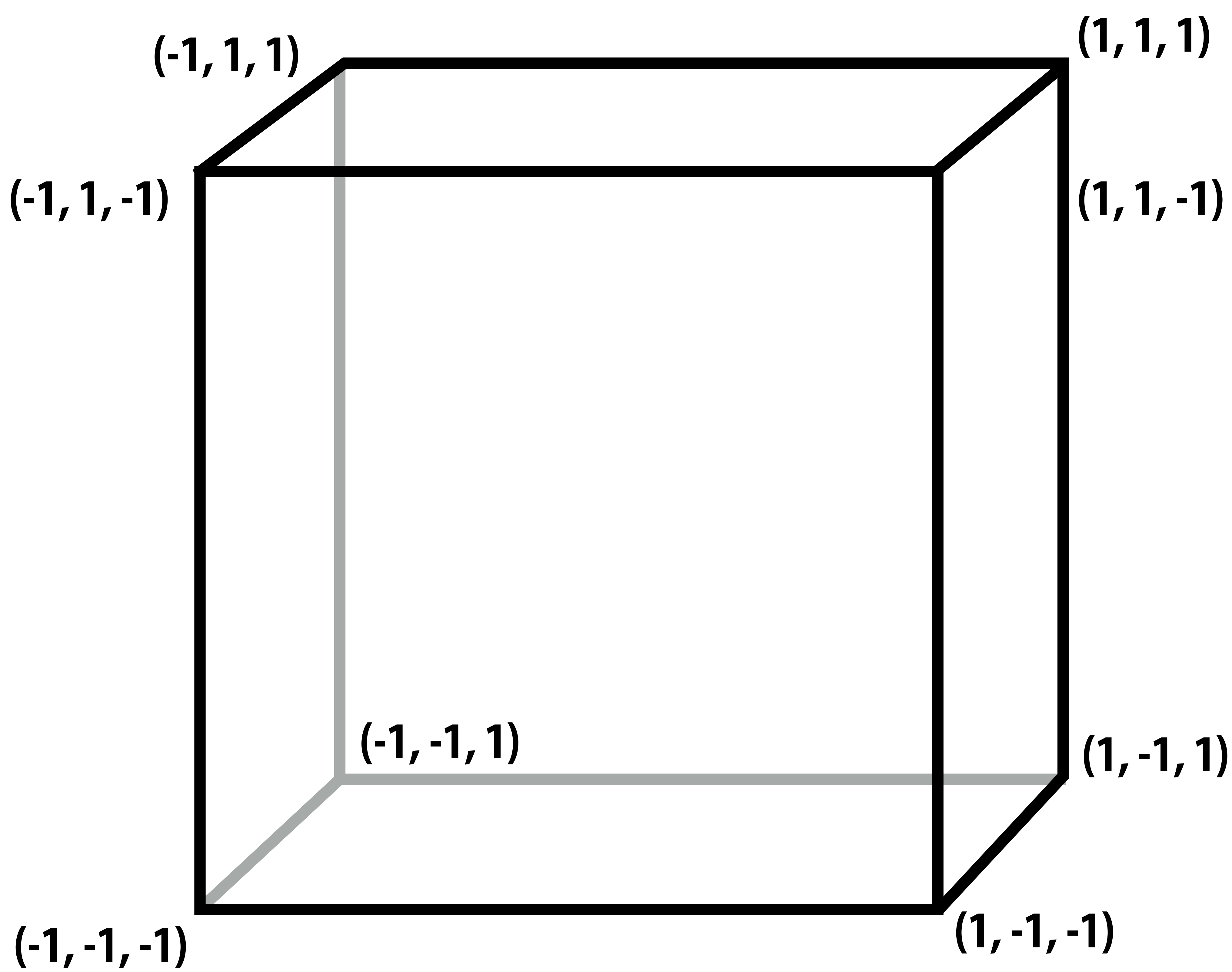
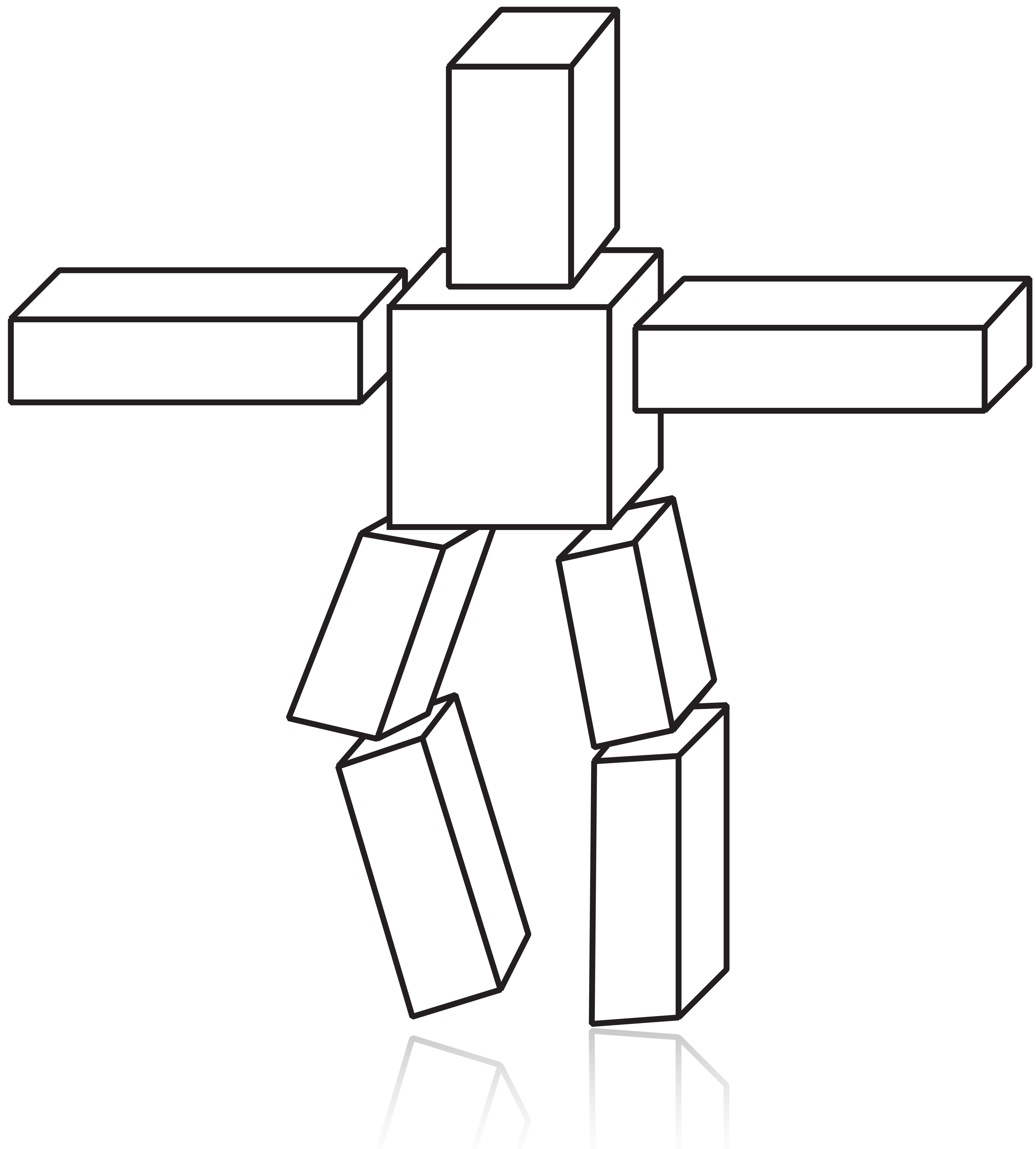
Coordinate Spaces and Transformations

Interactive Computer Graphics
Stanford CS248, Spring 2018

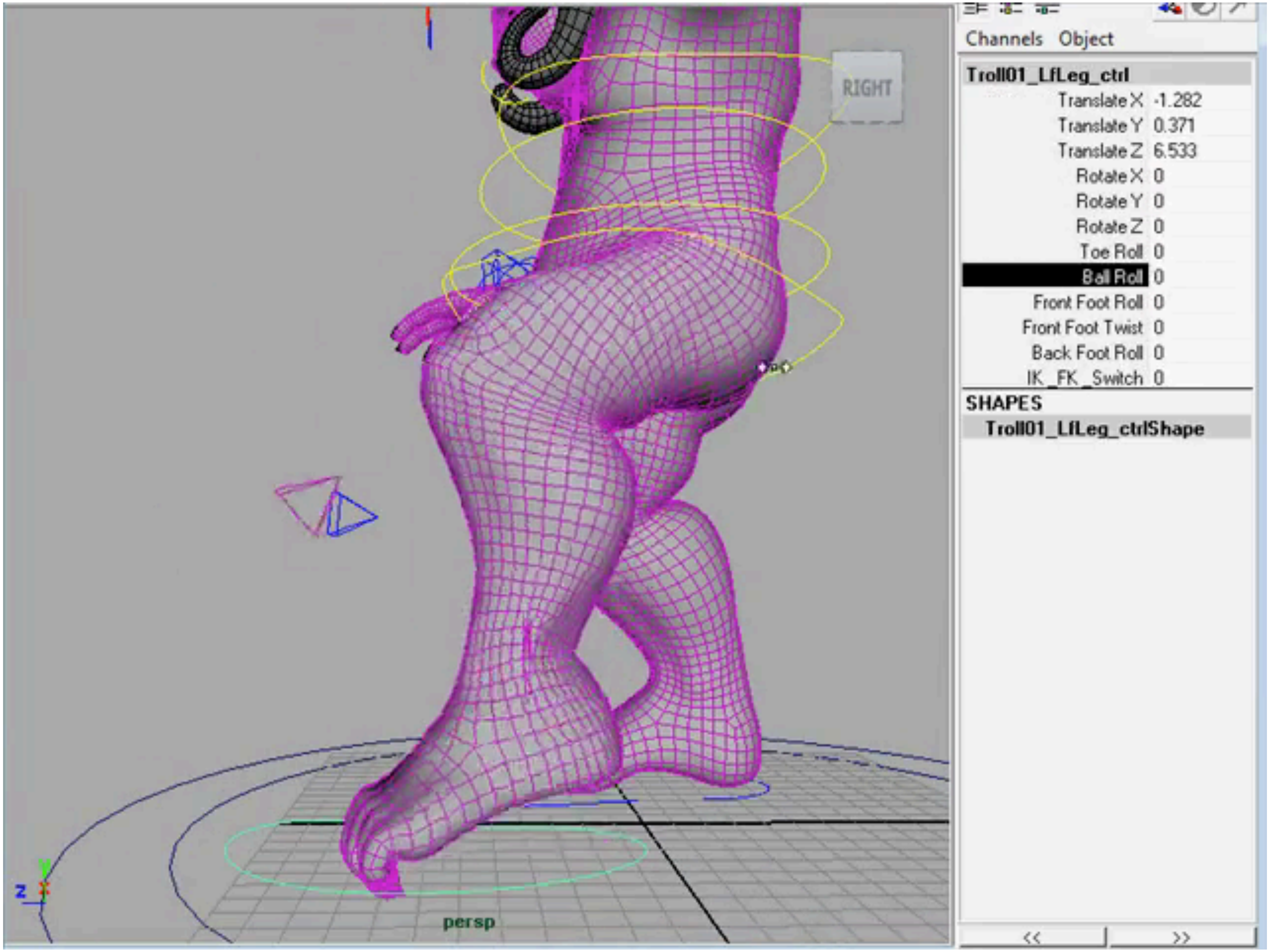
Cube



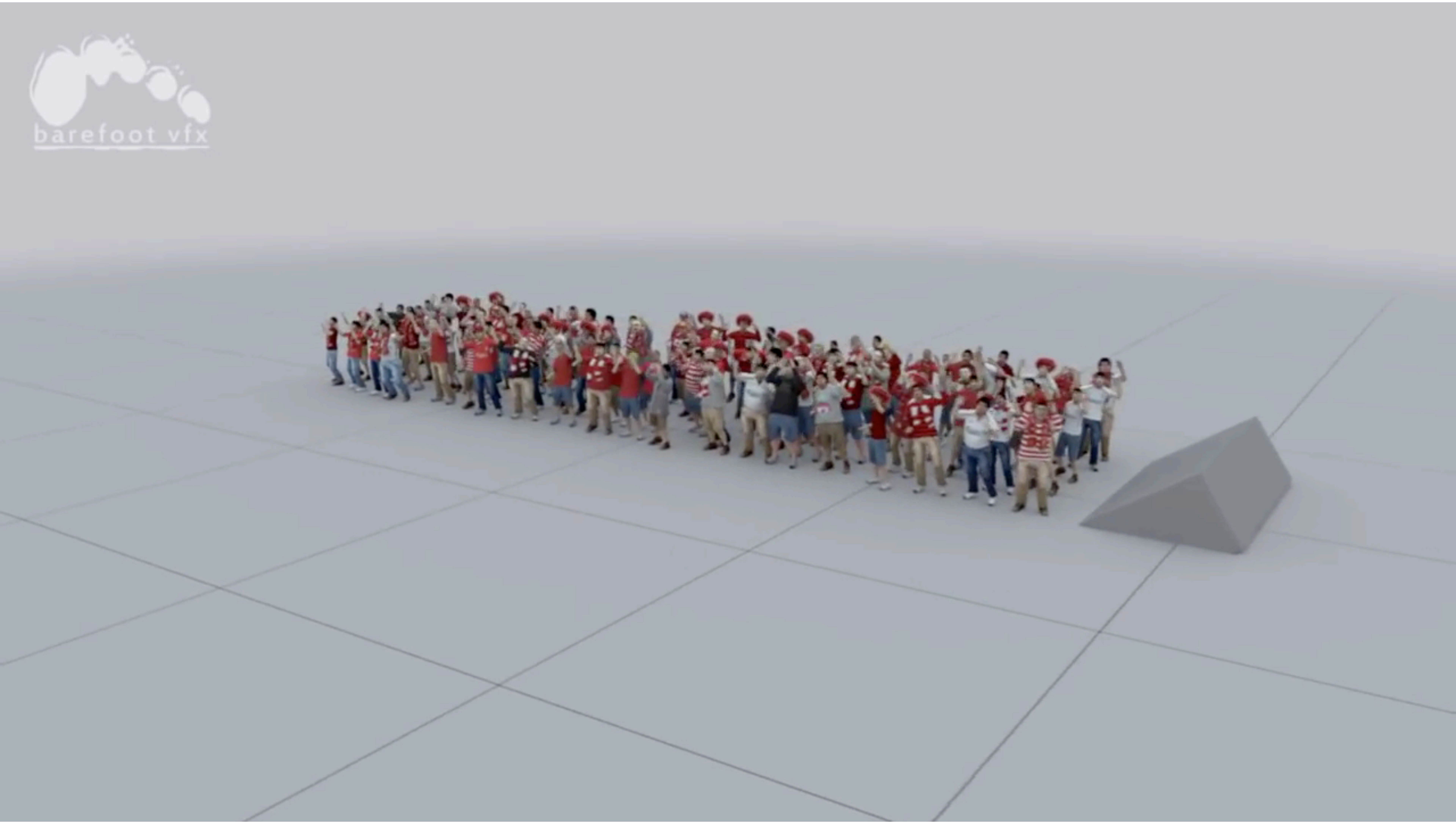
Consider drawing a cube man



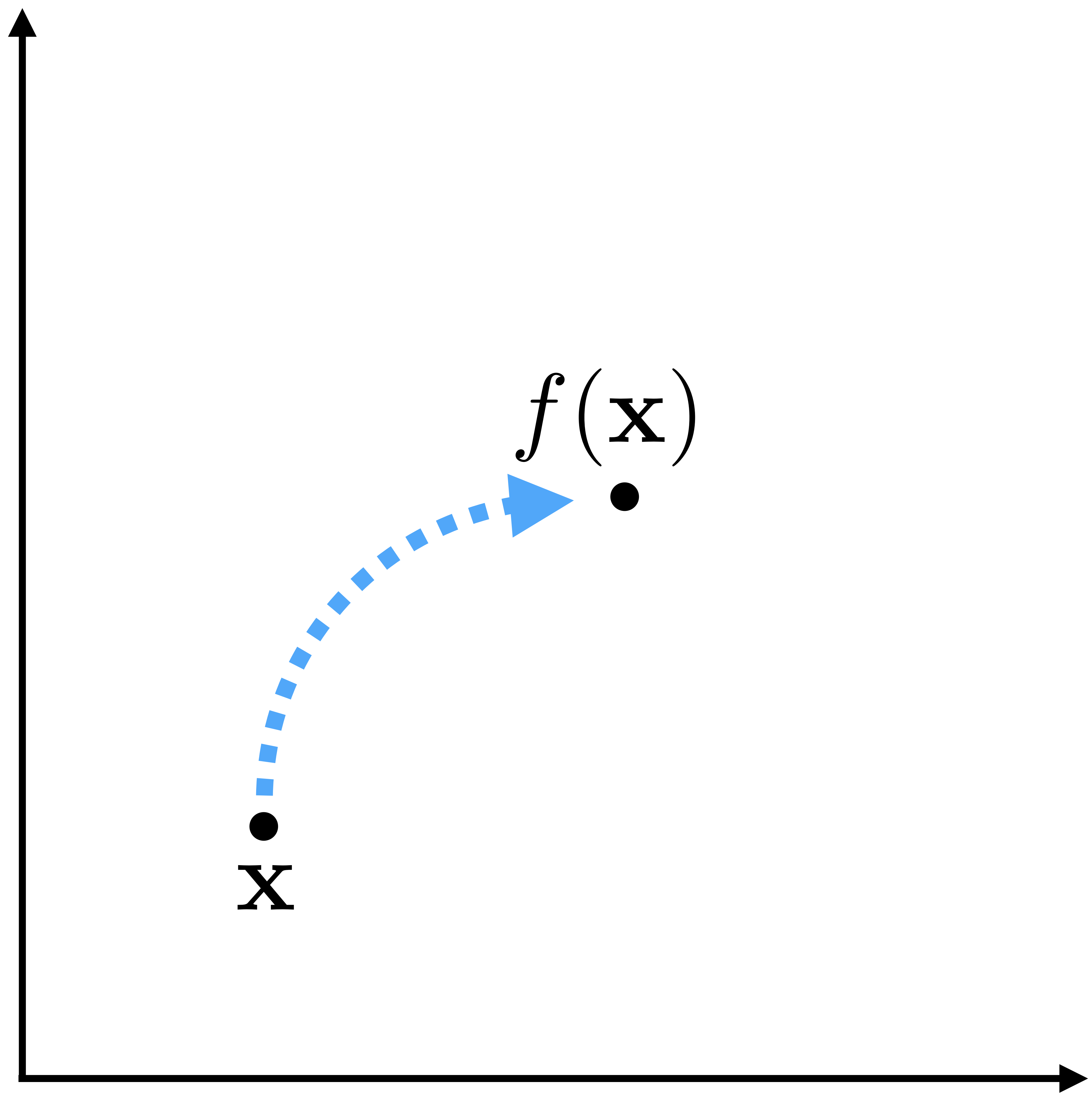
Transformations in character rigging



Transformations in instancing



Basic idea: f transforms $\mathbf{x}$ to $f(\mathbf{x})$



What can we do with *linear* transformations?

- What does *linear* mean?

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

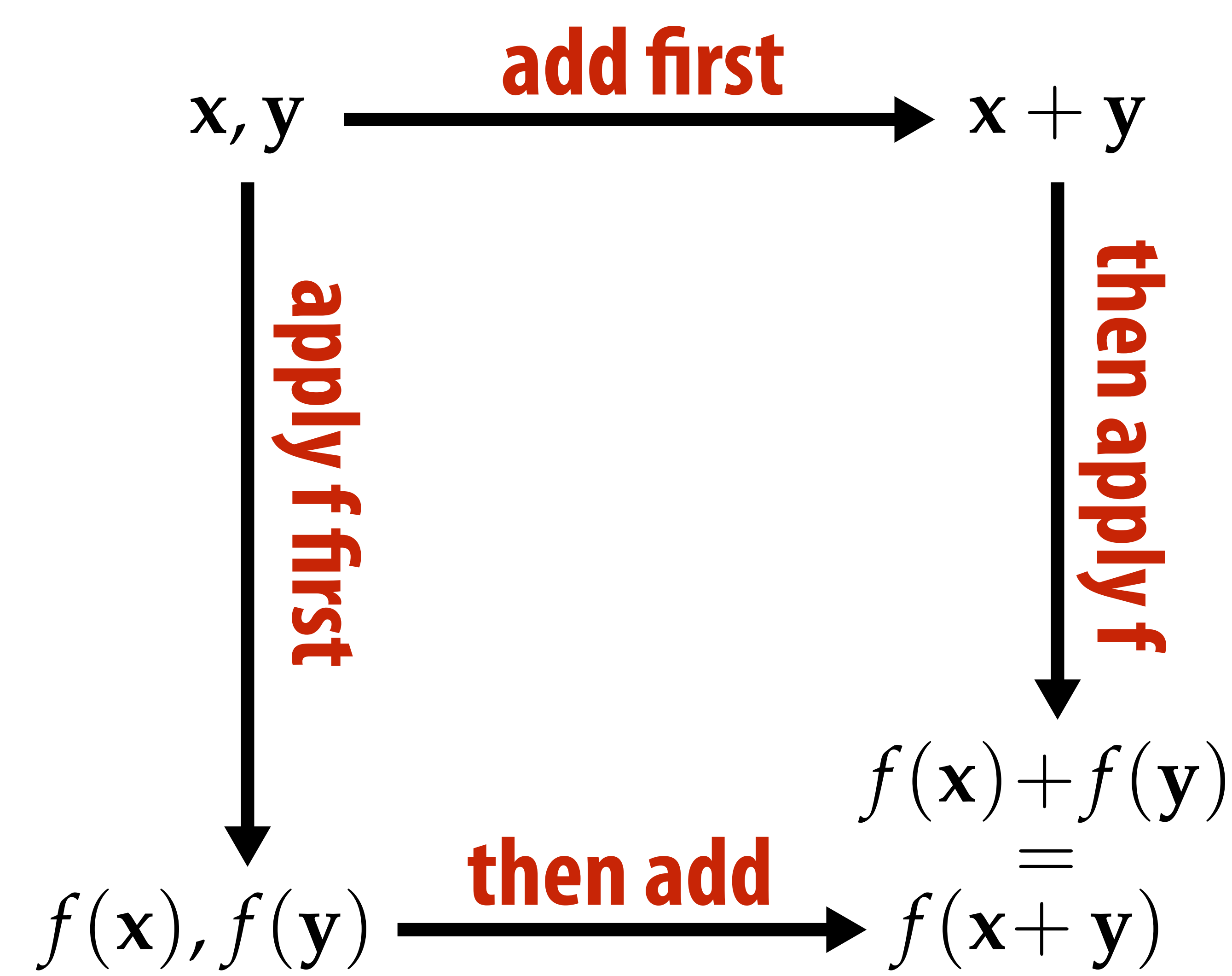
$$f(a\mathbf{x}) = af(\mathbf{x})$$

- Cheap to compute
- Composition of linear transformations is linear
  - Leads to uniform representation of transformations

