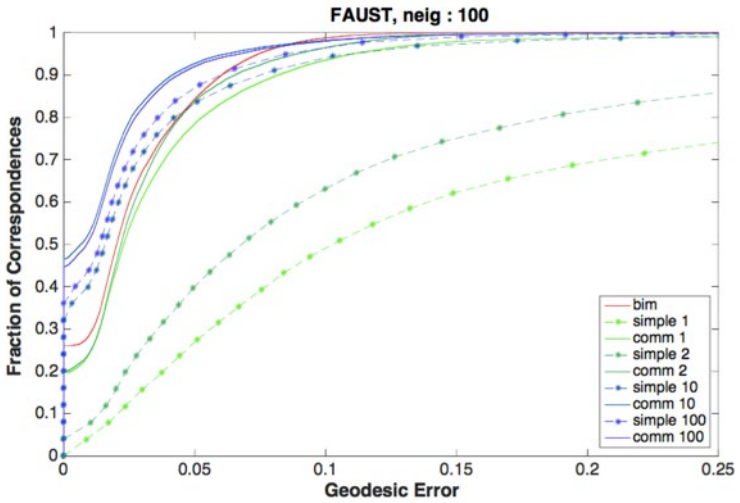
Given a pair of shapes $\mathcal{M}, \mathcal{N}$:

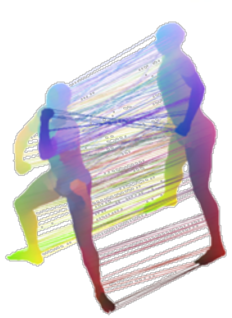
1. Compute the multi-scale bases for functions on the two shapes. Store them in matrices: $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$
2. Compute descriptor functions (e.g., Gauss curvature) on $\mathcal{M}, \mathcal{N}$. Express them in $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$ as columns of : $\mathbf{A}, \mathbf{B}$
3. Solve $$C_{\text{opt}} = \underset{C}{\arg\min} \|C\mathbf{A} - \mathbf{B}\|^2 + \|C\Delta_{\mathcal{M}} - \Delta_{\mathcal{N}} C\|^2 + \sum_k \|CS_{f_k} - S_{g_k} C\|^2$$
4. Convert the functional map C_{opt} to a point to point map *T*.

Informative Descriptor Preservation via Commutativity for Shape Matching, Nogneng, O., Eurographics 2017

Results with Extended Basic Pipeline

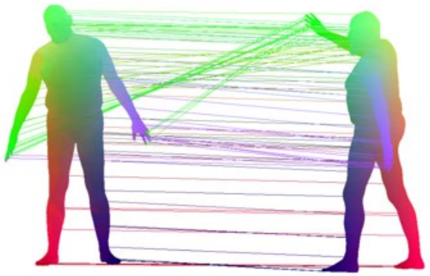




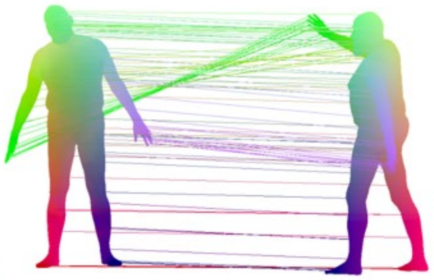


before

after



before



after

Incorporating multiplicative operators improves results significantly.

Informative Descriptor Preservation via Commutativity for Shape Matching,
Nogneng, O., Eurographics 2017

Segmentation Transport

Application: Segmentation Transfer

Map Visualization

Map Visualization

Even given a map $T : M \to N$, it is often hard to visualize it.