Linear transformation

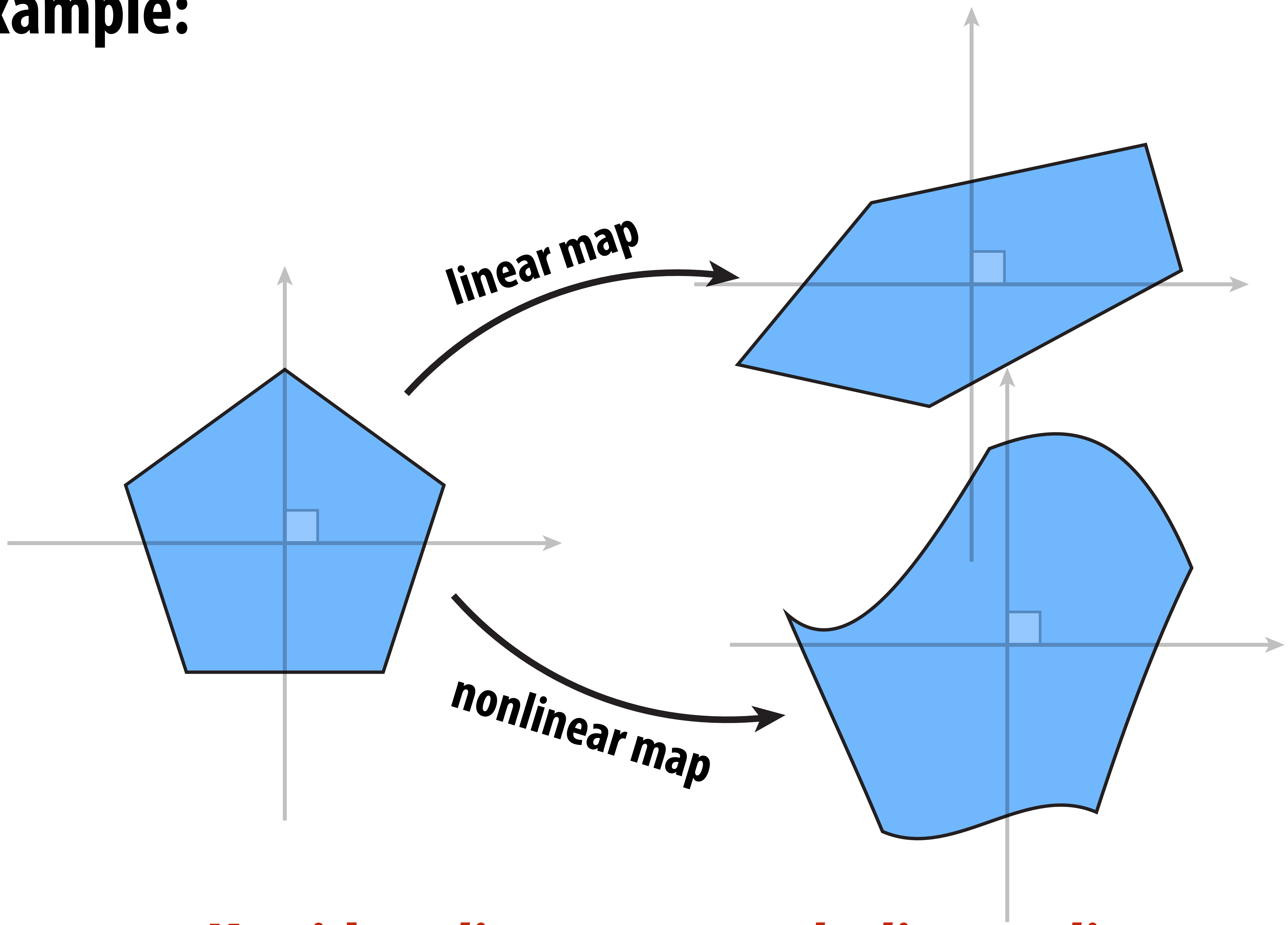
$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$
$$f(a\mathbf{u}) = af(\mathbf{u})$$

- **In other words: if it doesn't matter whether we add the vectors and then apply the map, or apply the map and then add the vectors (and likewise for scaling):**



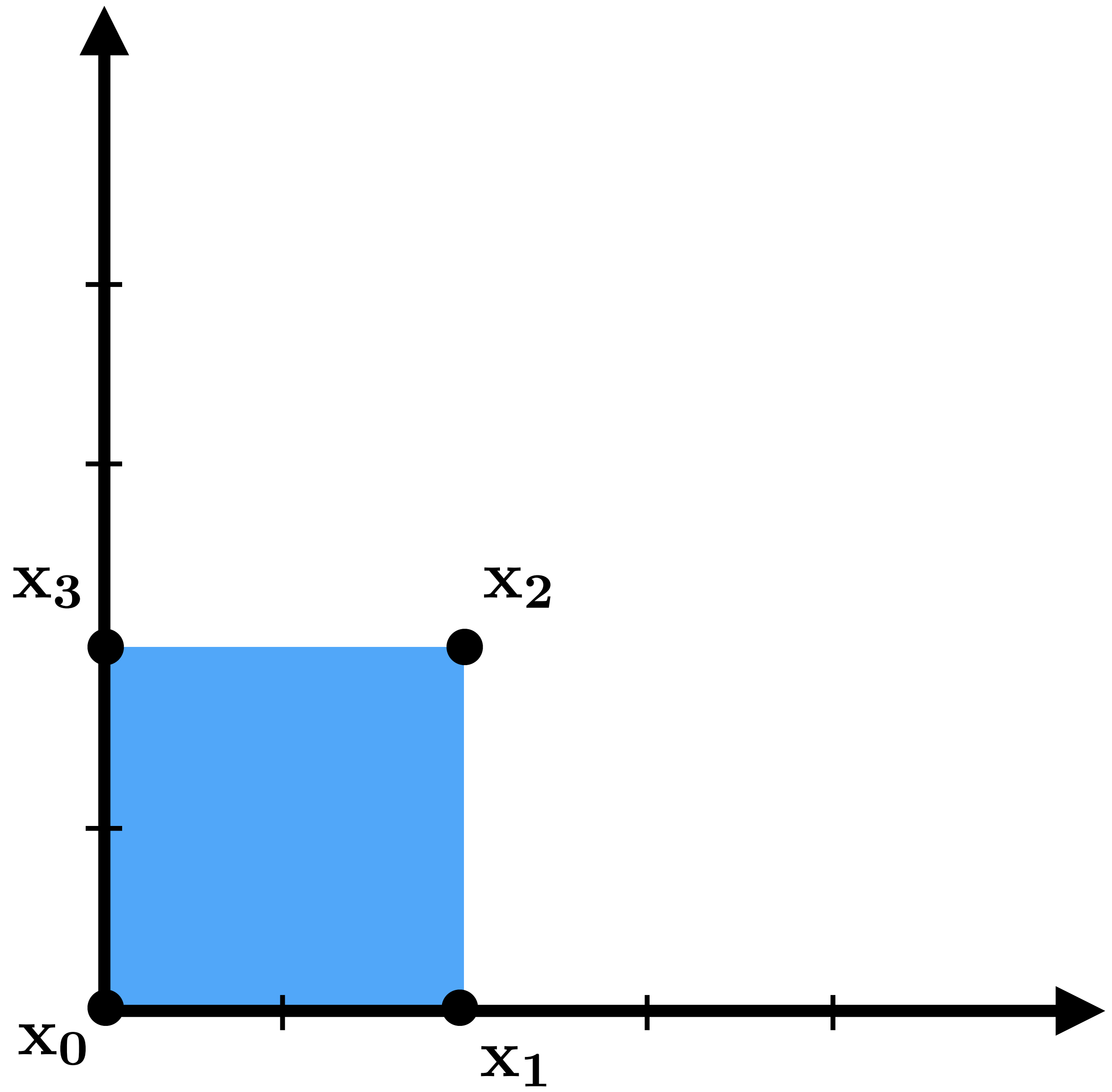
Linear transforms/maps—visualized

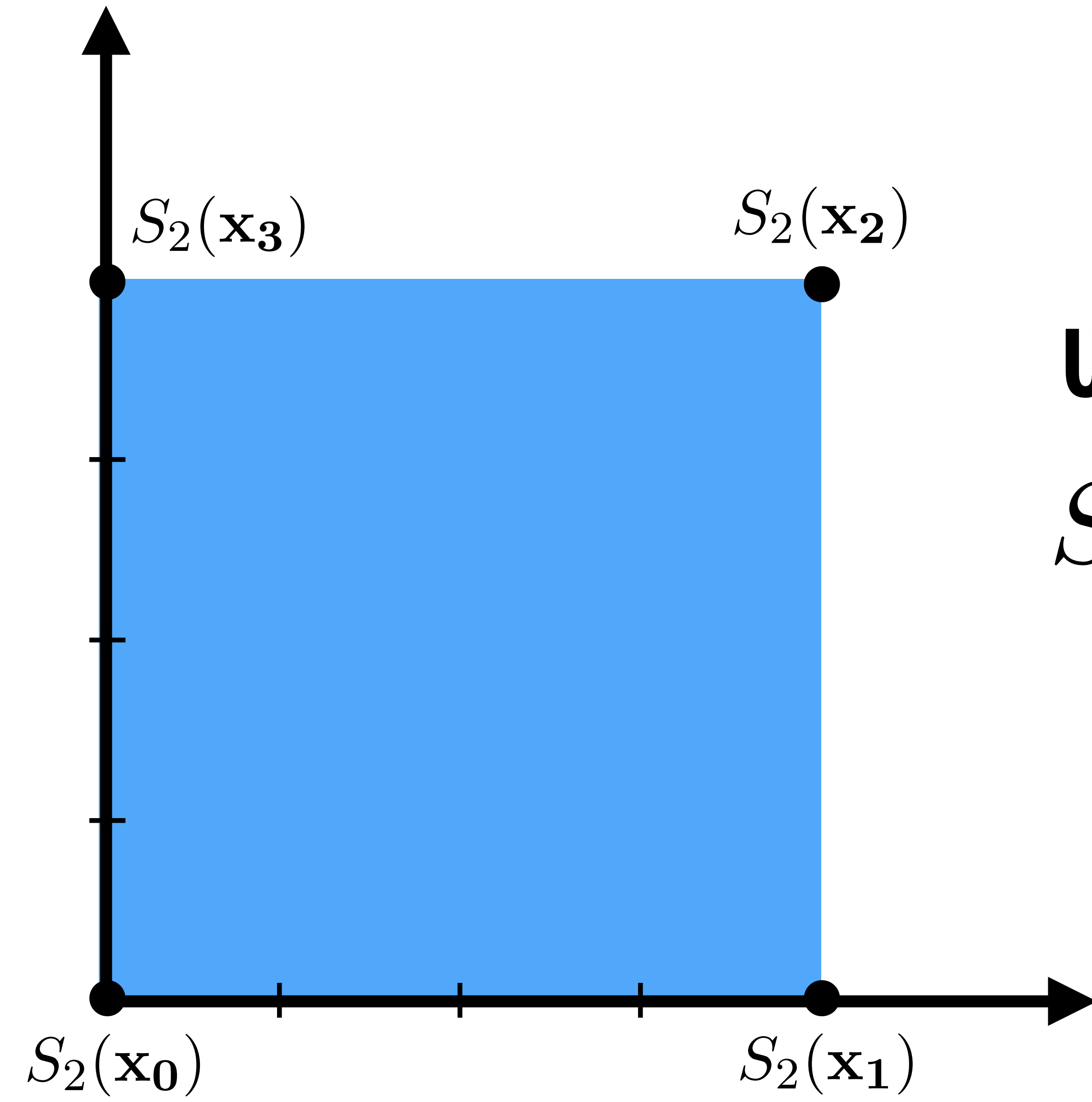
- **Example:**



Key idea: ***linear maps take lines to lines***

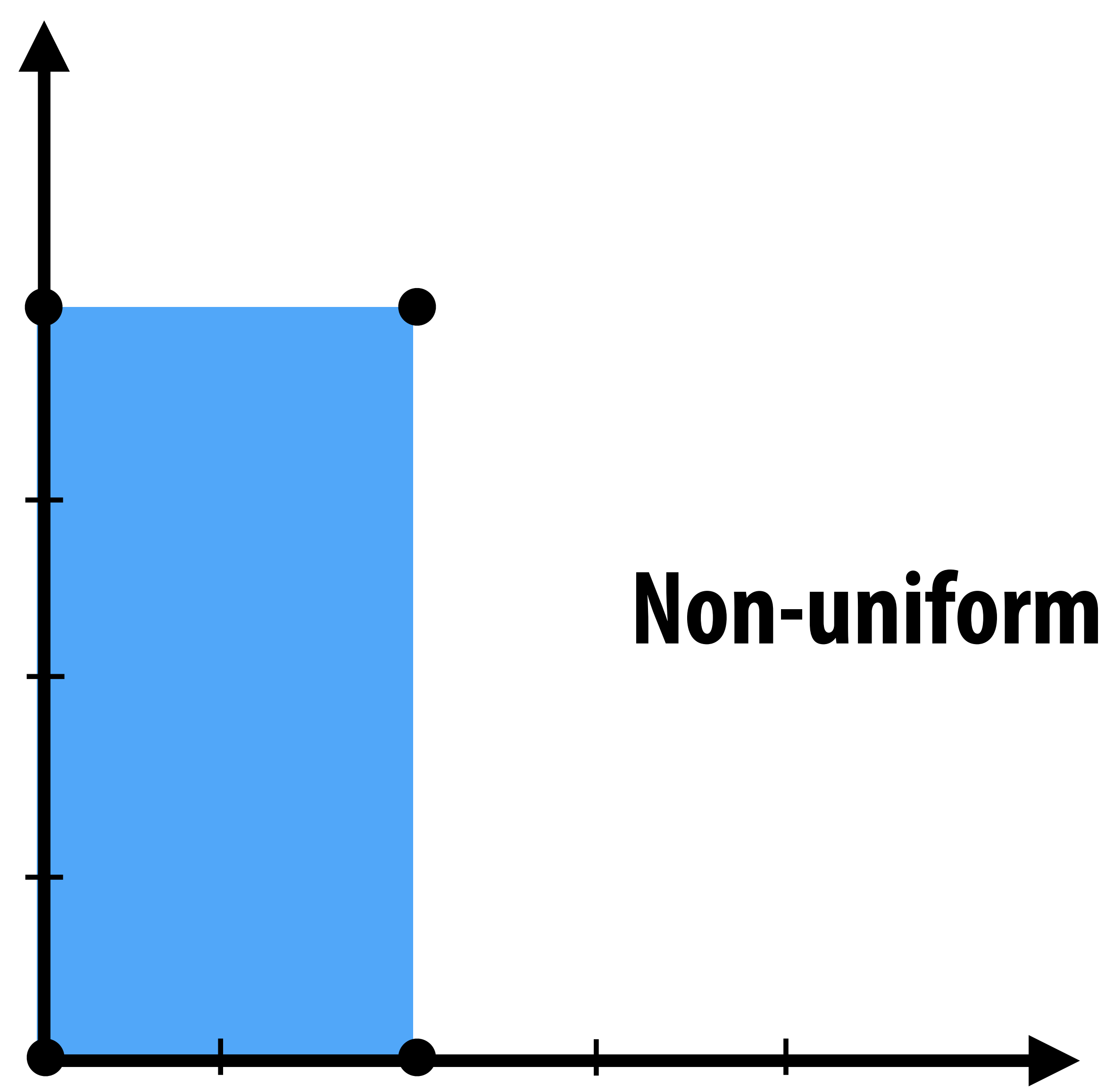
Scale





Uniform scale:

$$S_a(\mathbf{x}) = a\mathbf{x}$$



Non-uniform scale??

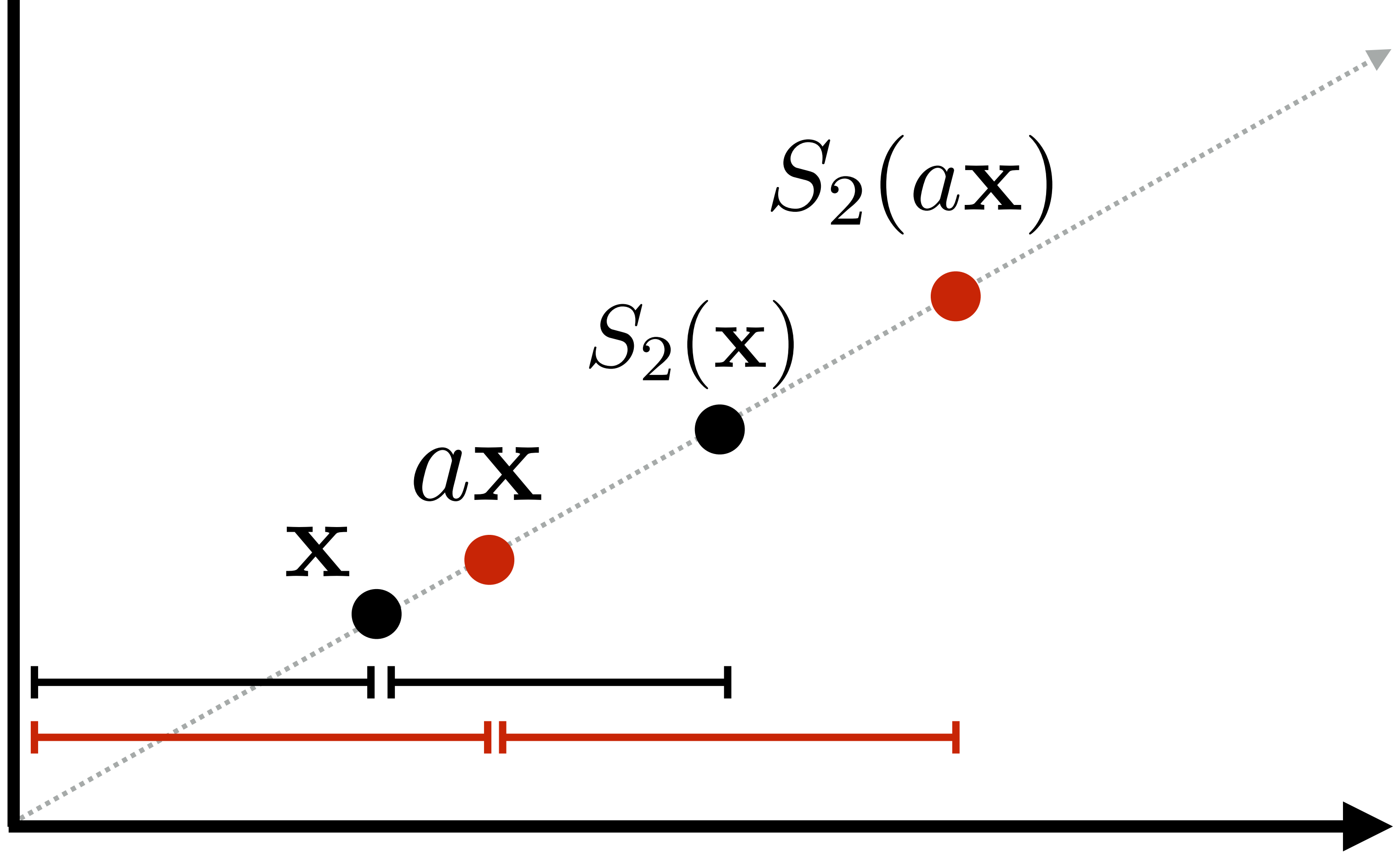
Is scale a linear transform?

$$S_2(\mathbf{x}) = 2\mathbf{x}$$

$$aS_2(\mathbf{x}) = 2a\mathbf{x}$$

$$S_2(a\mathbf{x}) = 2a\mathbf{x}$$

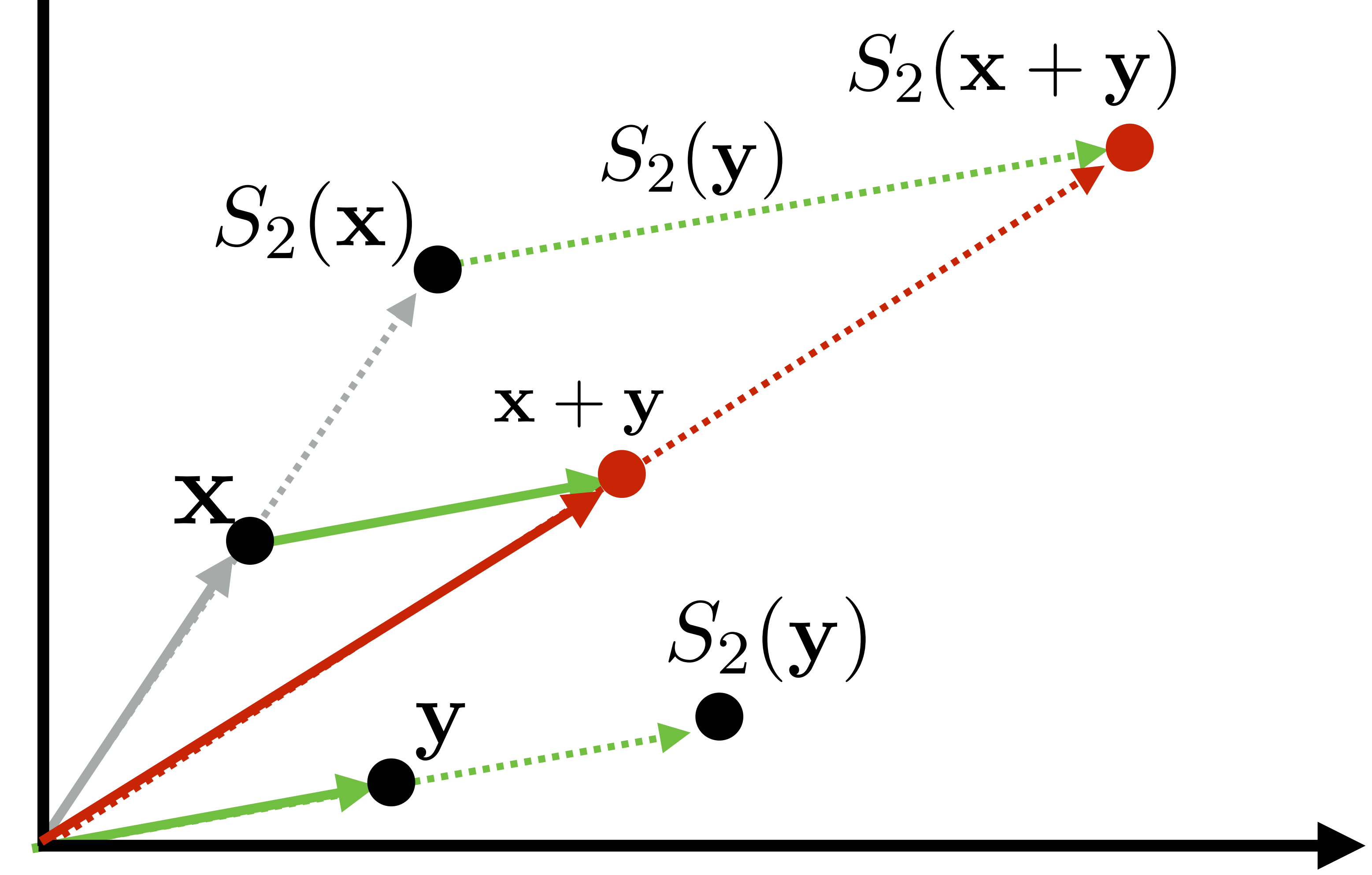
$$S_2(a\mathbf{x}) = aS_2(\mathbf{x})$$



$$S_2(\mathbf{x}+\mathbf{y}) = 2(\mathbf{x}+\mathbf{y})$$

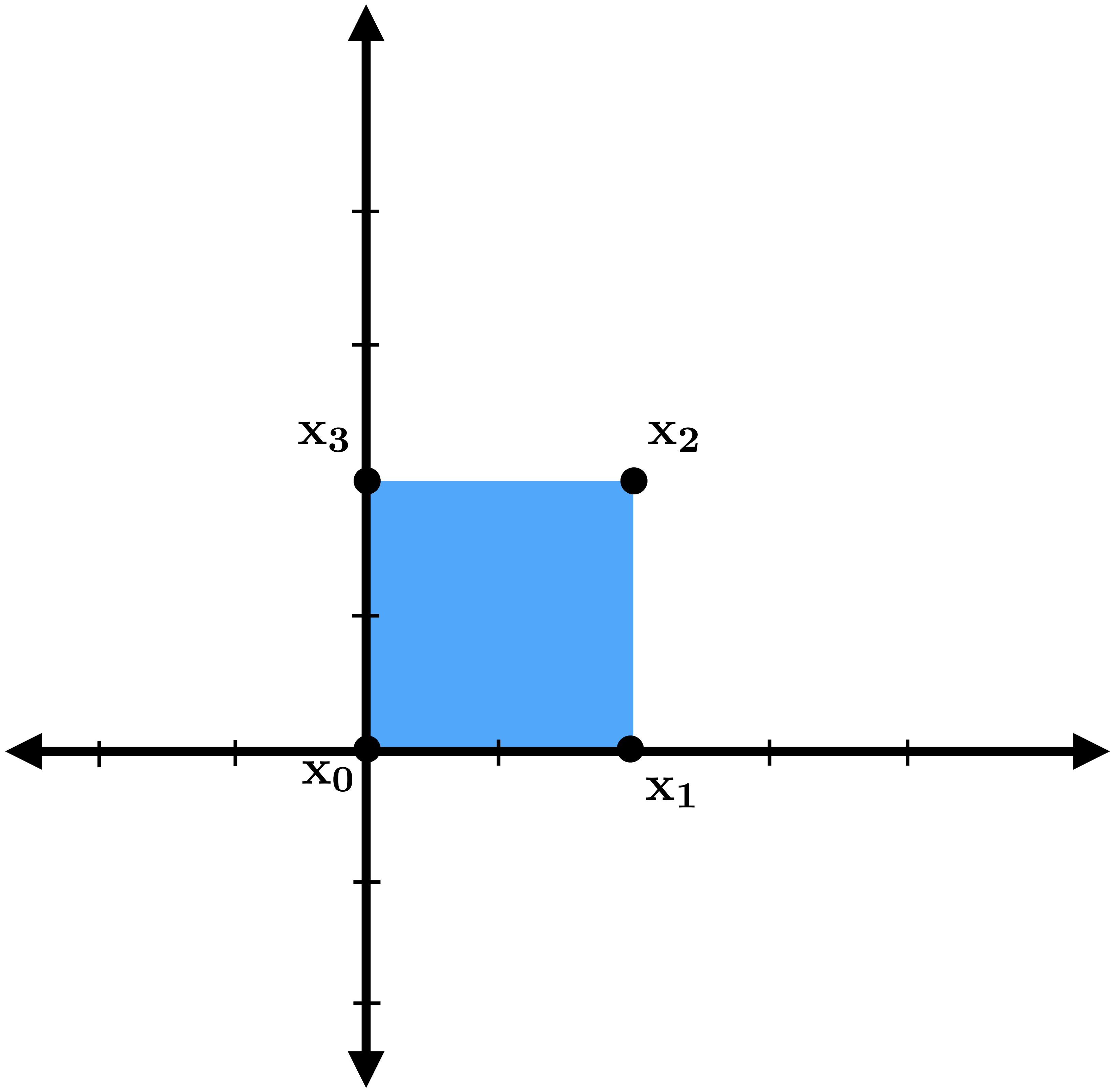
$$S_2(\mathbf{x}) + S_2(\mathbf{y}) = 2\mathbf{x} + 2\mathbf{y}$$

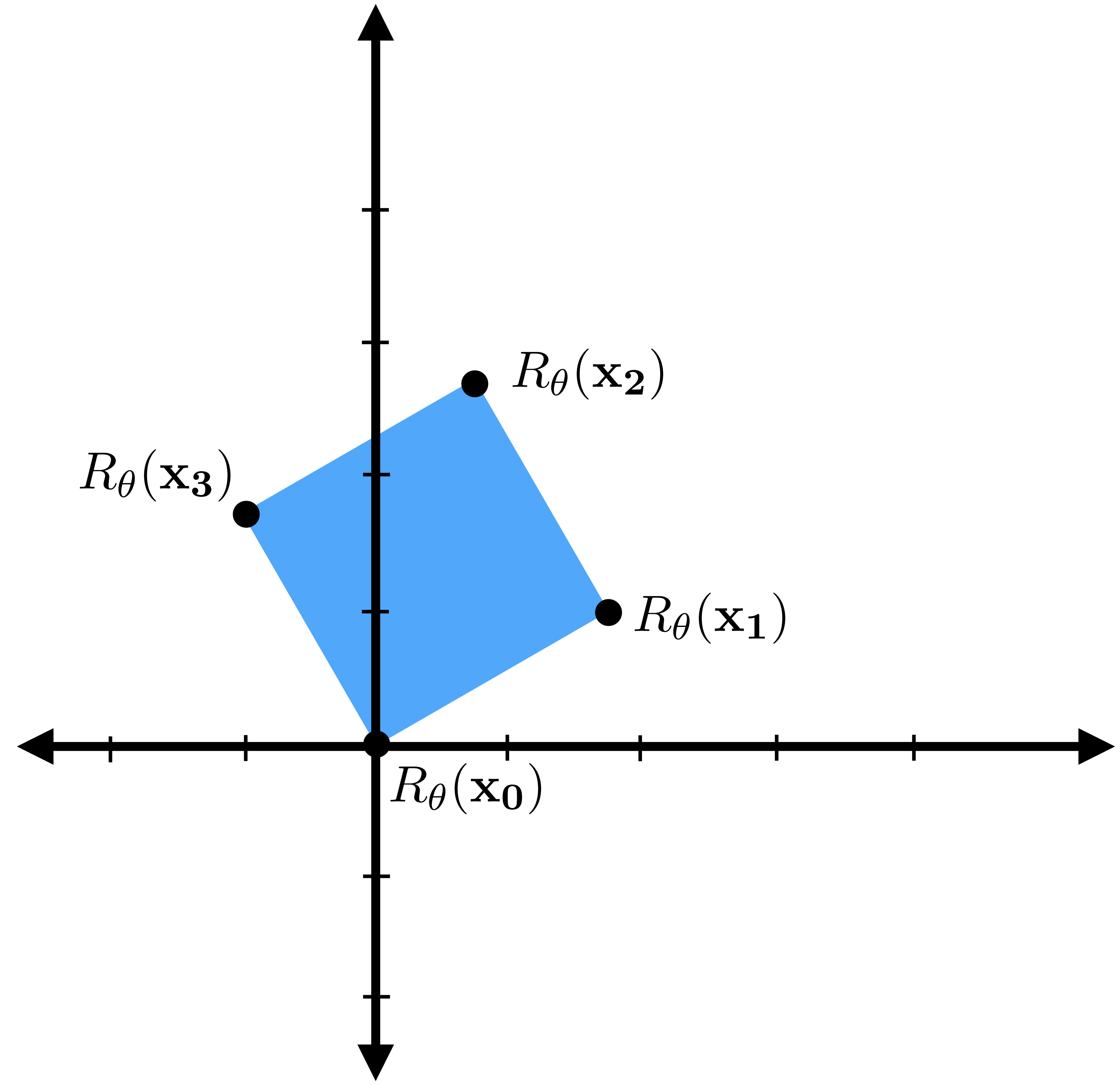
$$S_2(\mathbf{x}+\mathbf{y}) = S_2(\mathbf{x}) + S_2(\mathbf{y})$$



Yes!

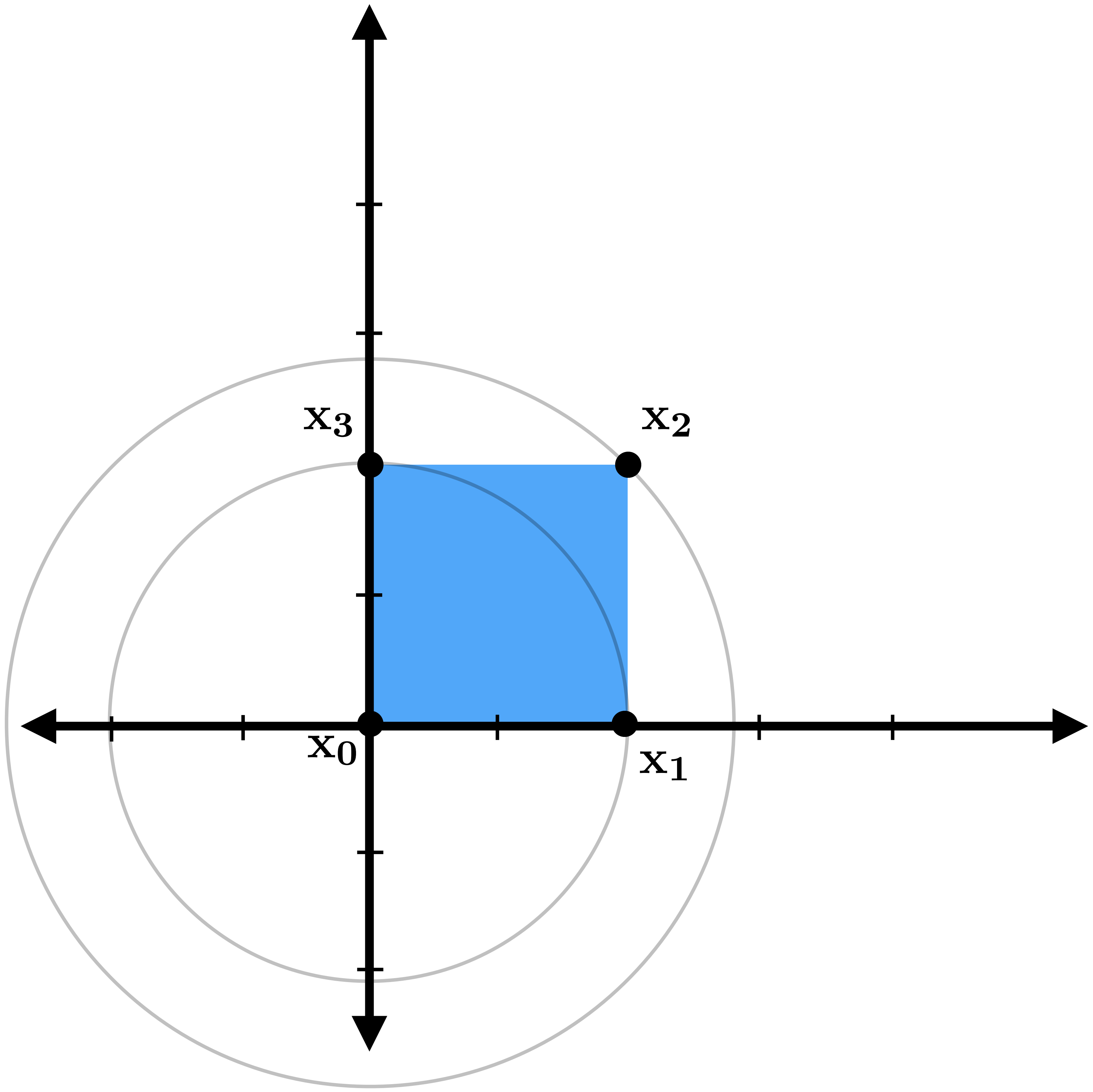
Rotation

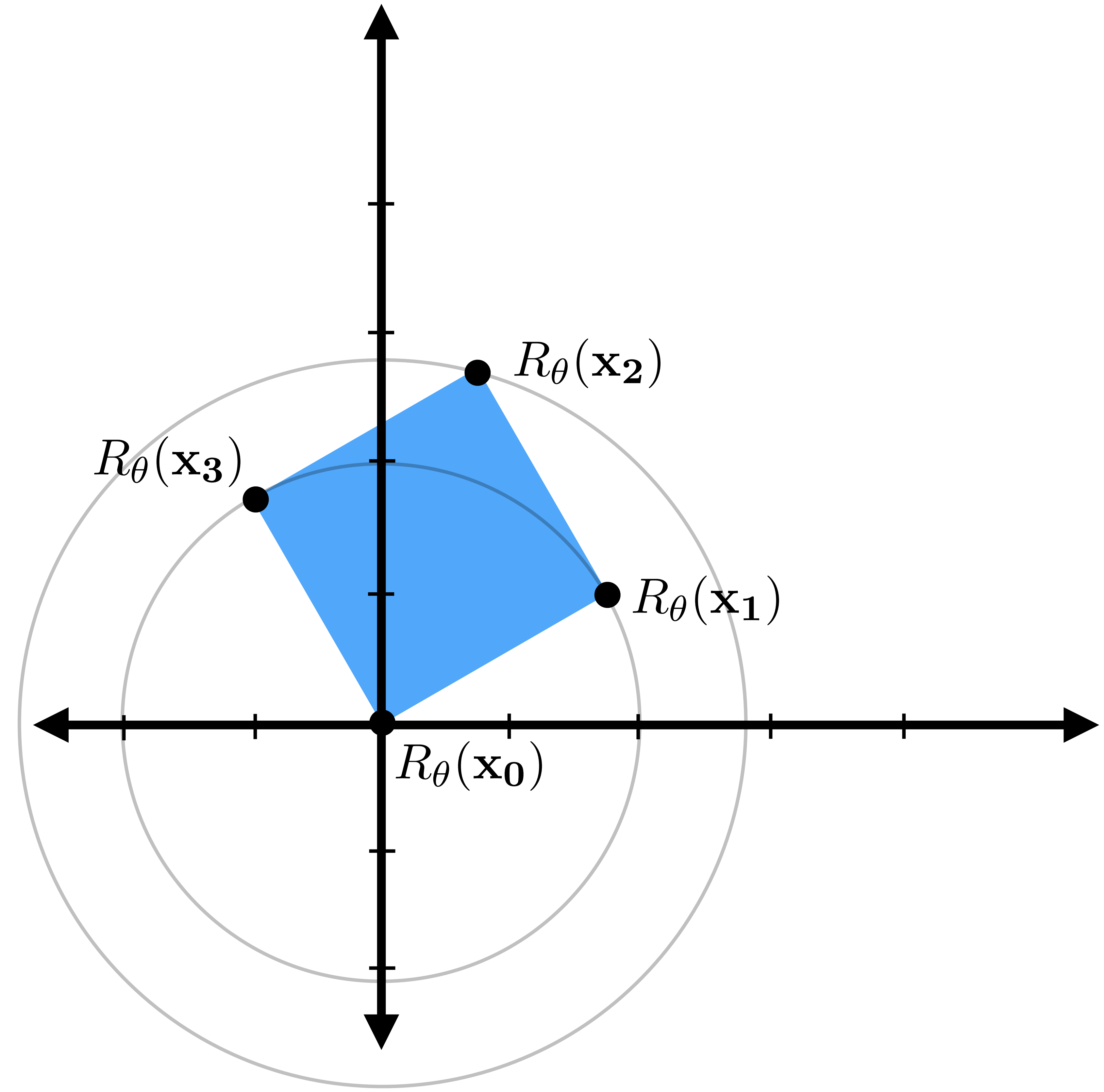




R_θ = **rotate counter-clockwise by** θ

Rotation as circular motion



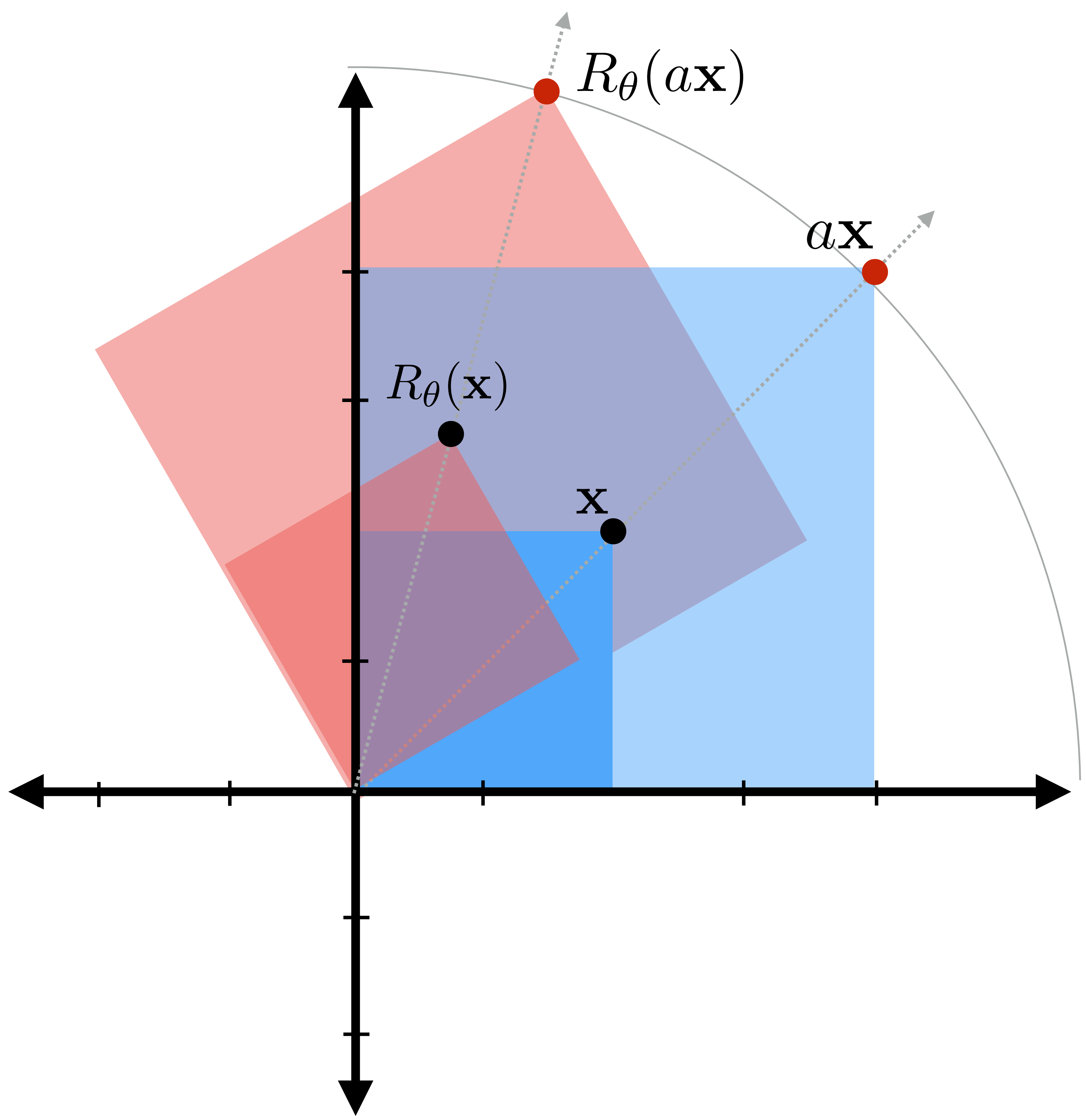


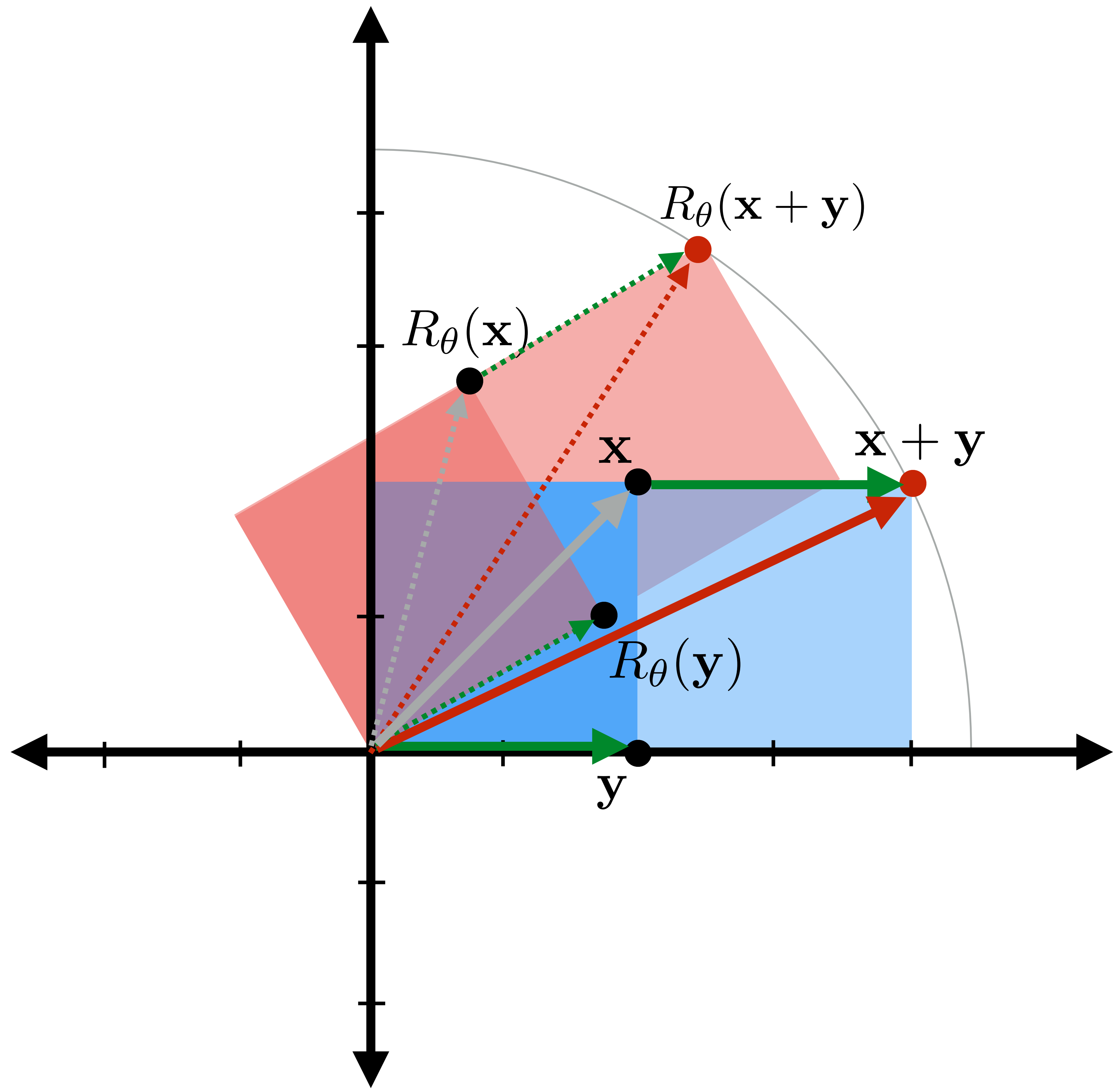
R_θ **= rotate counter-clockwise by** θ

As angle changes, points move along *circular* trajectories.