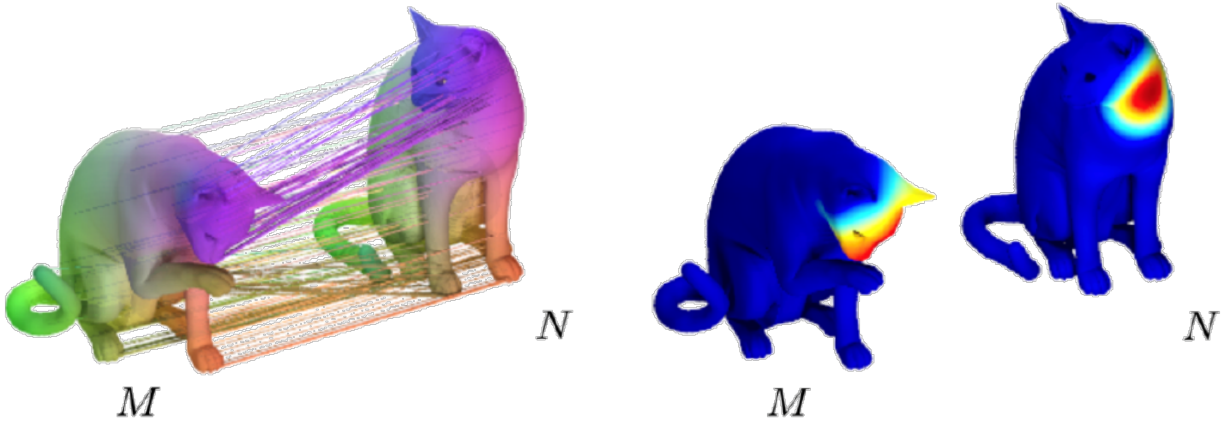
Common visualizations:

- Connecting (some) points by lines
- Plotting a function f on N and $f \circ T$ on M.

Question: how to pick a “good” function f.

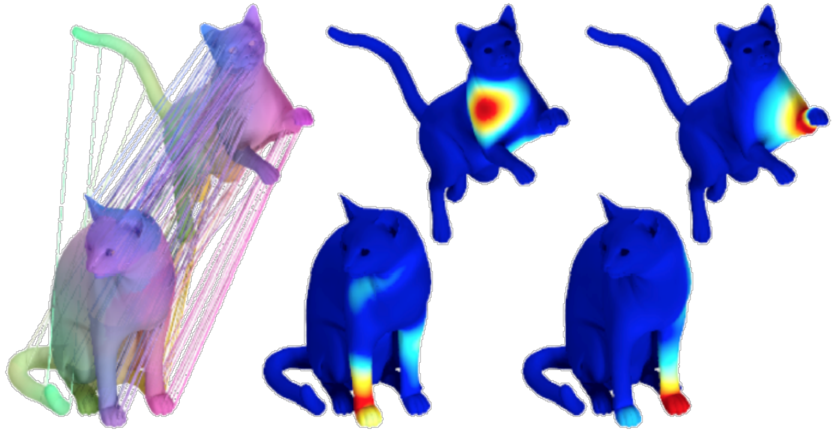
Map Visualization

Singular vectors of the functional representation C of T identify most distorted regions in a multi-scale way.



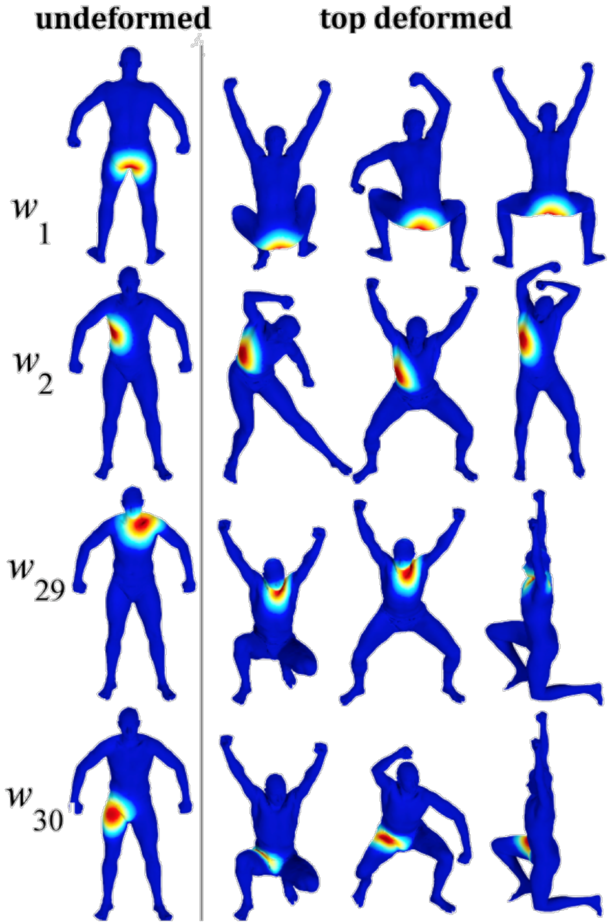
Map Visualization

Can show that singular vectors of the functional representation C of T identify most distorted regions in a multi-scale way.



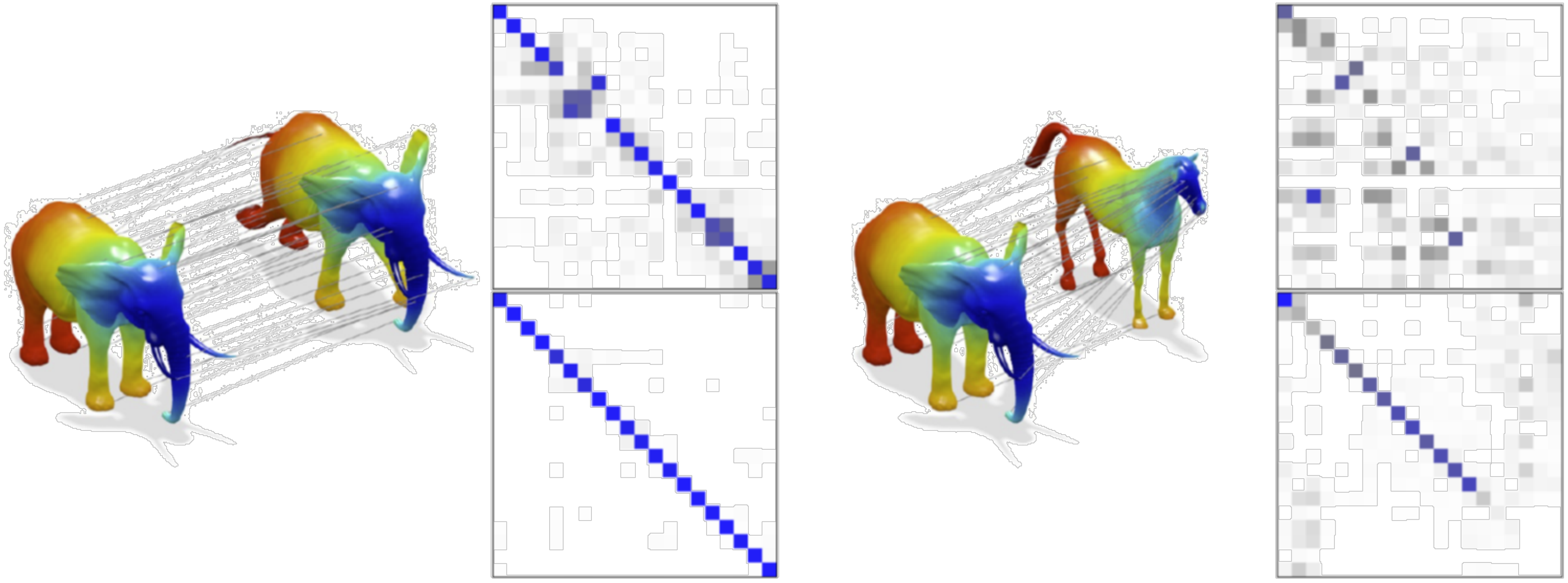
Multiple Shapes

With same method, can visualize maps to *multiple* shapes.