Hence, rotations preserve length of vectors: $|R_\theta(\mathbf{x})| = |\mathbf{x}|$

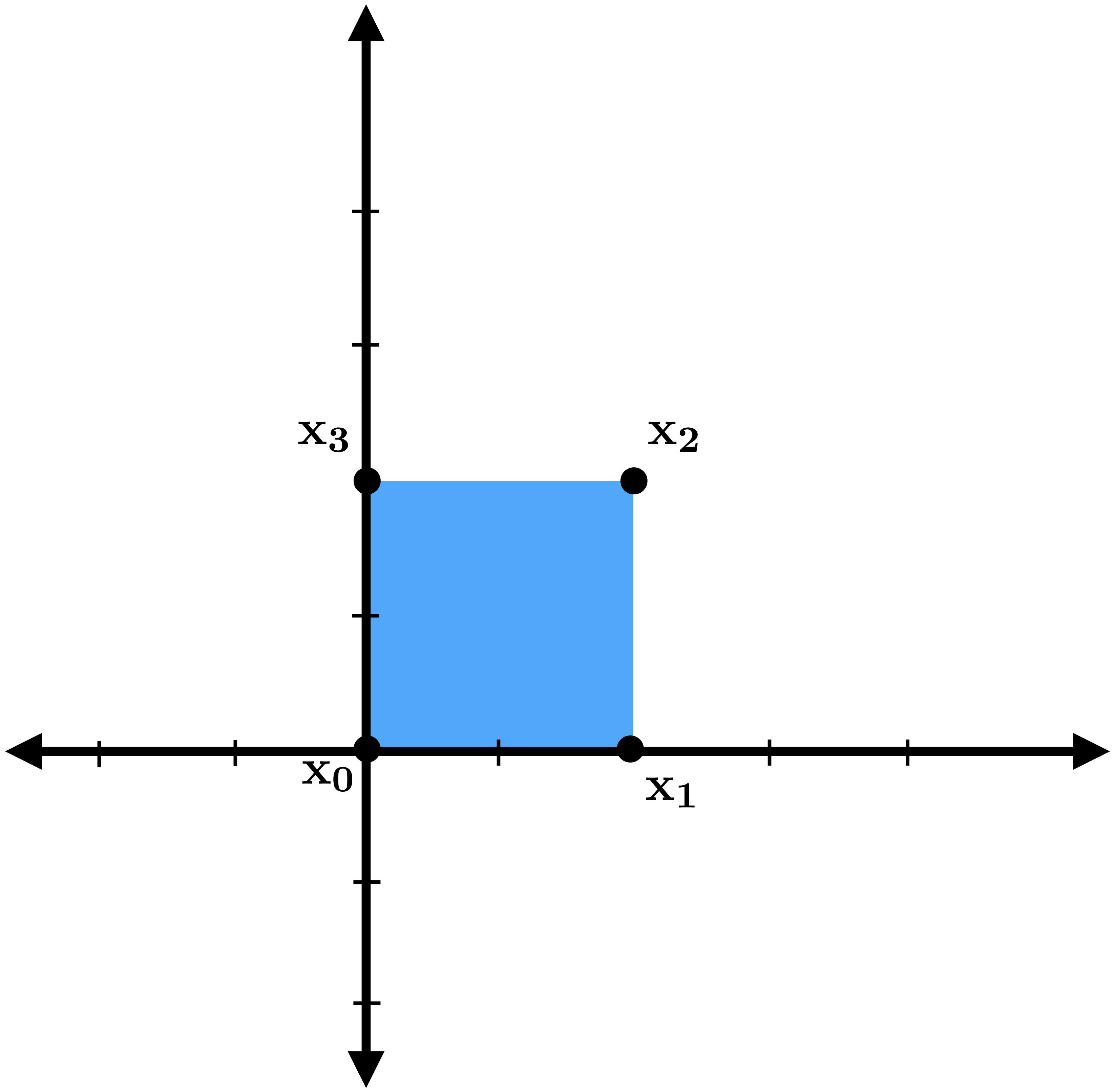
Is rotation linear?

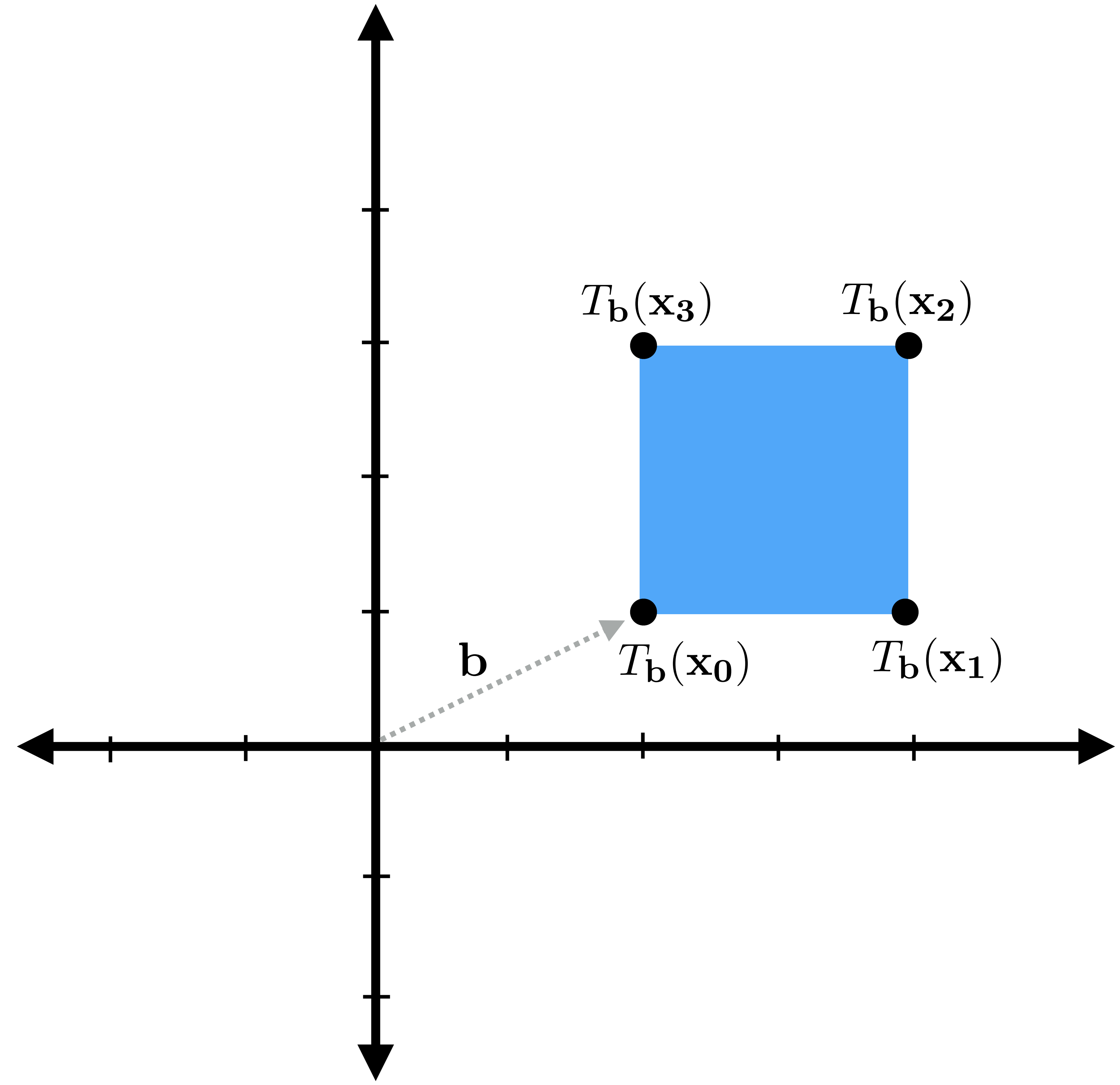




Yes!

Translation

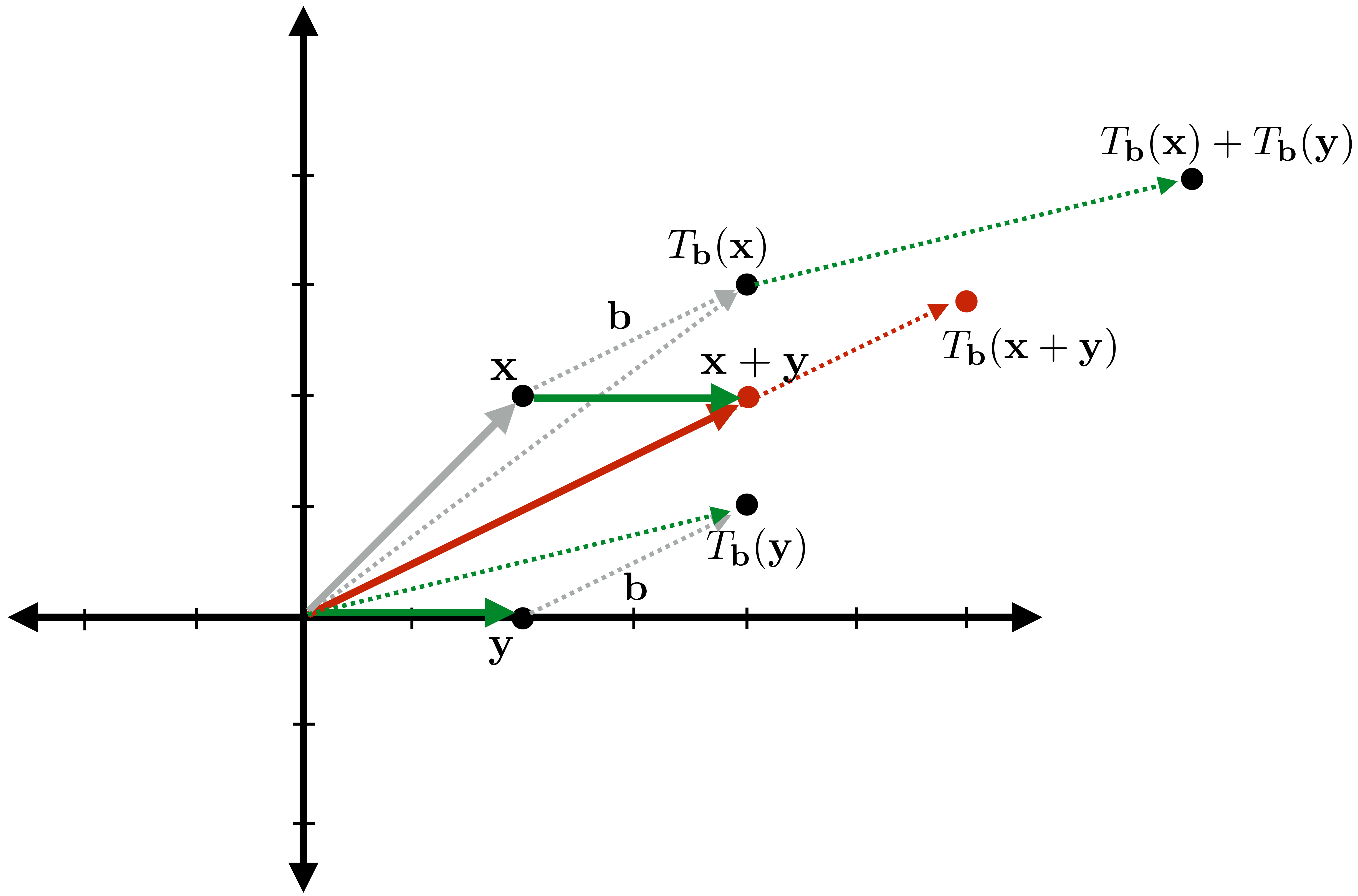




$T_{\mathbf{b}}$ — **"translate by b"**

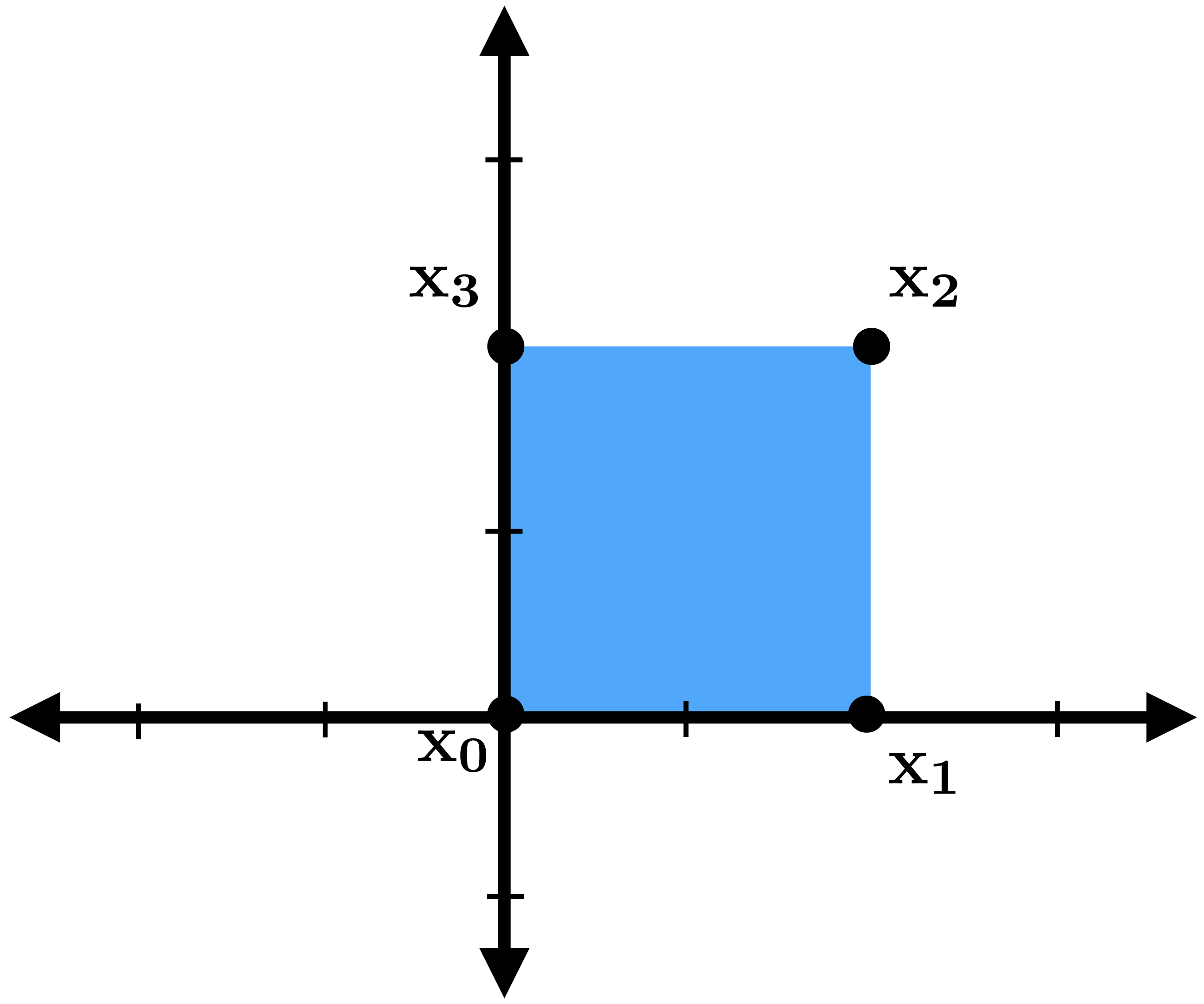
$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$

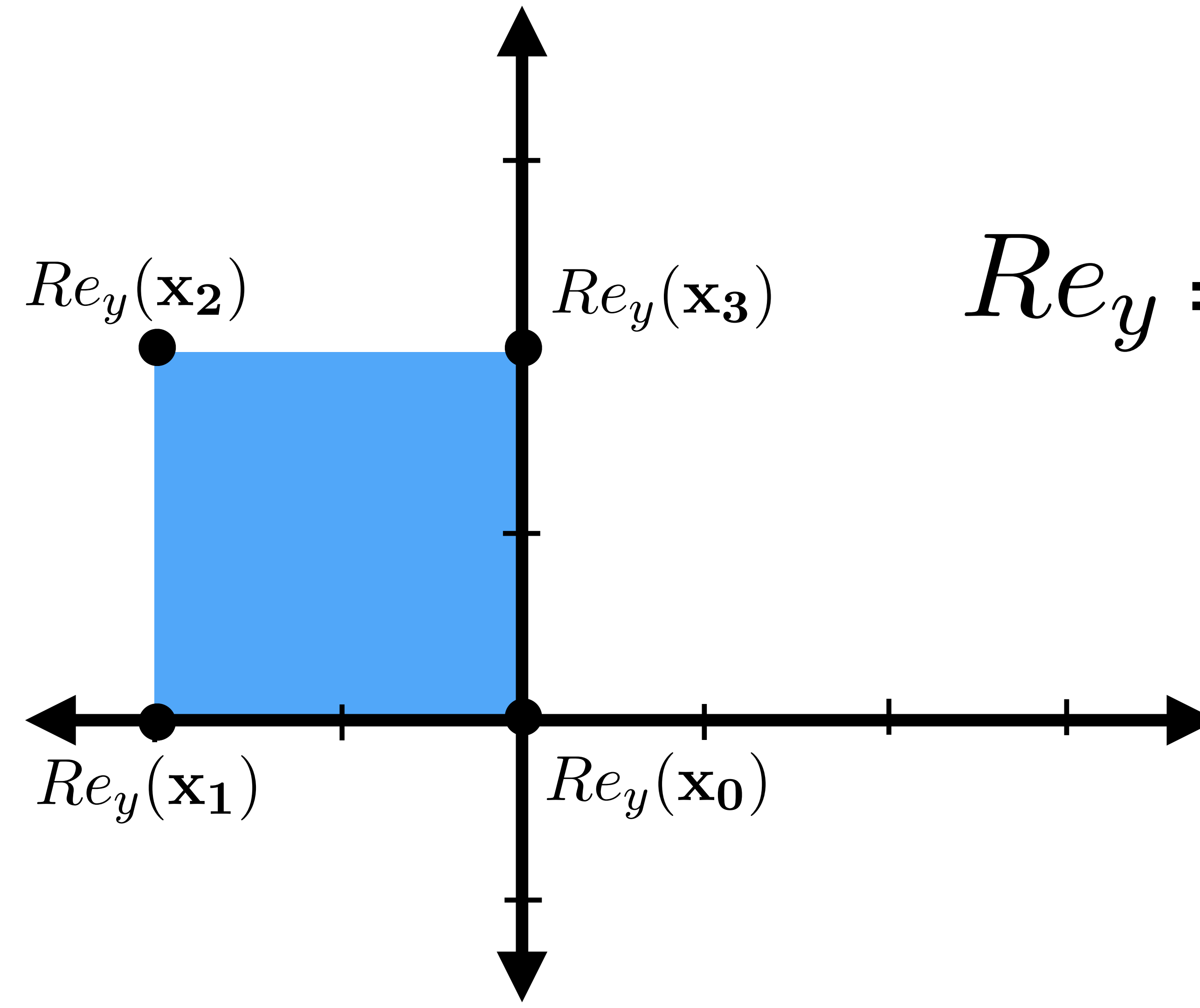
Is translation linear?



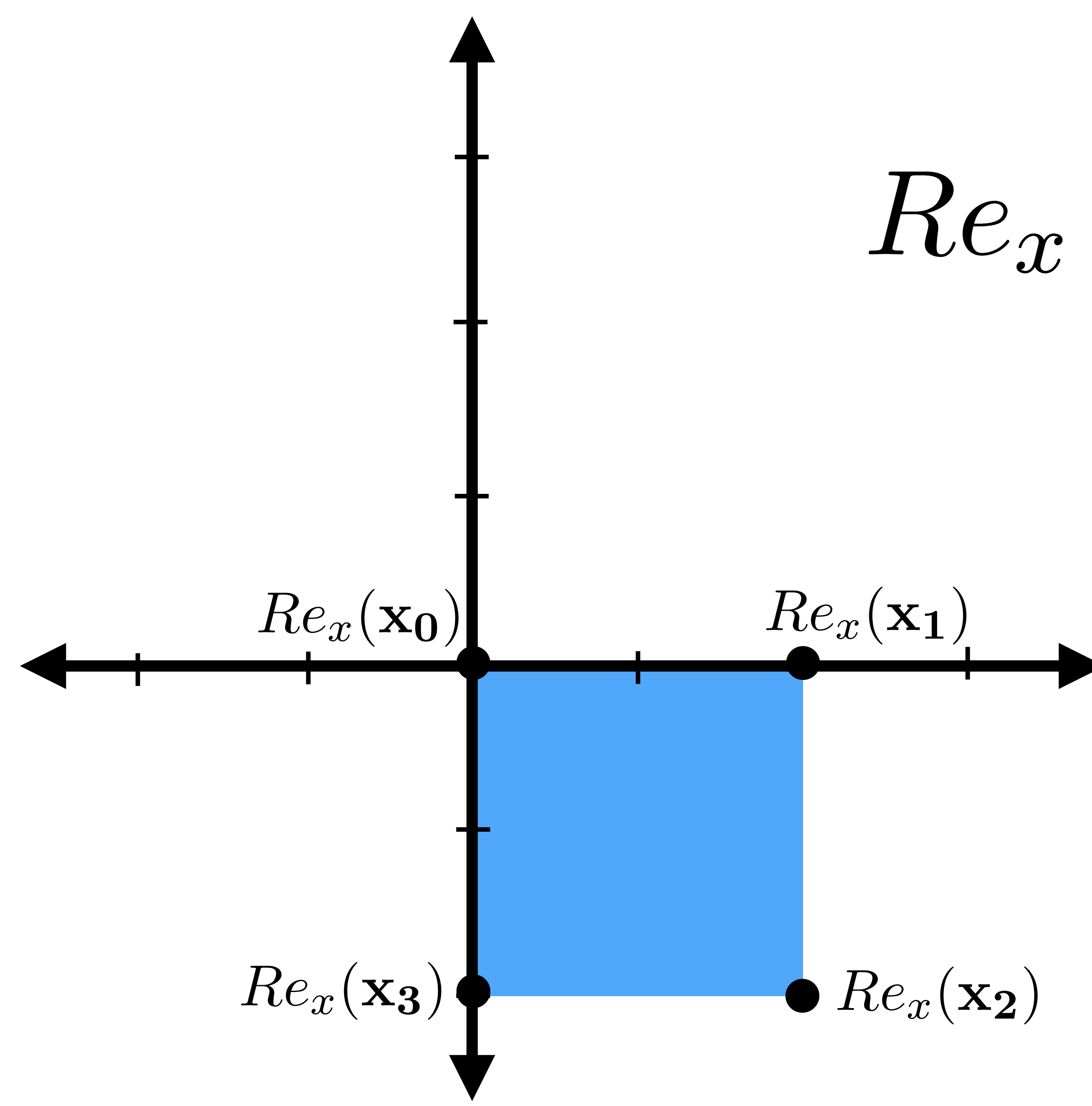
No. Translation is affine.