Diagonalizing the Map

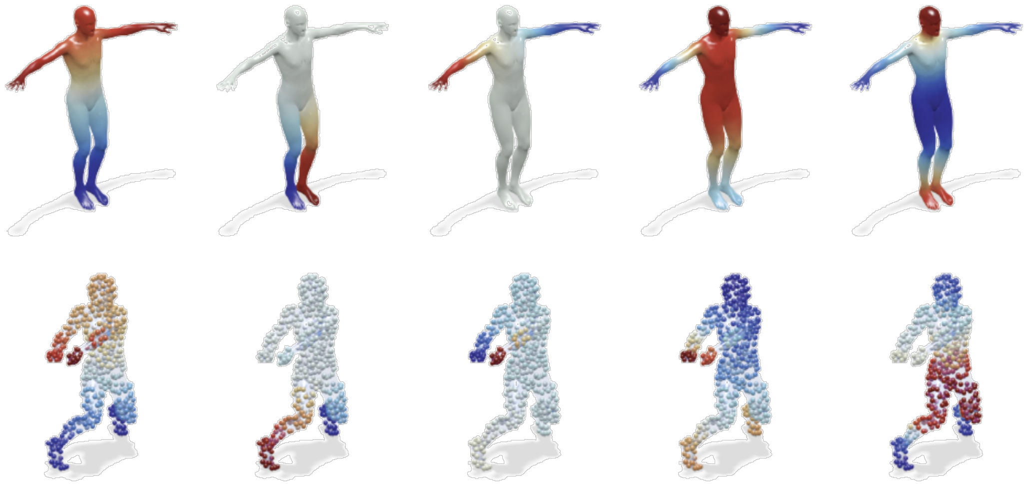
Coupled Bases



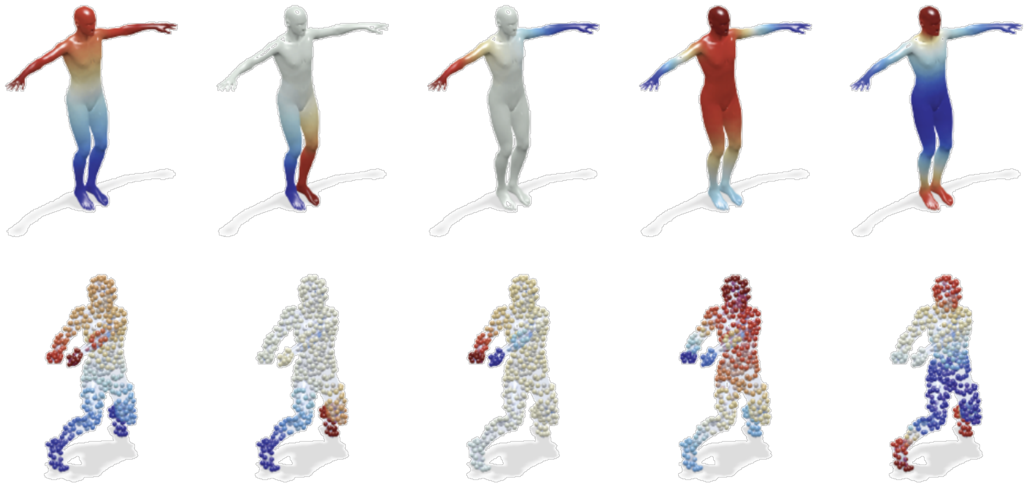
$$\min_{\mathbf{P},\mathbf{Q}} \quad \mathrm{trace}(\mathbf{P}^\top \boldsymbol{\Lambda}_X \mathbf{P}) + \mathrm{trace}(\mathbf{Q}^\top \boldsymbol{\Lambda}_Y \mathbf{Q}) + \mu \|\mathbf{P}^\top \mathbf{A} - \mathbf{Q}^\top \mathbf{B}\|$$

$$\text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}, \mathbf{Q}^\top \mathbf{Q} = \mathbf{I}$$

Coupled Bases



Laplacian eigenbases

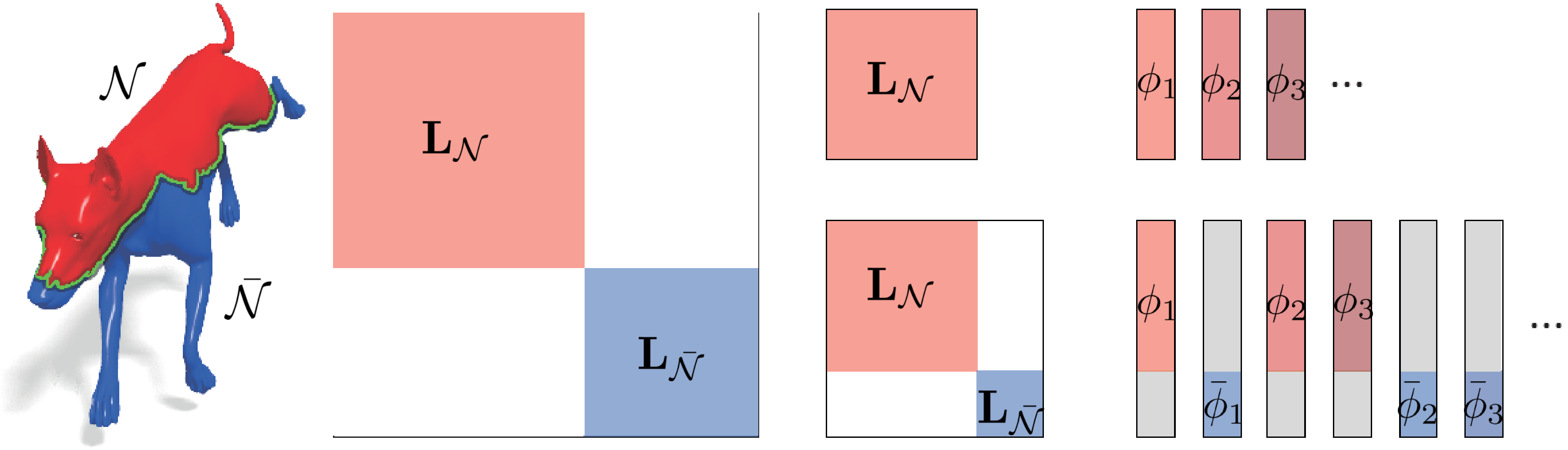


Coupled bases

Partial Functional Maps

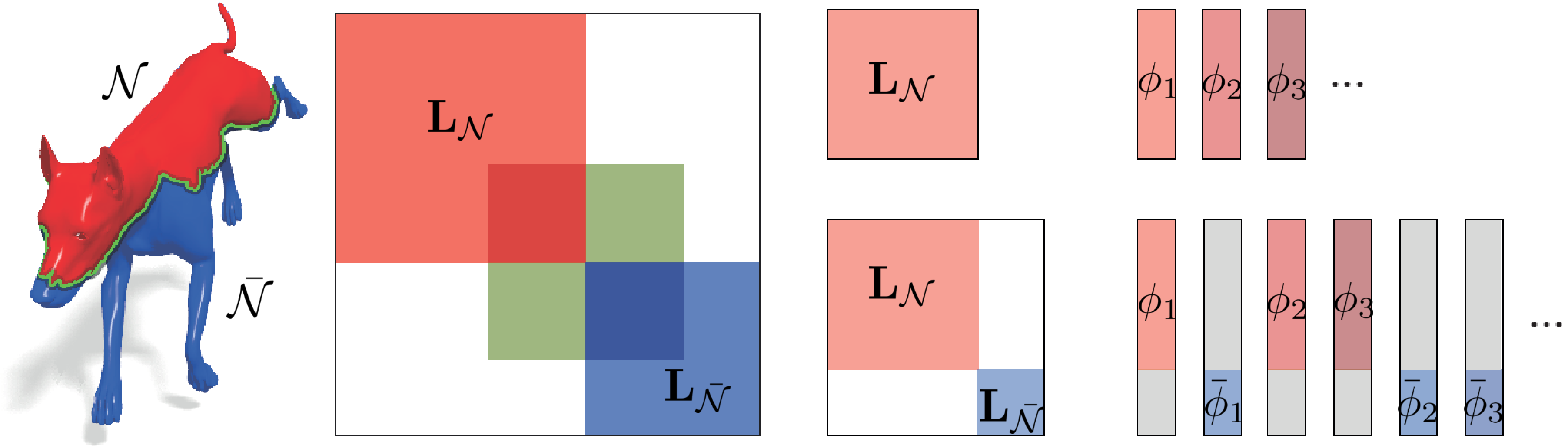
Partial Functional Maps

Block diagonal case