Reflection



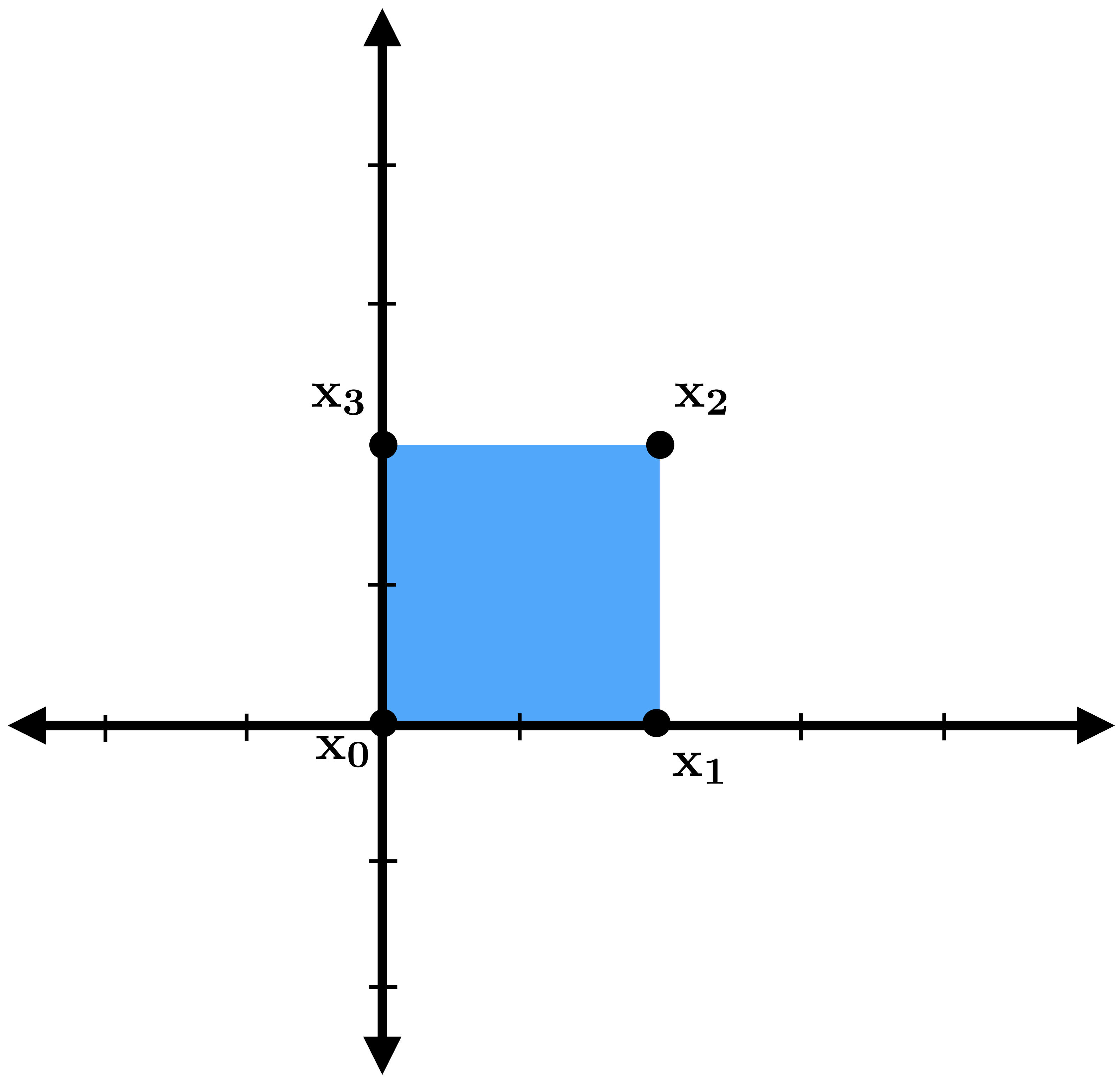


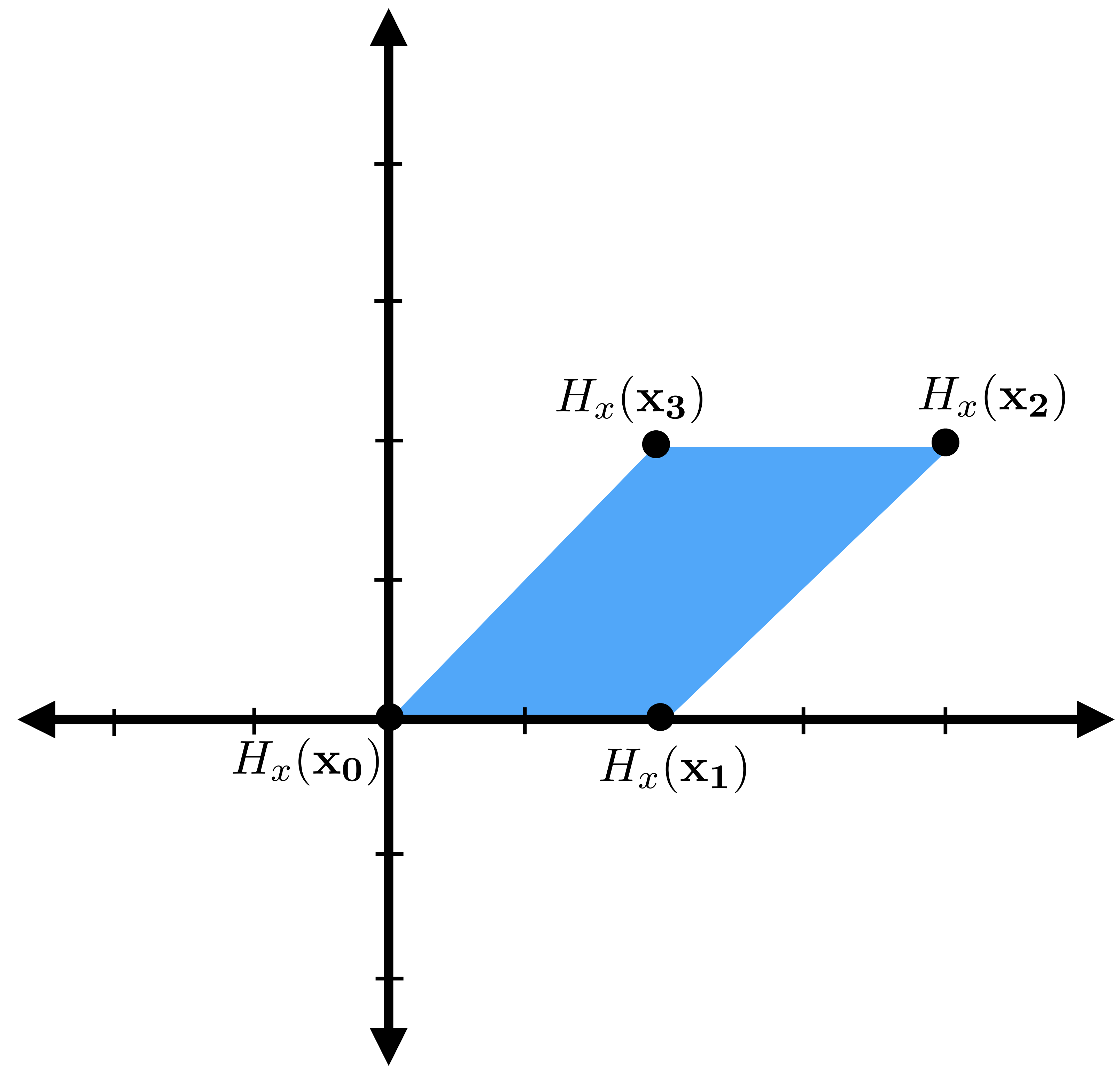
Re_y = **reflection about *y***



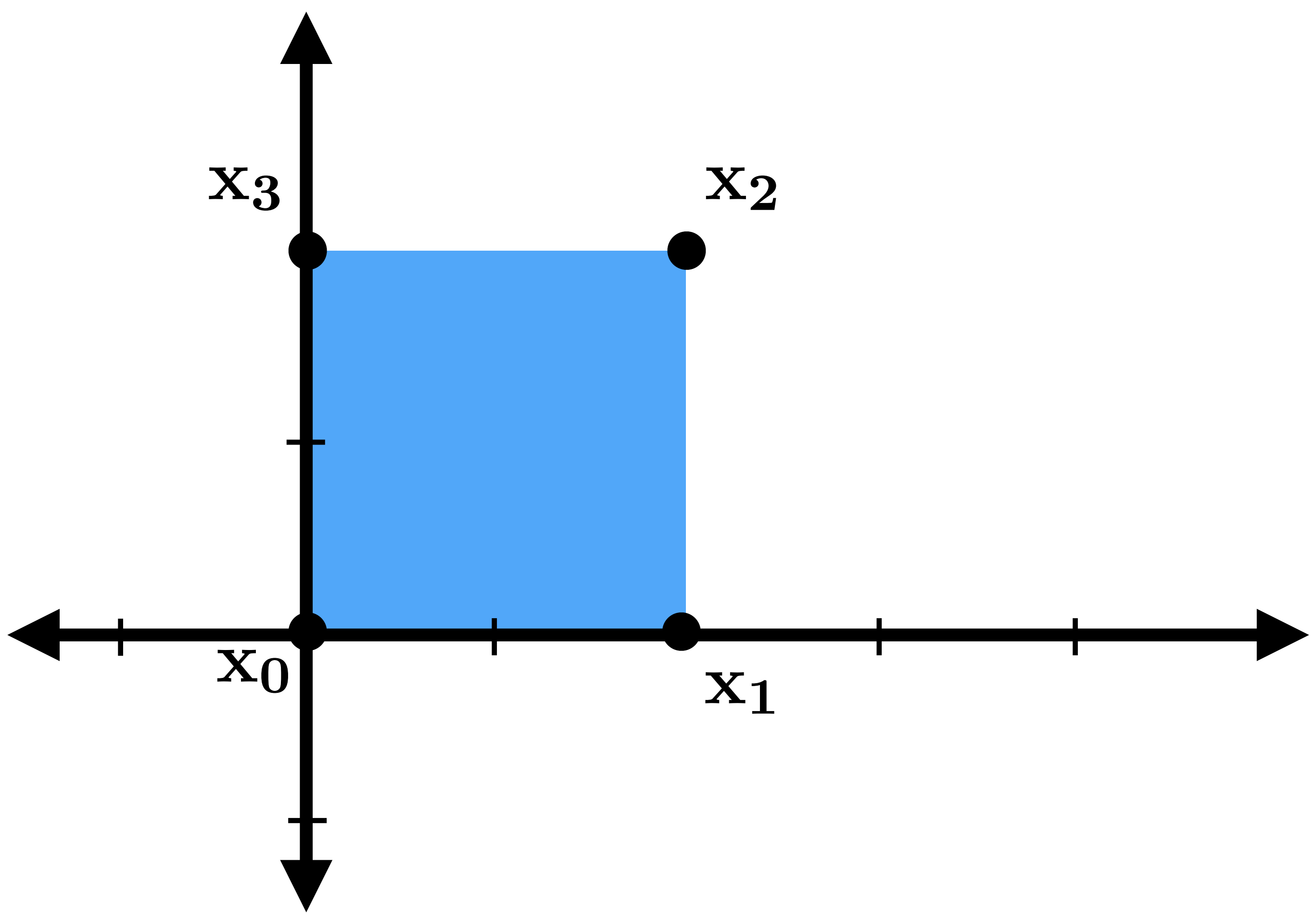
Re_x = **reflection about *x***

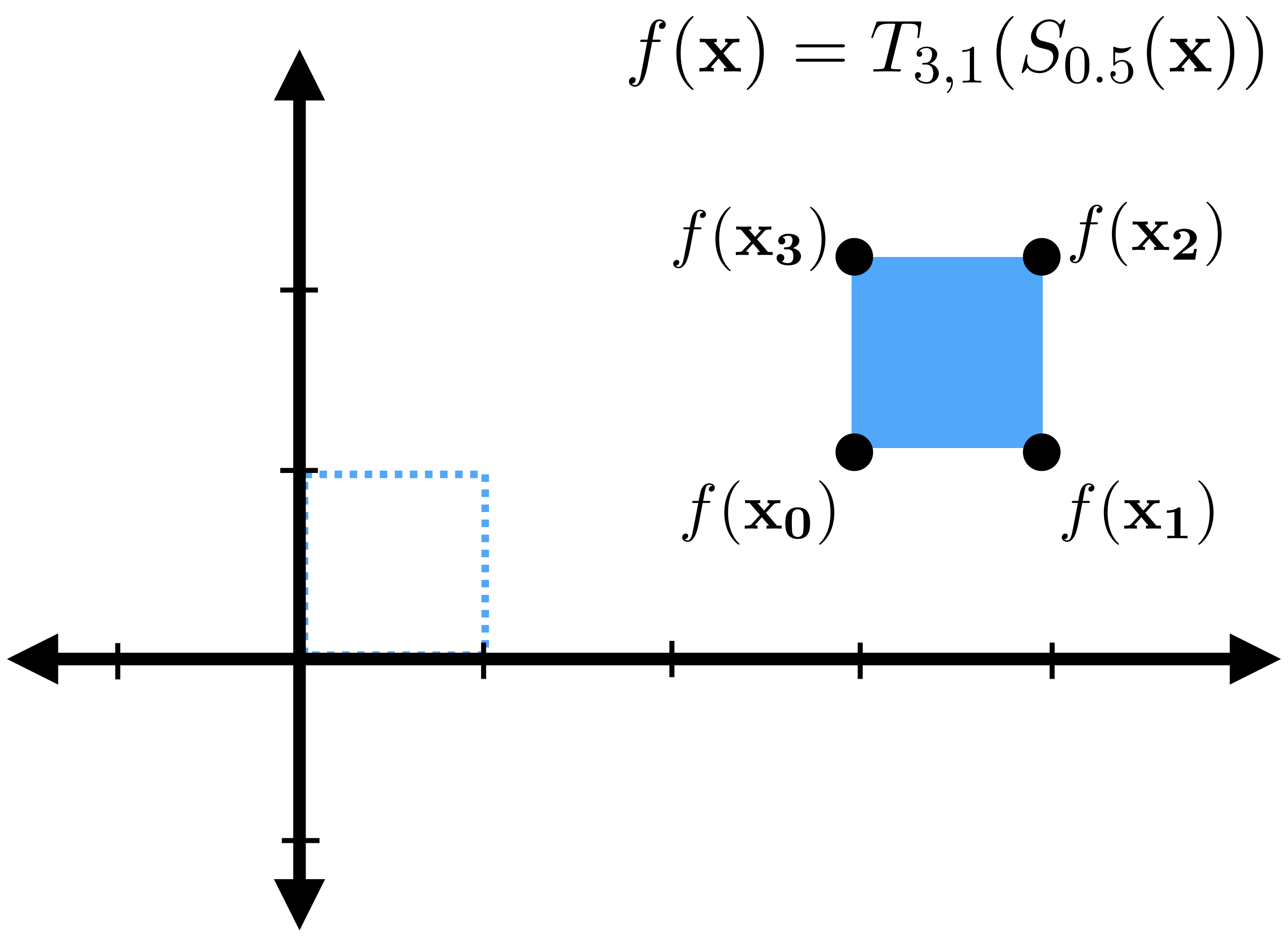
Shear (in x direction)