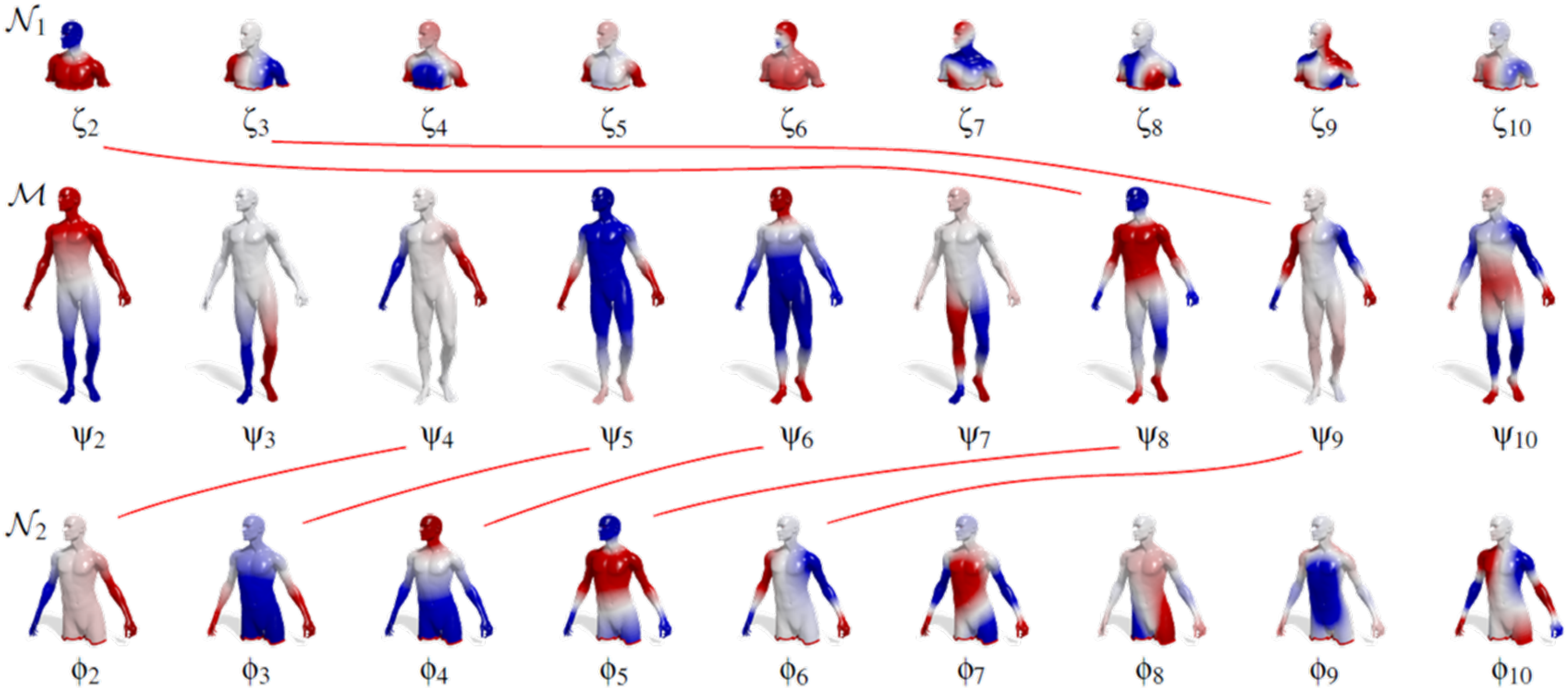


Partial Functional Maps

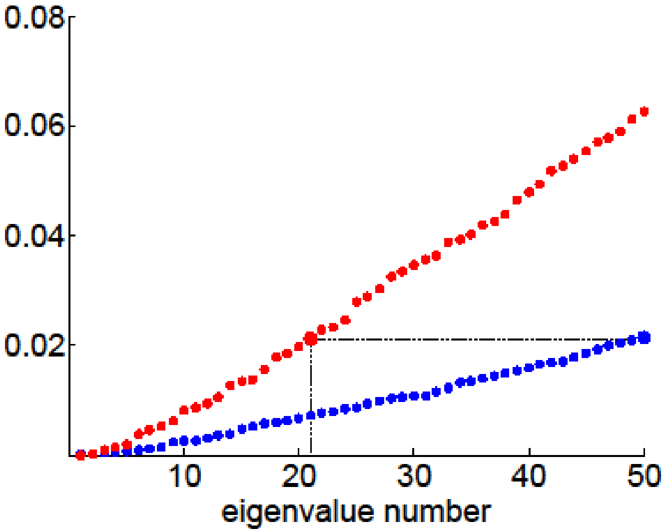
Weak coupling: eigenfunctions of the partial shape show up among those of the full shape, up to some bounded perturbation

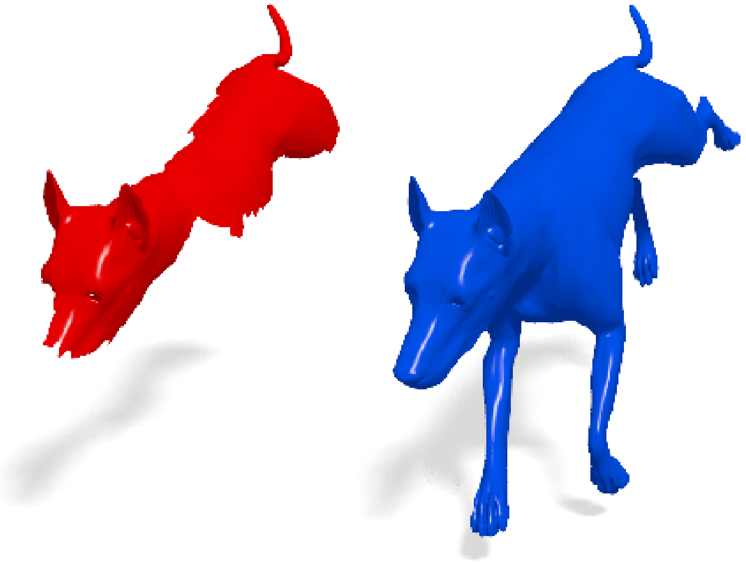


Eigenfunction Preservation



The Slope Rule



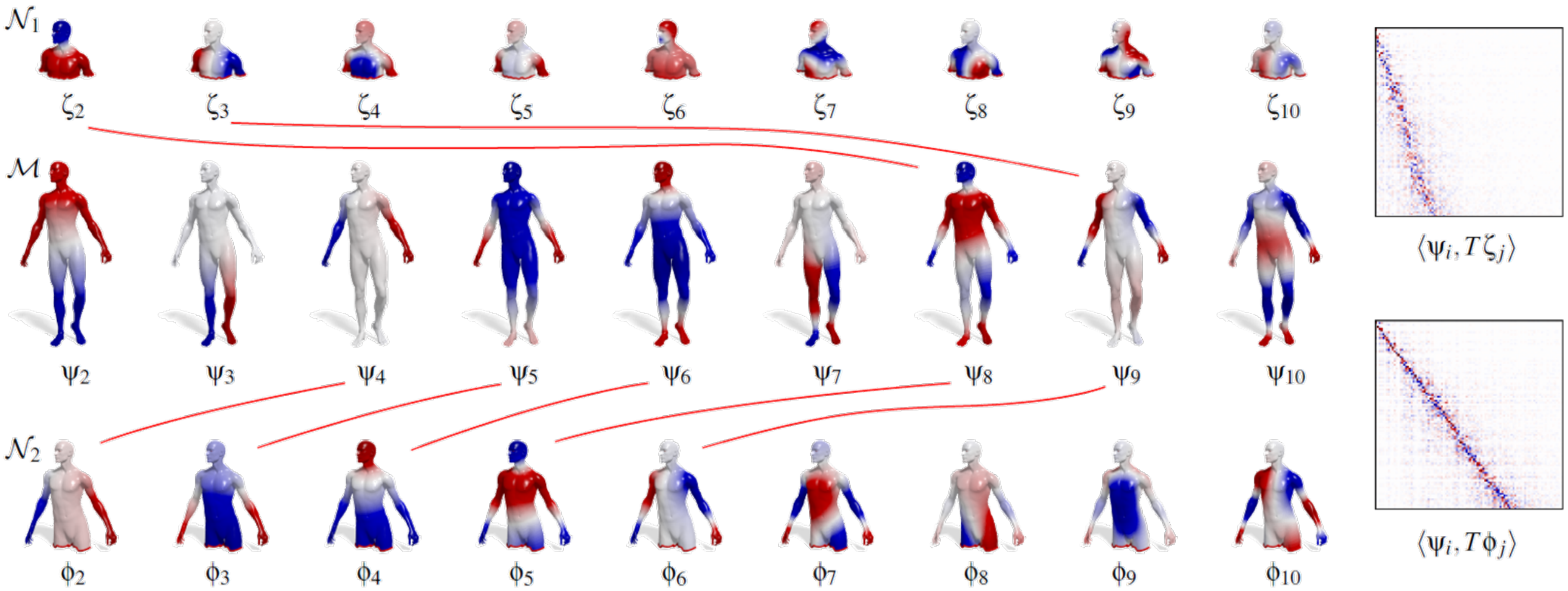


Weyl's law:

$$\lambda_j \approx \frac{1}{|S|} j$$

The Laplacian spectrum has slope inversely proportional to the surface area.

The Slope Rule



Partial Functional Maps

That’s All