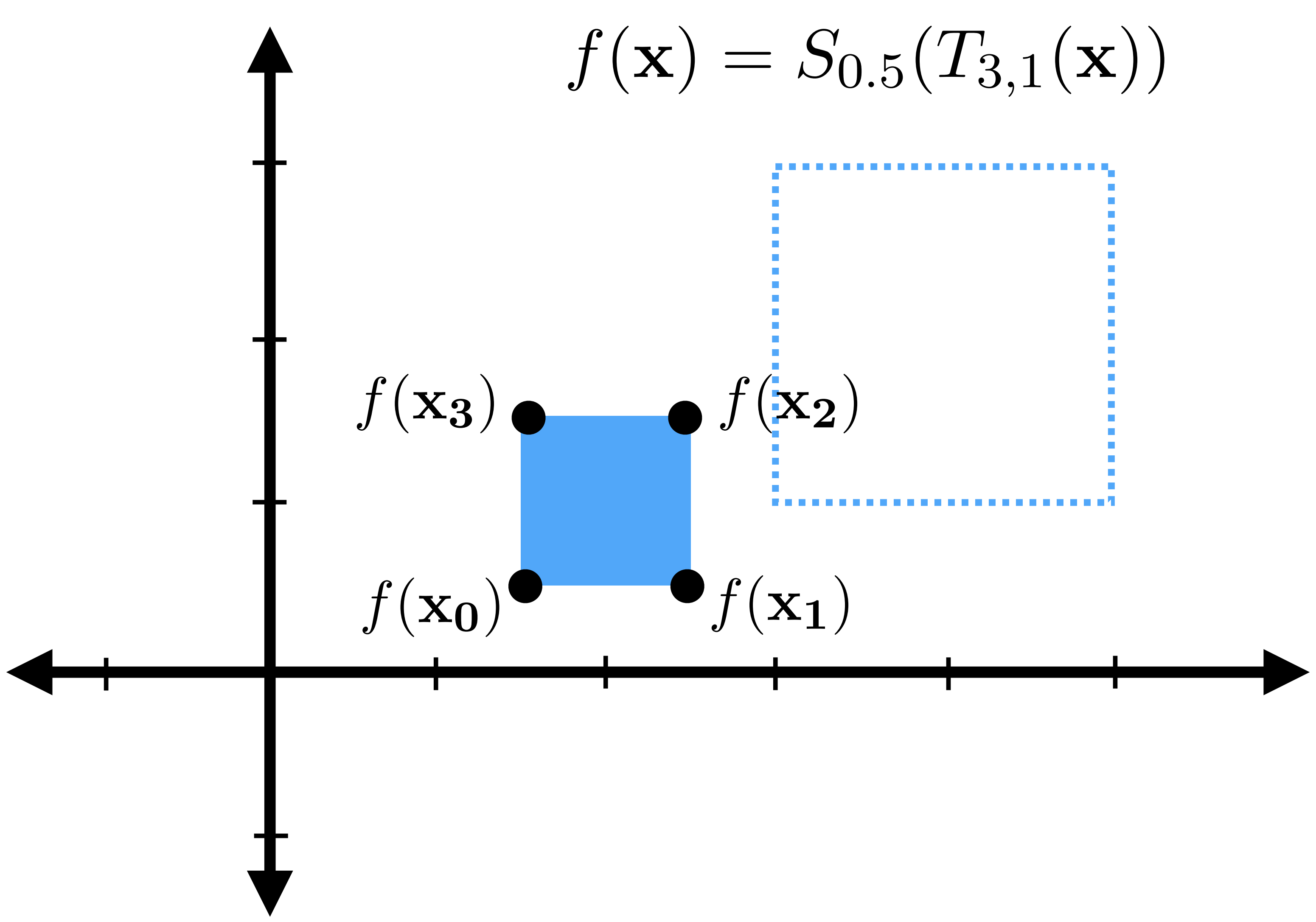




Compose basic transformations to construct more complicated ones



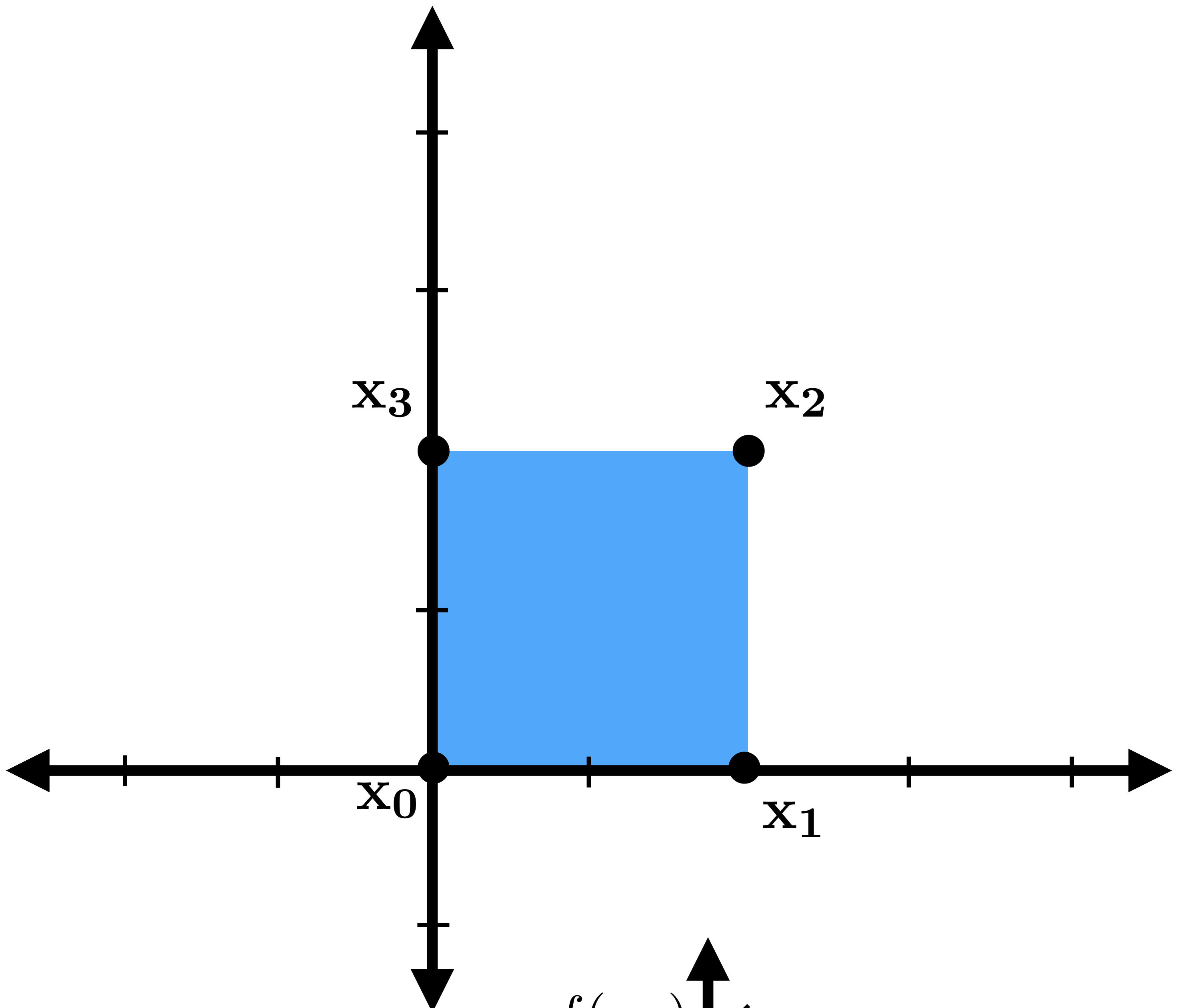


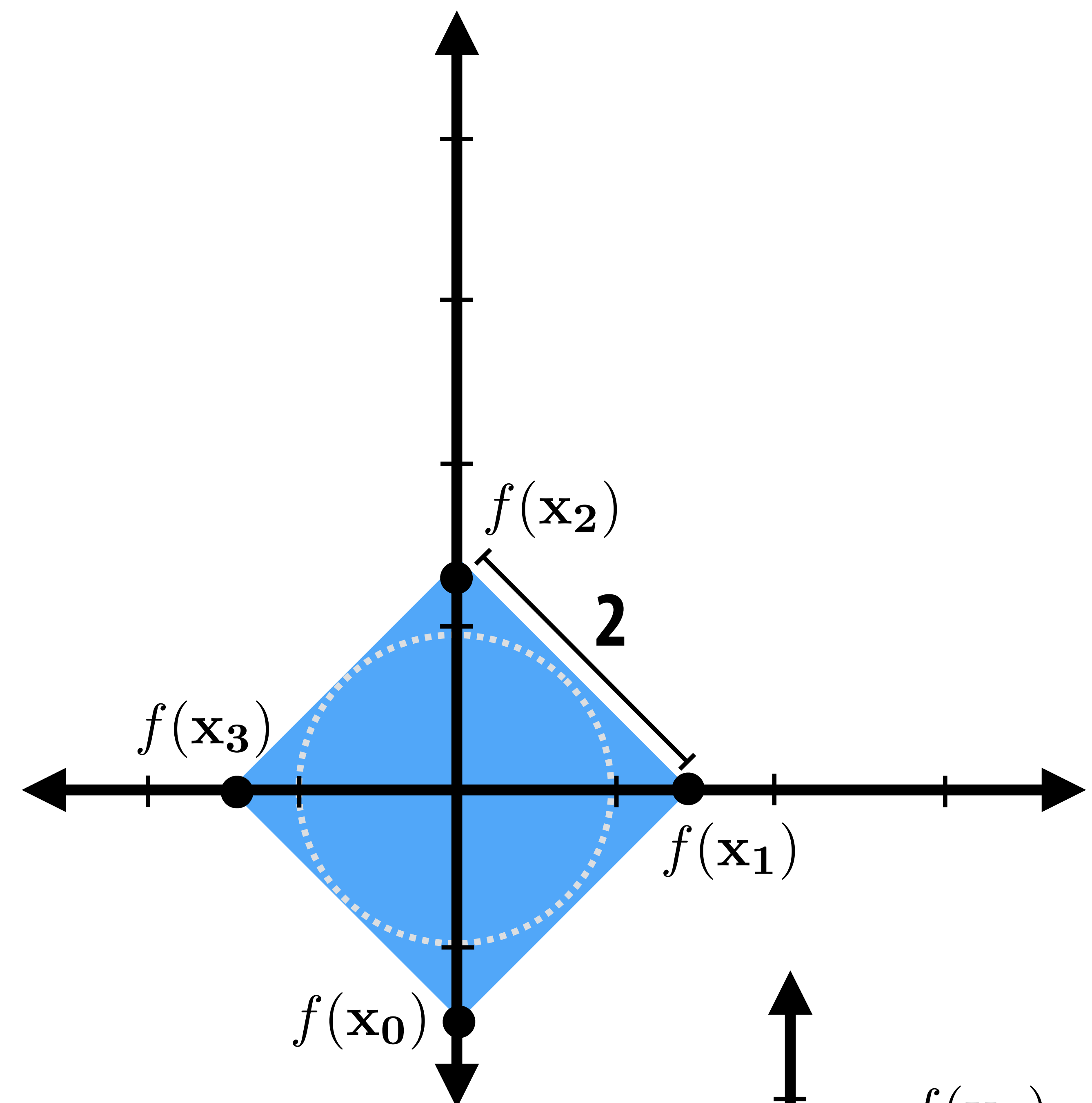


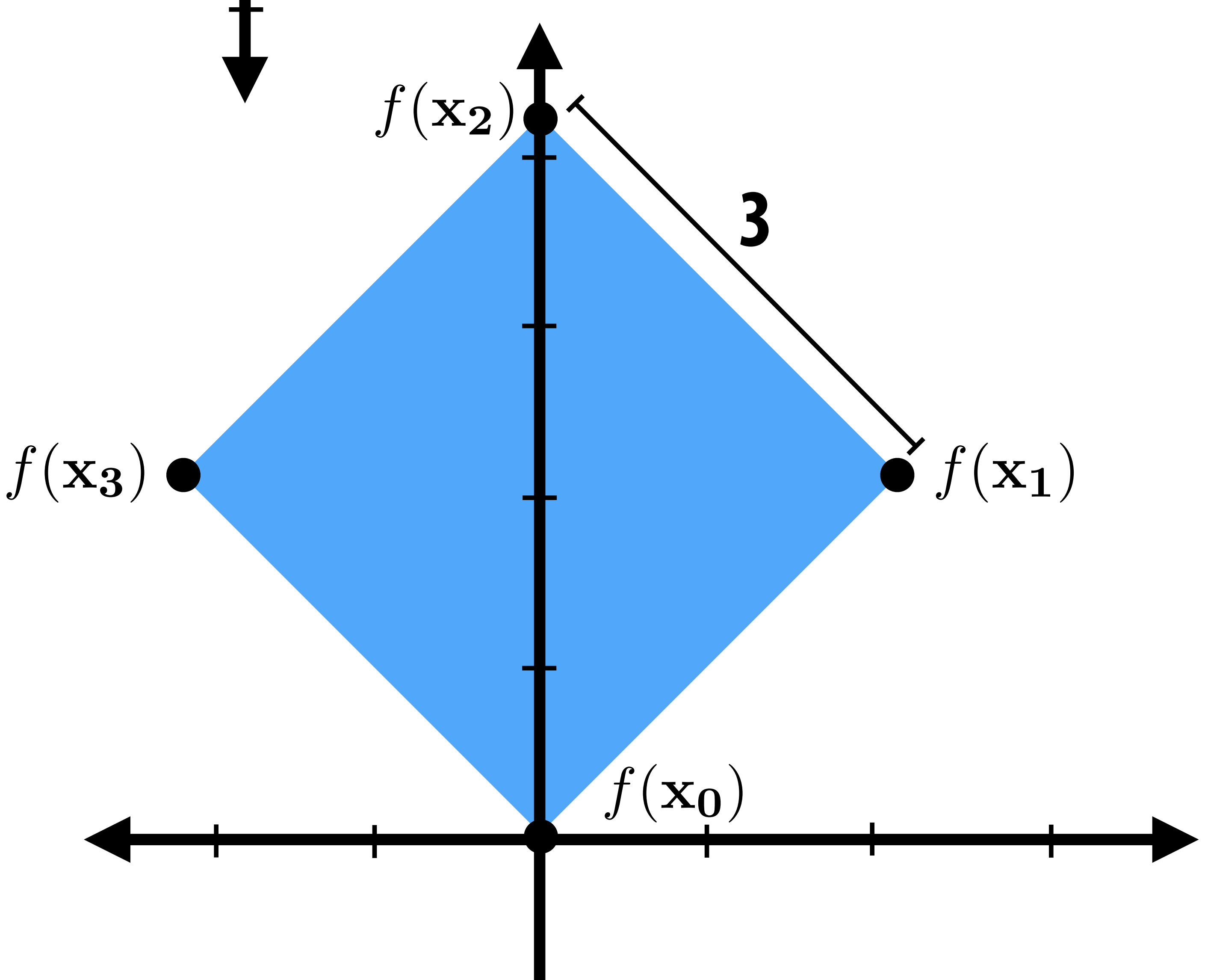
Note: order of composition matters

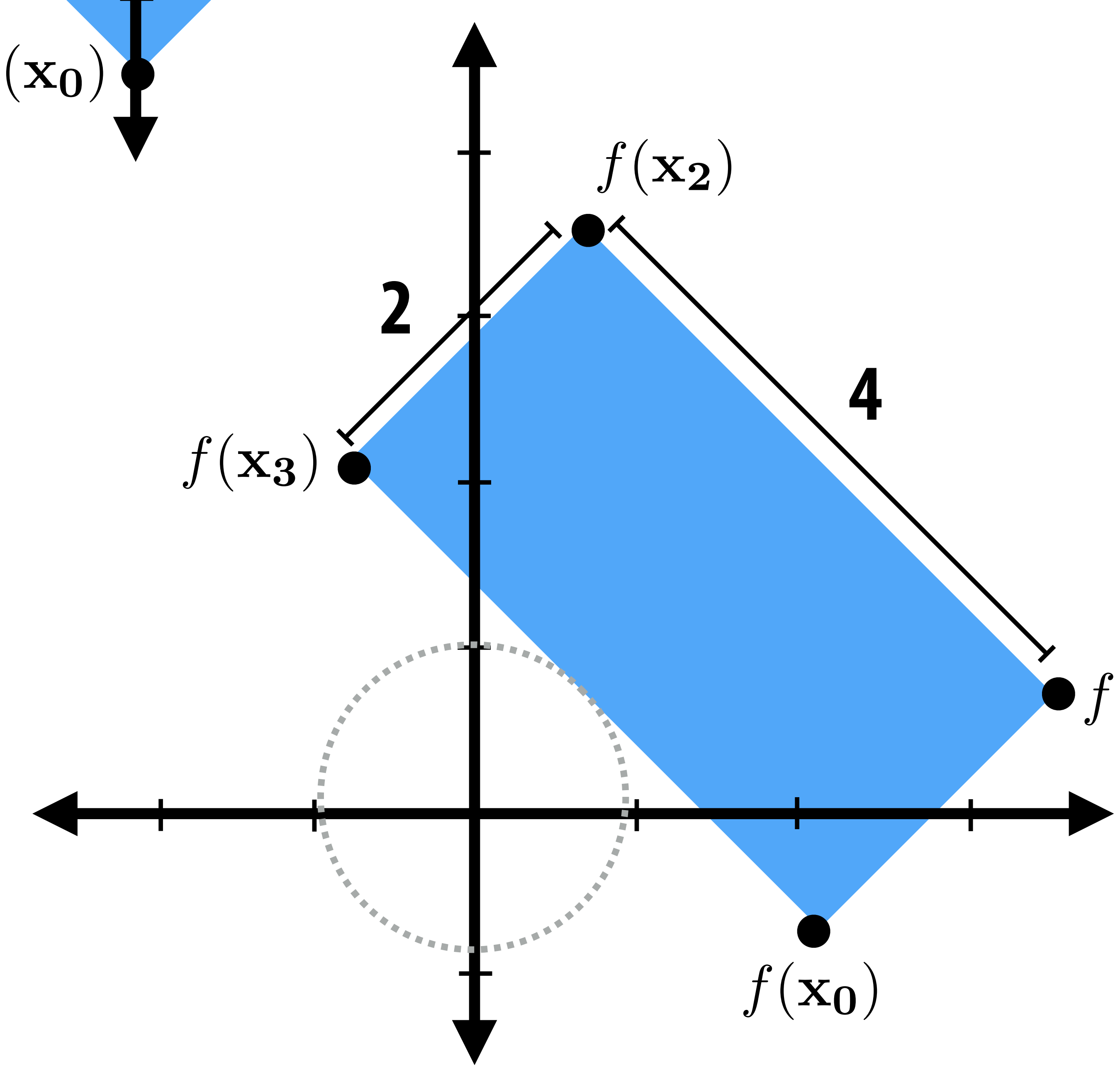
Top-right: scale, then translate
Bottom-right: translate, then scale

How would you perform these transformations?



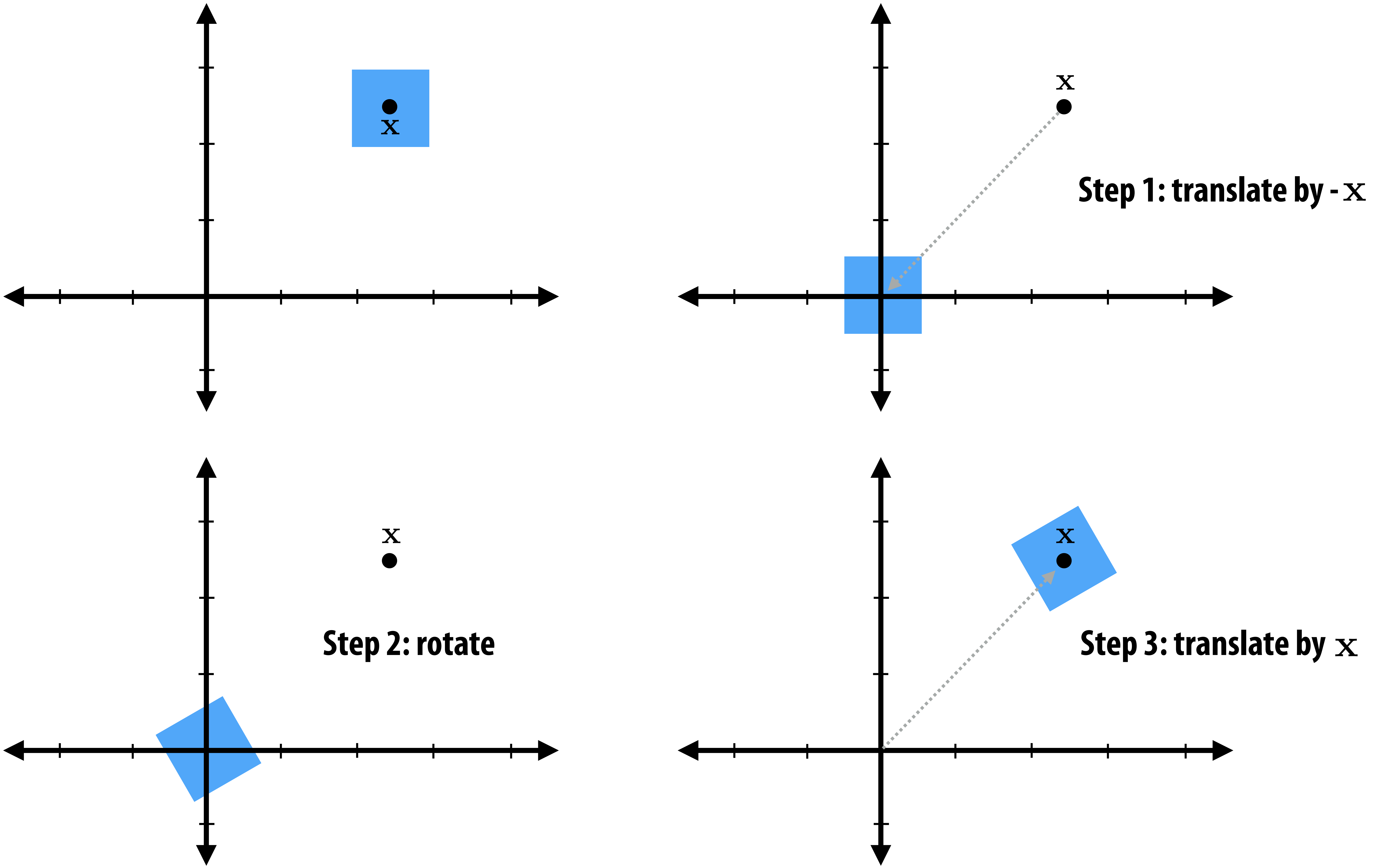






Usually more than one way to do it!

Common task: rotate about a point x



Summary of basic transformations

Linear:

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$
$$f(a\mathbf{x}) = af(\mathbf{x})$$

Scale
Rotation
Reflection
Shear

Not linear:

Translation

Affine:

Composition of linear transform + translation
(all examples on previous two slides)

$$f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$$

Not affine: perspective projection (will discuss later)

Euclidean: (Isometries)

Preserve distance between points (preserves length)

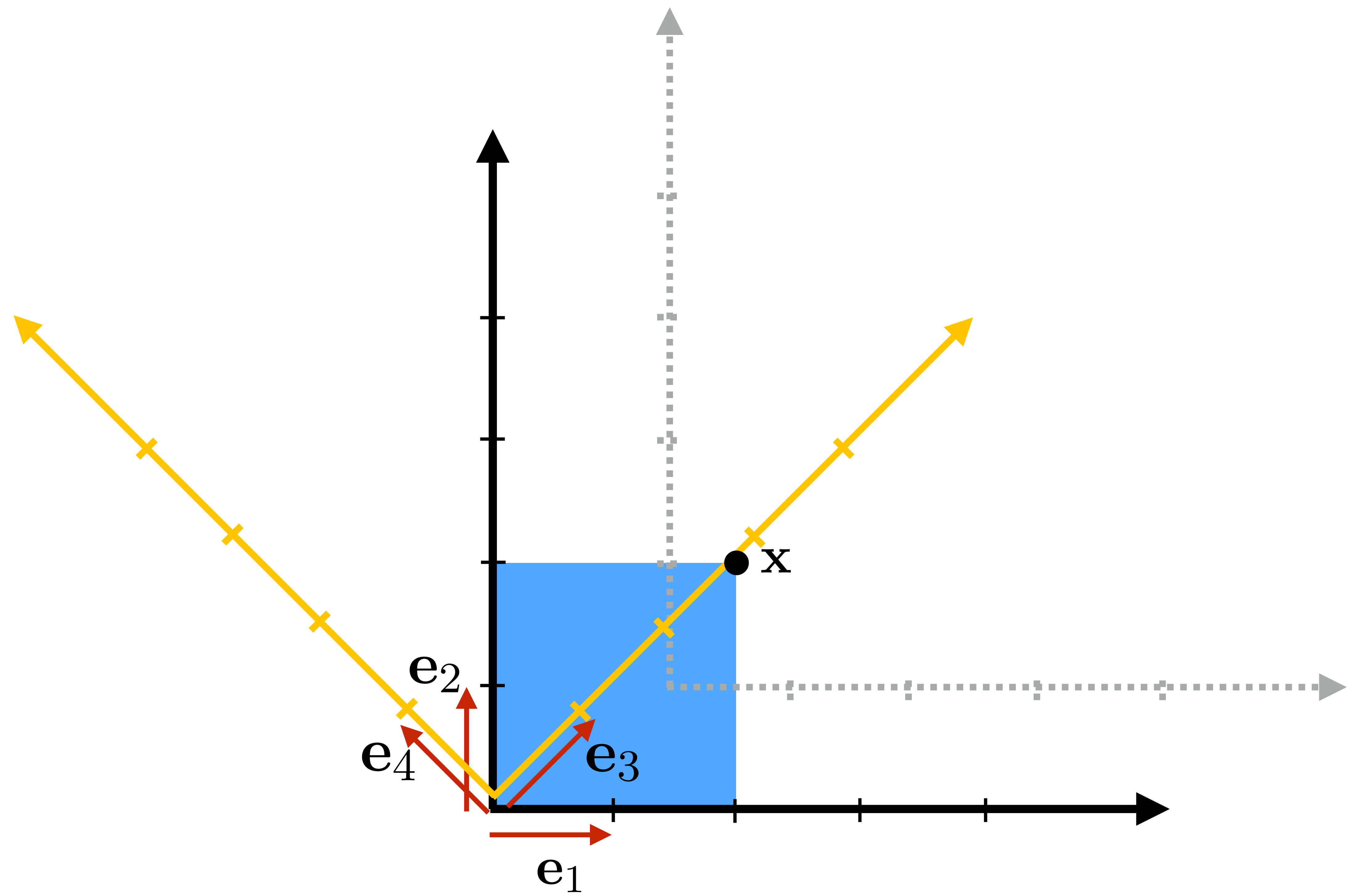
$$|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Translation
Rotation
Reflection

"Rigid body" transformations are distance-preserving motions that also preserve *orientation* (i.e., does not include reflection)

Representing Transformations in Coordinates

Review: representing points in a coordinate space



Consider coordinate space defined by orthogonal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$

$\mathbf{x} = 2\mathbf{e}_1 + 2\mathbf{e}_2$

$\mathbf{x} = \begin{bmatrix} 2 & 2 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, with origin at (1.5, 1)**

$\mathbf{x} = \begin{bmatrix} \sqrt{8} & 0 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_3$ and $\mathbf{e}_4$, with origin at (0, 0)**

Review: 2D matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} =$$

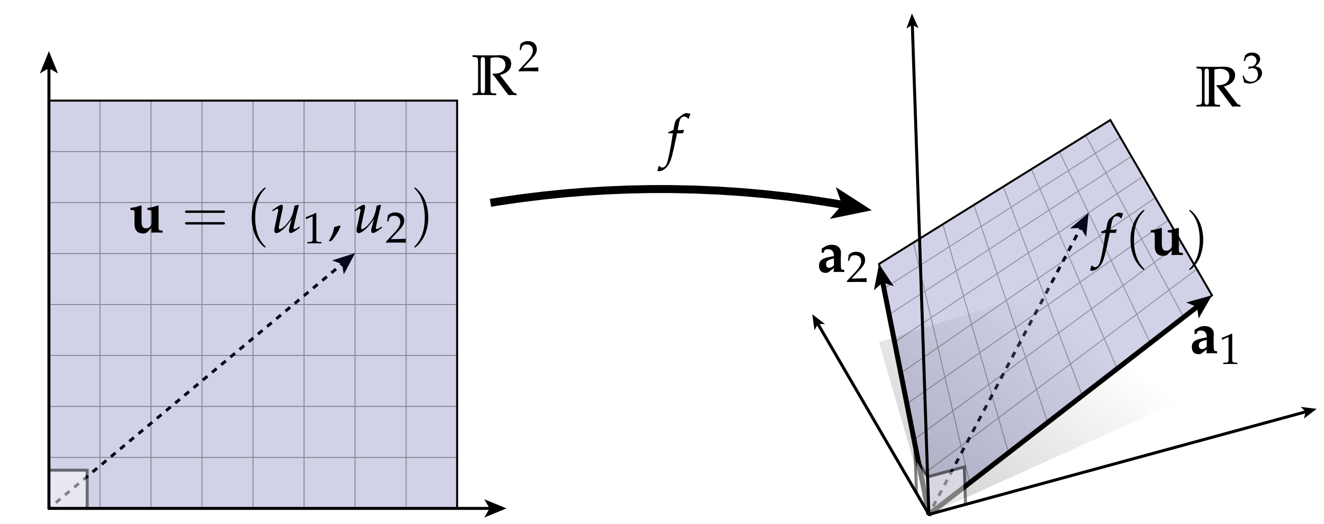
$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- **Matrix multiplication is linear combination of columns**
- **Encodes a linear map!**

Linear maps via matrices

- **Example: suppose I have a linear map**

$$f(\mathbf{u}) = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$



- **Encoding as a matrix: "a" vectors become matrix columns:**

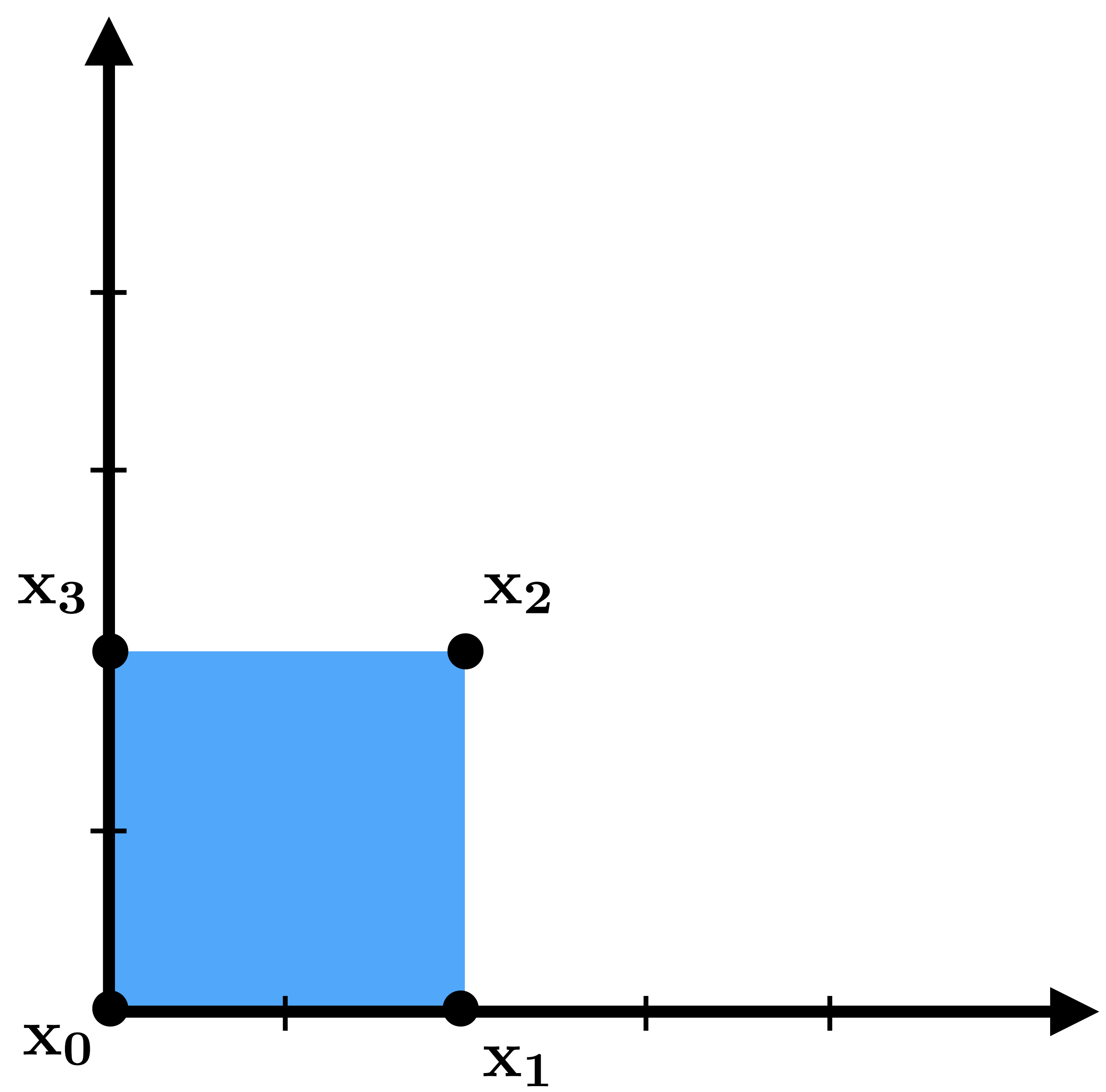
$$A := \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix}$$

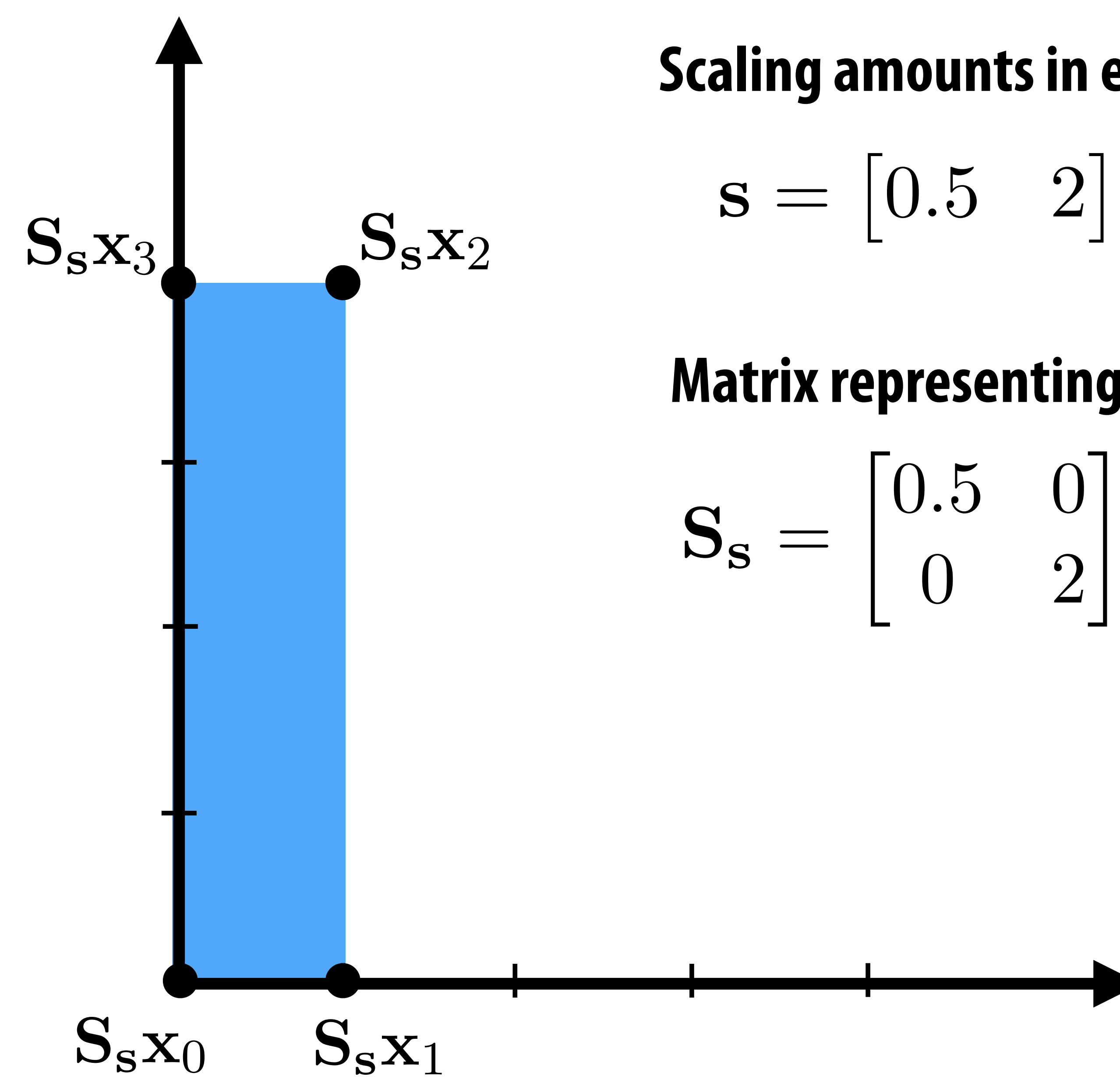
- **Matrix-vector multiply computes same output as original map:**

$$\begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{1,x}u_1 + a_{2,x}u_2 \\ a_{1,y}u_1 + a_{2,y}u_2 \\ a_{1,z}u_1 + a_{2,x}u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$

Linear transformations in 2D can be represented as 2x2 matrices

Consider non-uniform scale: $\mathbf{S_s} = \begin{bmatrix} \mathbf{s}_x & 0 \\ 0 & \mathbf{s}_y \end{bmatrix}$





Scaling amounts in each direction:

$$\mathbf{s} = \begin{bmatrix} 0.5 & 2 \end{bmatrix}^T$$

Matrix representing scale transform:

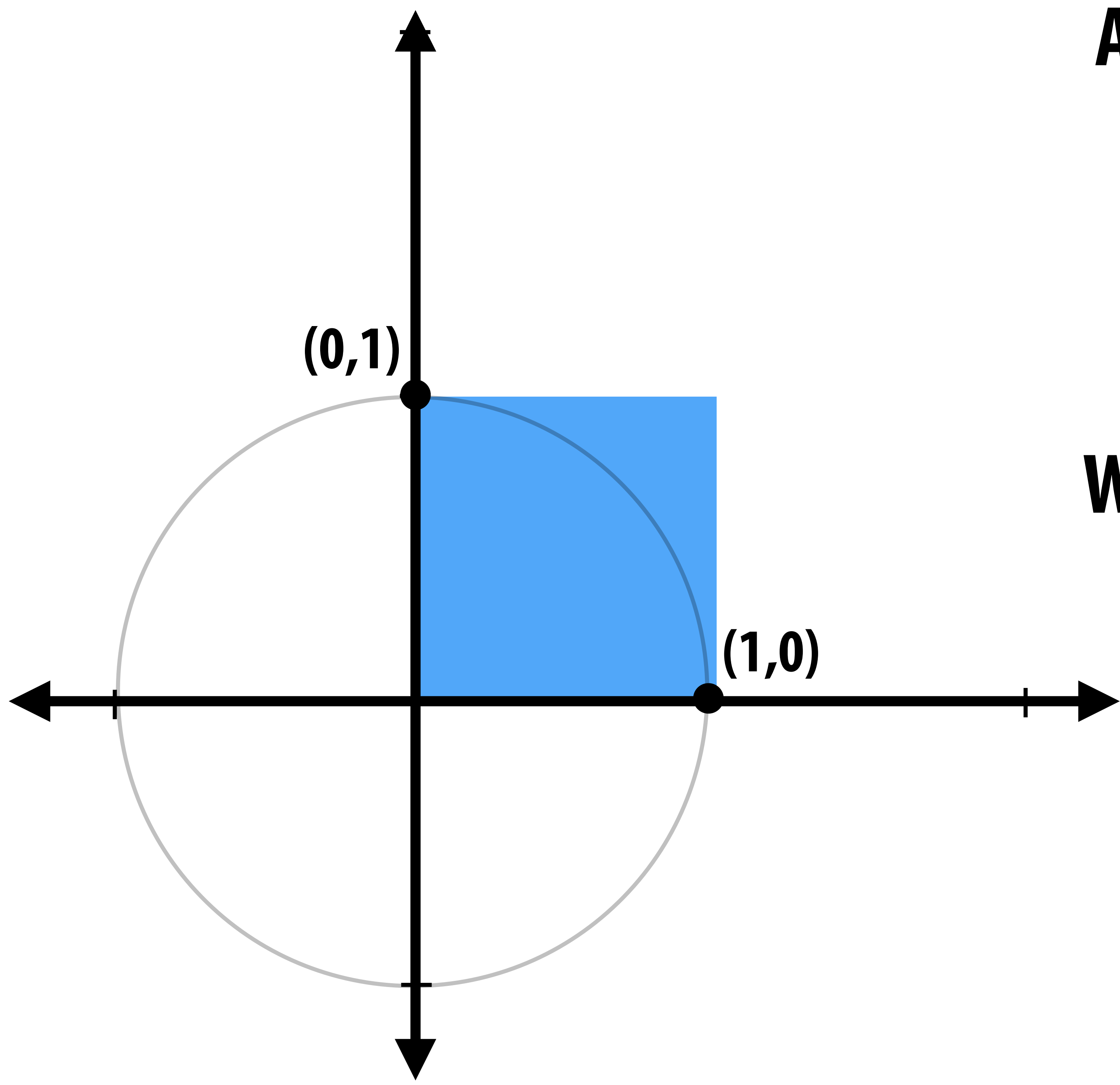
$$\mathbf{S_s} = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

Rotation matrix (2D)

Question: what happens to (1, 0) and (0,1) after rotation by θ?

Reminder: rotation moves points along circular trajectories.

(Recall that $\cos\theta$ and $\sin\theta$ are the coordinates of a point on the unit circle.)



Answer:

$$R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

$$R_\theta(0,1) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$$

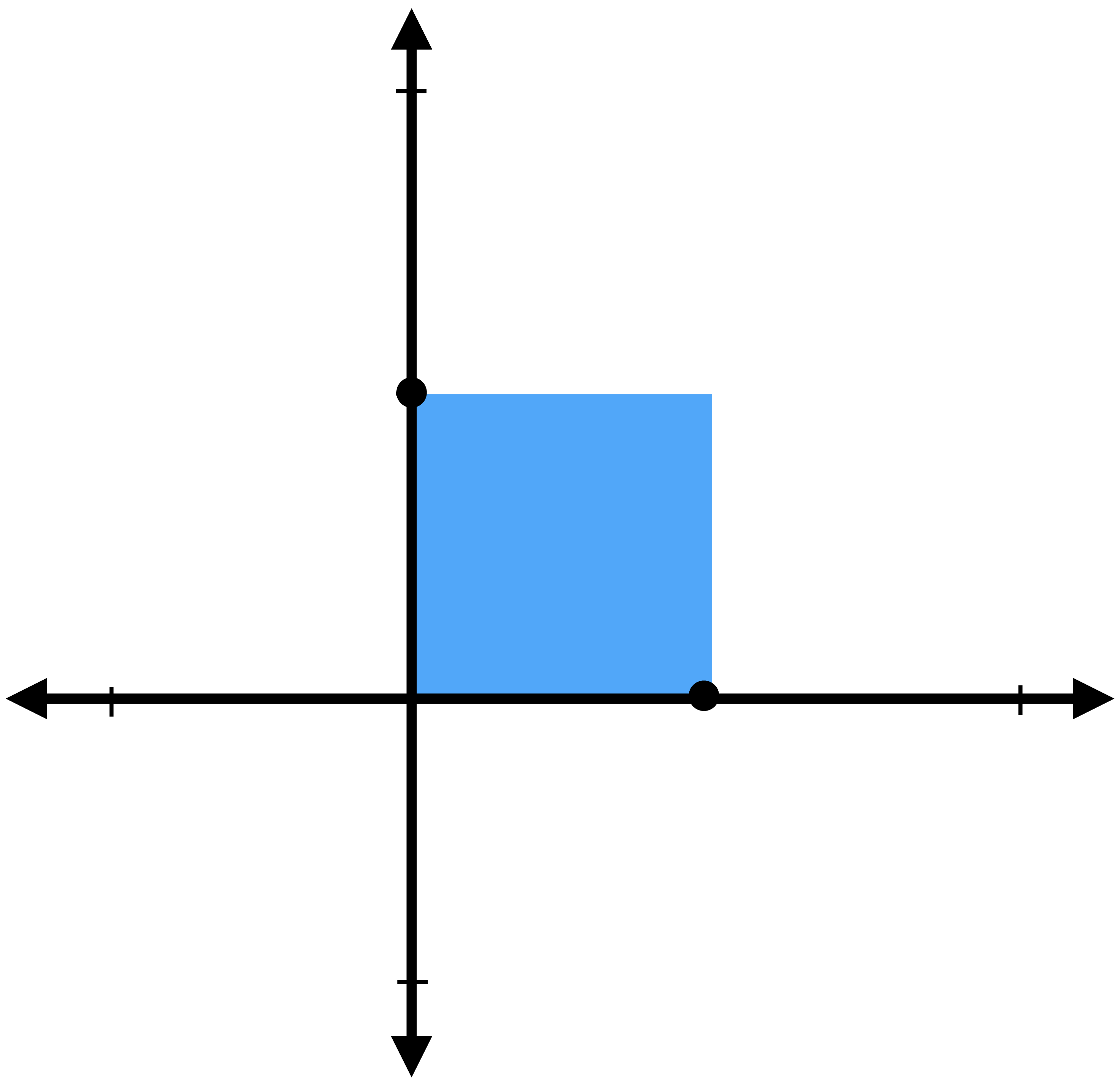
Which means the matrix must look like:

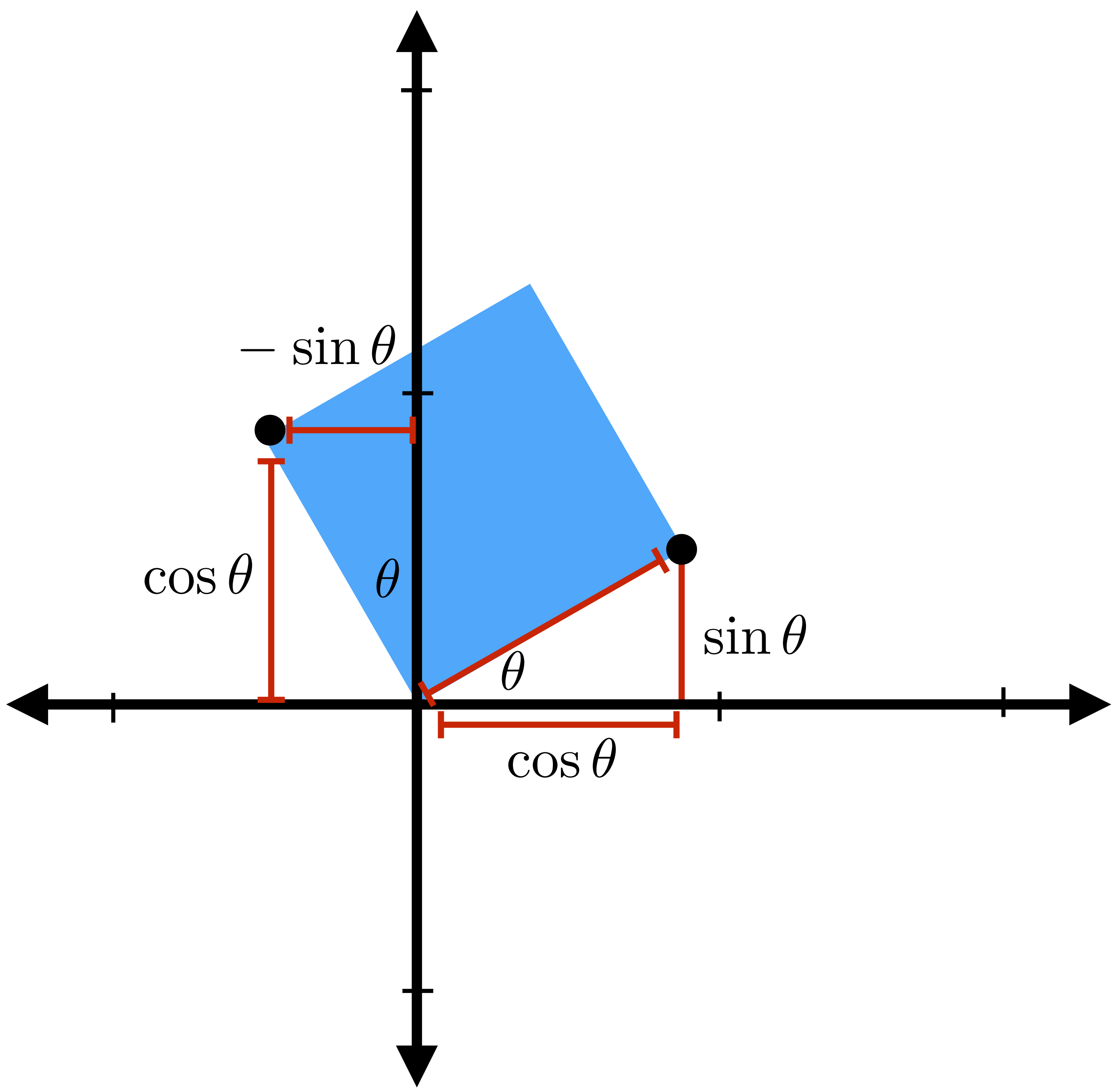
$$R_\theta = \begin{bmatrix} \cos(\theta) & \cos(\theta + \pi/2) \\ \sin(\theta) & \sin(\theta + \pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

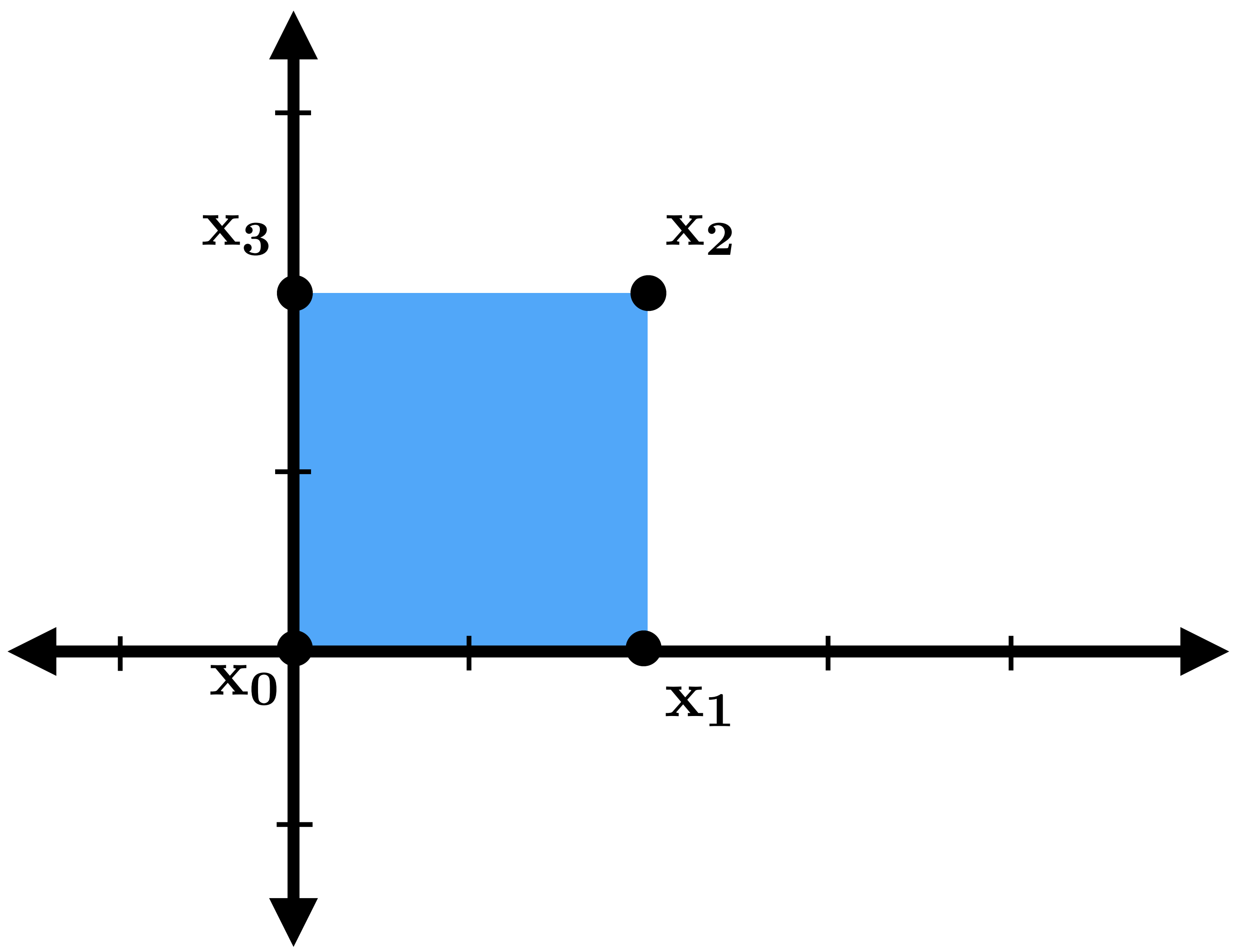
Rotation matrix (2D): another way...

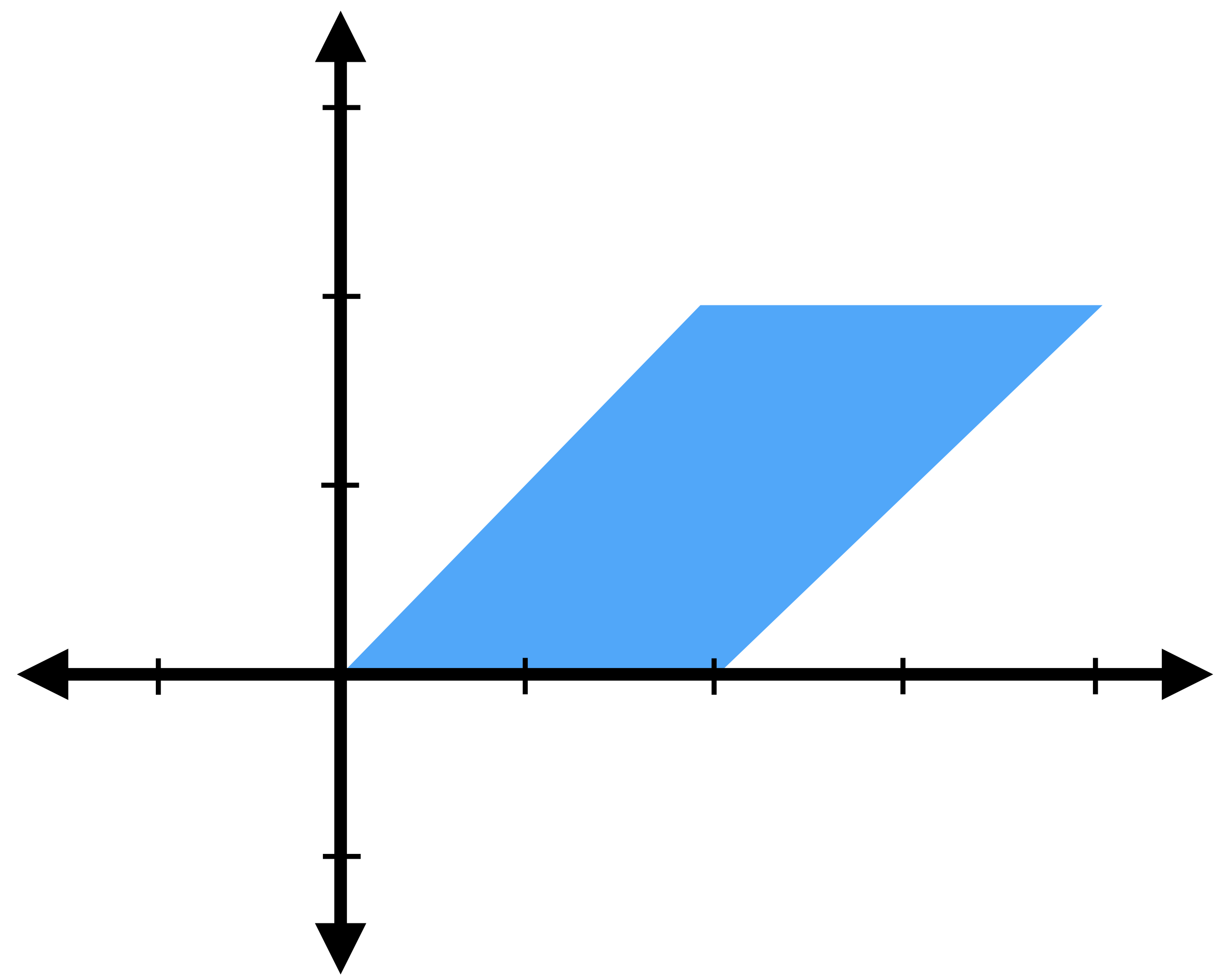
$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$





Shear



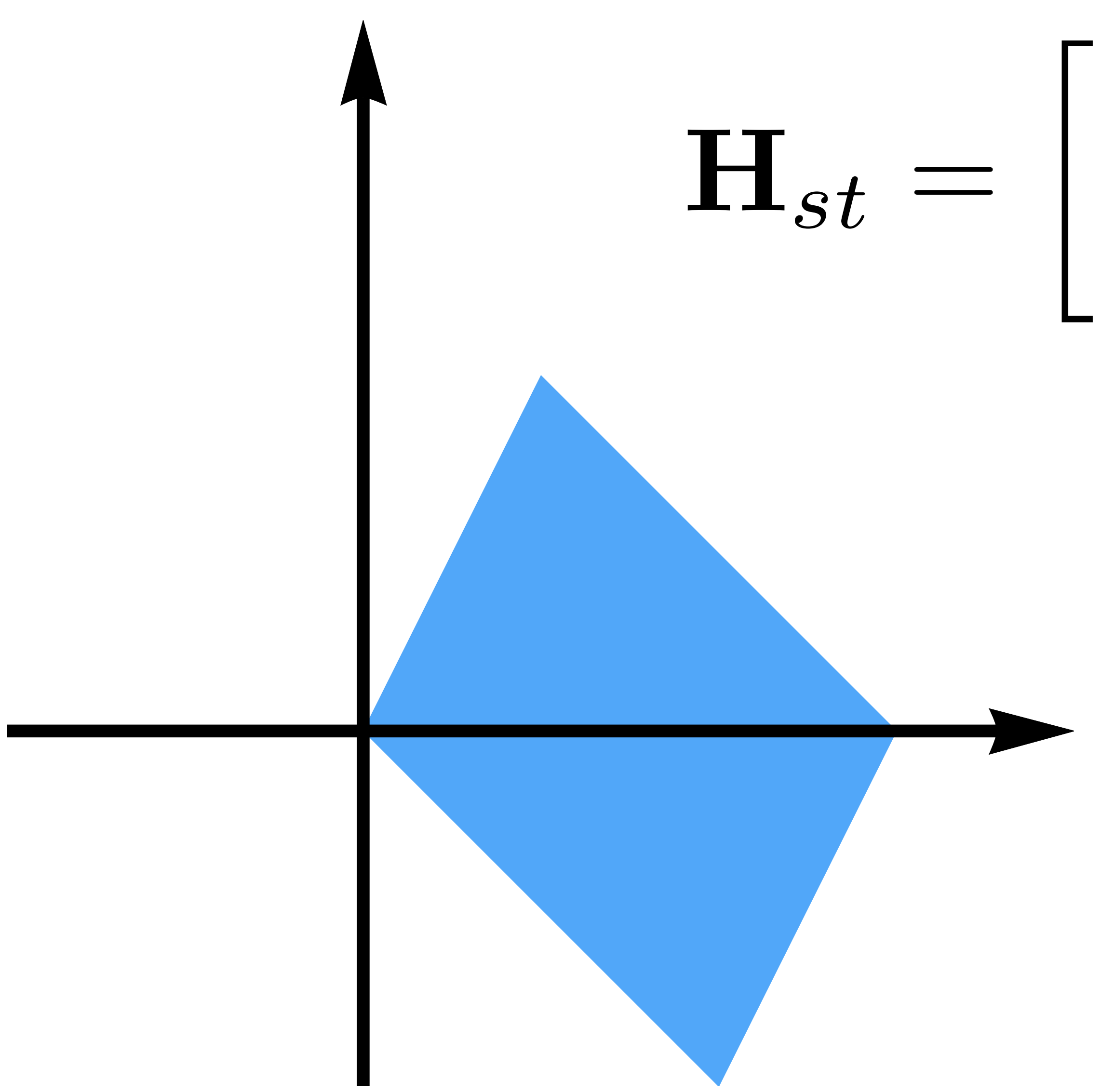


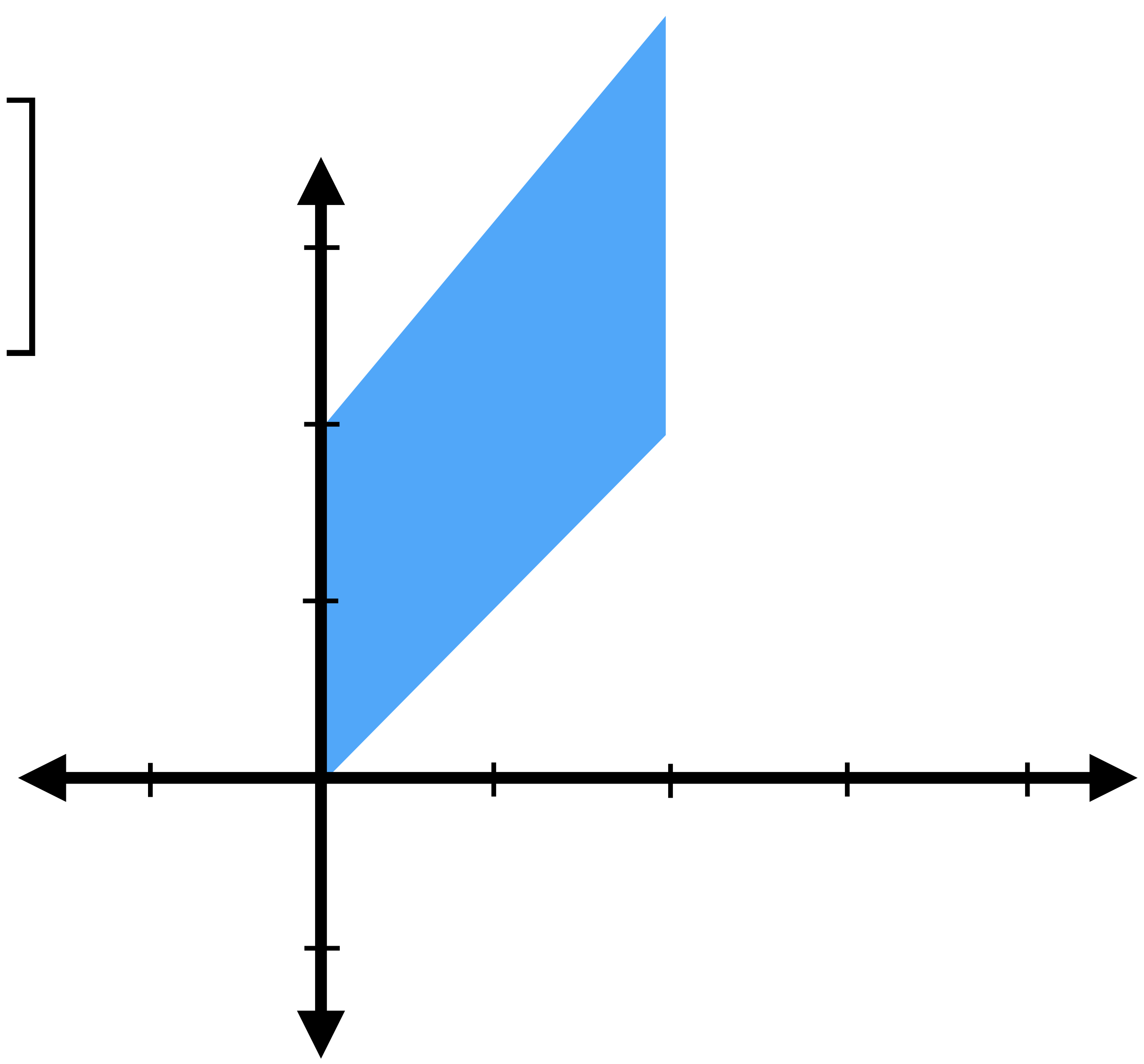
Shear in x:

$$\mathbf{H}_{xs} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Arbitrary shear:

$$\mathbf{H}_{st} = \begin{bmatrix} 1 & s \\ t & 1 \end{bmatrix}$$

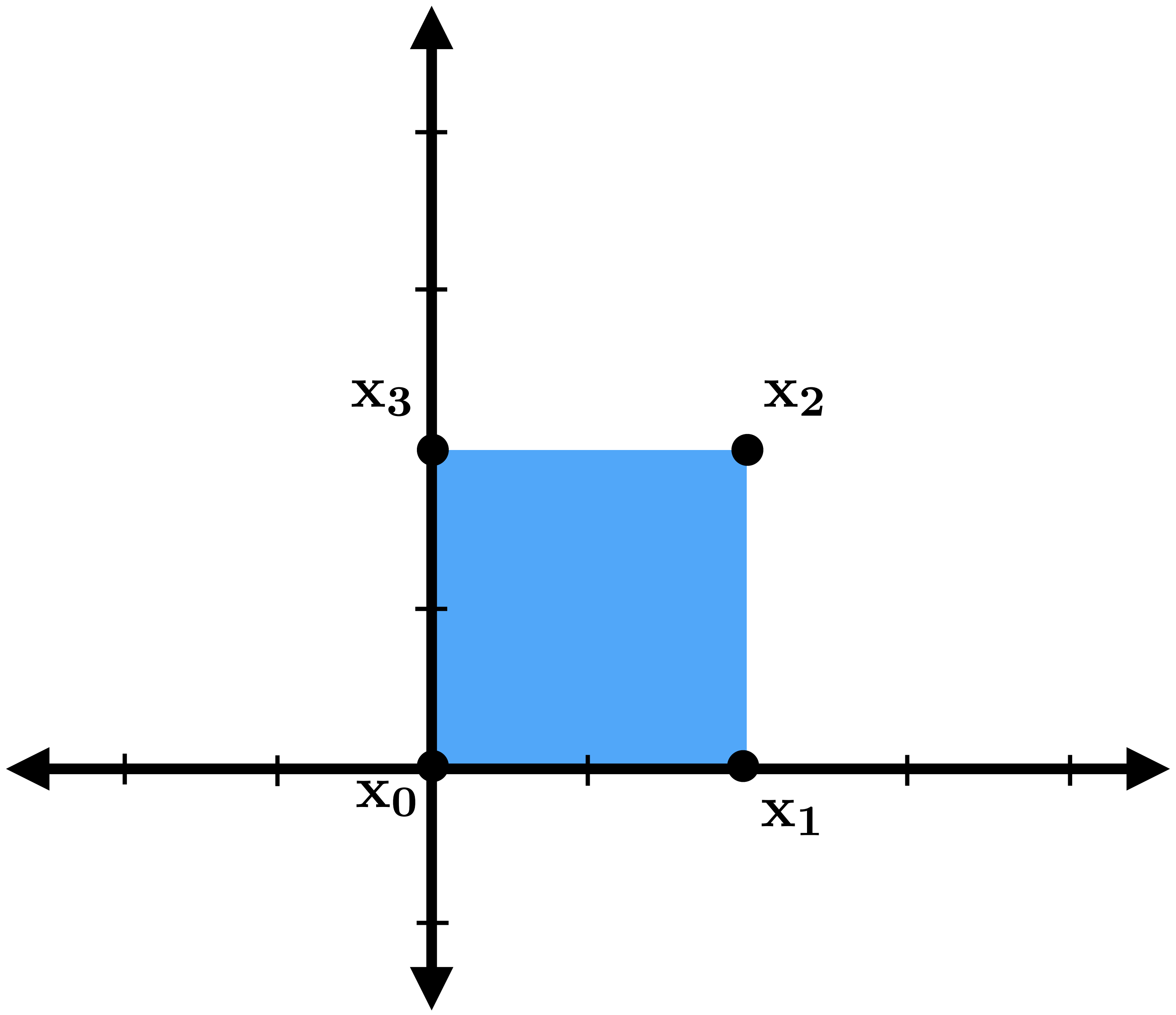


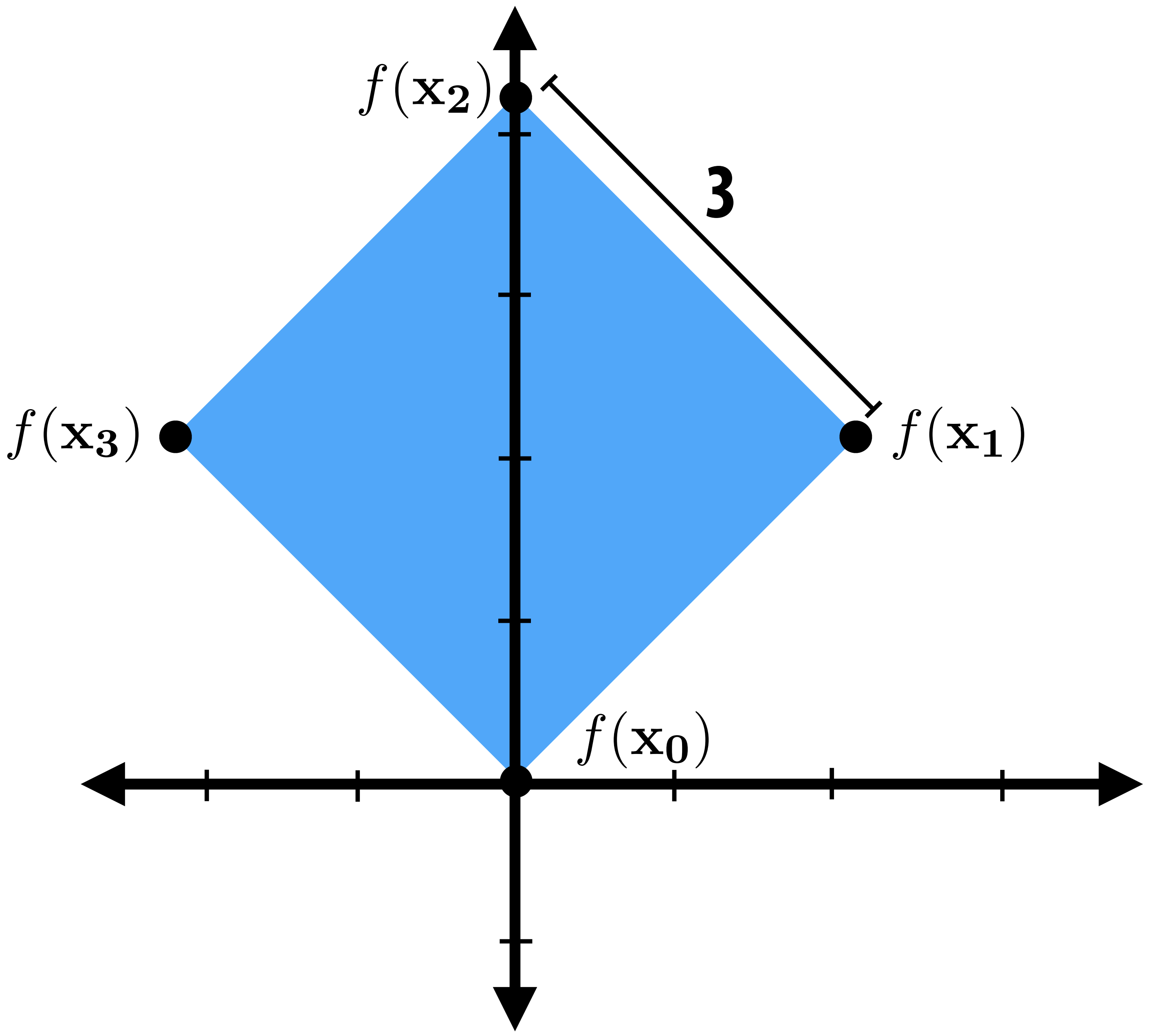


Shear in y:

$$\mathbf{H}_{ys} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

How do we compose linear transformations?





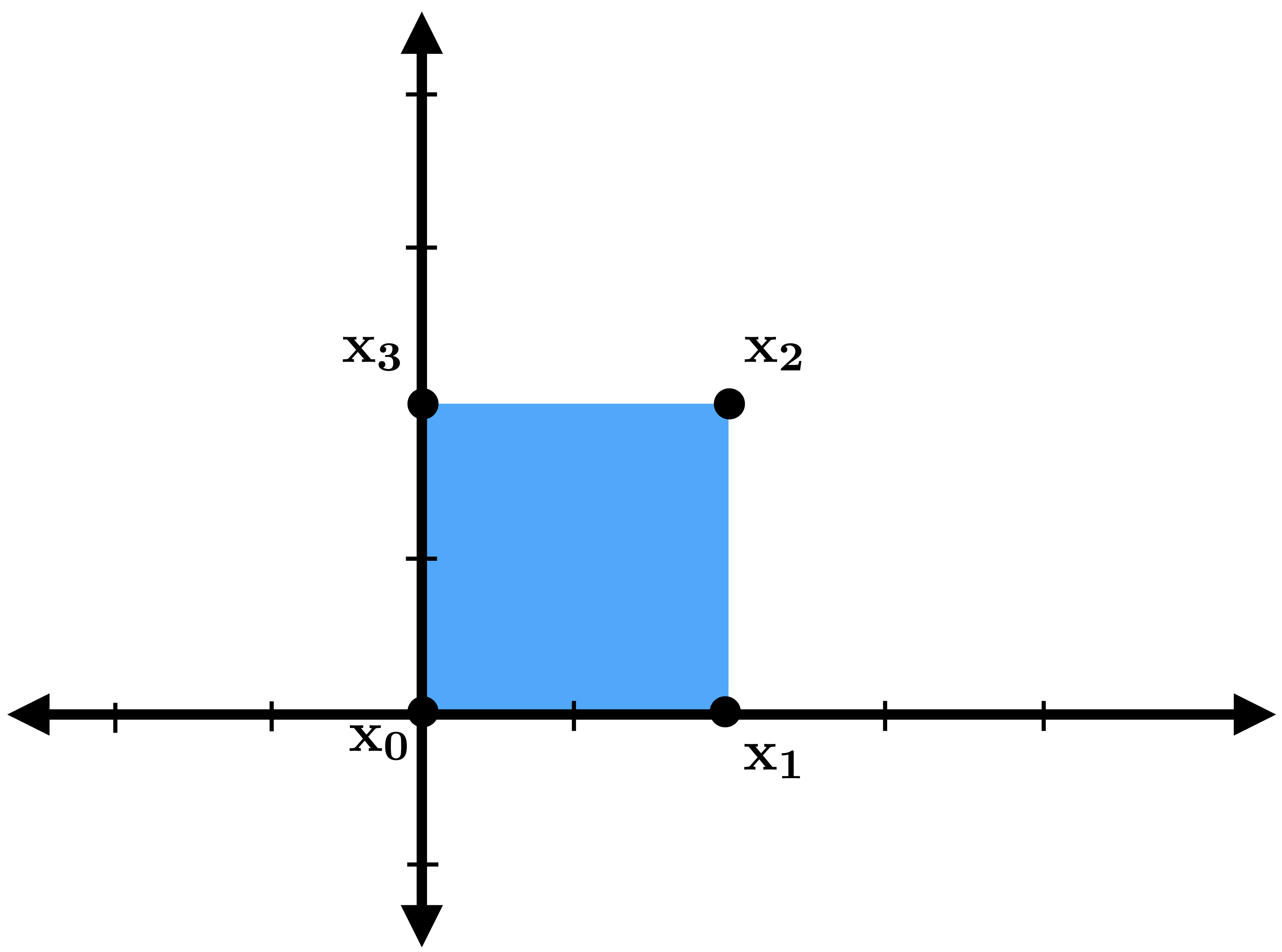
Compose linear transformations via matrix multiplication.
This example: uniform scale, followed by rotation

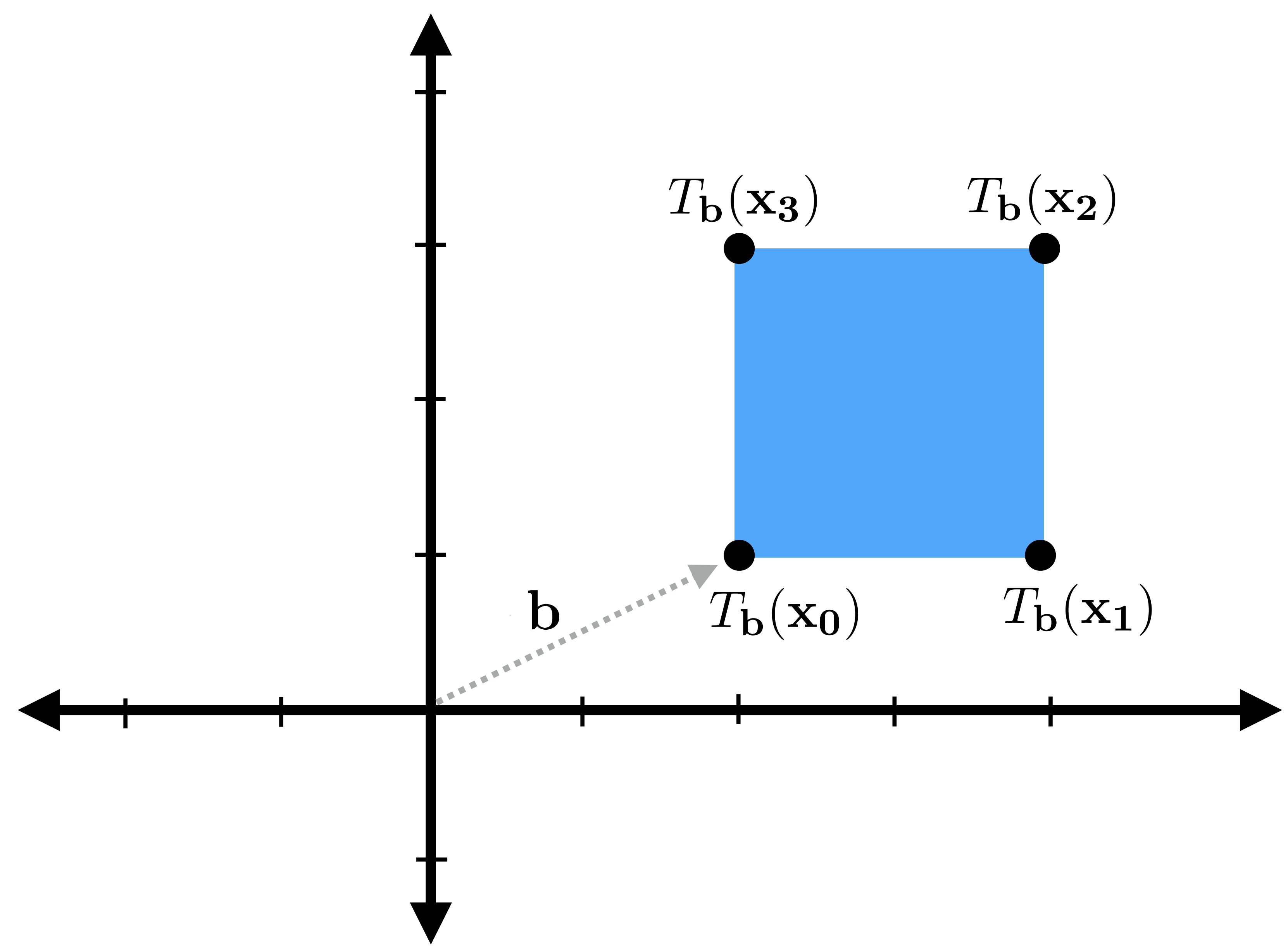
$$f(\mathbf{x}) = R_{\pi/4}\mathbf{S}_{[1.5,1.5]}\mathbf{x}$$

Enables simple, efficient implementation: reduce complex chain of transformations to a single matrix multiplication.

How do we deal with translation? (Not linear)

$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$





Recall: translation is not a linear transform

→ Output coefficients are not a linear combination of input coefficients

→ Translation operation cannot be represented by a 2x2 matrix

$$\mathbf{x_{out}}_x = \mathbf{x}_x + \mathbf{b}_x$$

$$\mathbf{x_{out}}_y = \mathbf{x}_y + \mathbf{b}_y$$

Translation math

2D homogeneous coordinates (2D-H)

Interesting idea: represent 2D points with THREE values ("homogeneous coordinates")

So the point (x, y) **is represented as the 3-vector:** $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$

And transformations are represented a 3x3 matrices that transform these vectors.

Recover final 2D coordinates by dividing by "extra" (third) coordinate

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

(More on this later…)

Example: scale and rotation in 2D-H coords

- **For transformations that are already linear, not much changes:**

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the last row/column doesn't do anything interesting. E.g., for scaling:

$$\mathbf{S_s} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix}$$

Now we divide by the 3rd coordinate to get our final 2D coordinates (not too exciting!)

$$\begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{S}_x x/1 \\ \mathbf{S}_y y/1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \end{bmatrix}$$

(Will get more interesting when we talk about *perspective*…)

Translation in 2D homogeneous coordinates

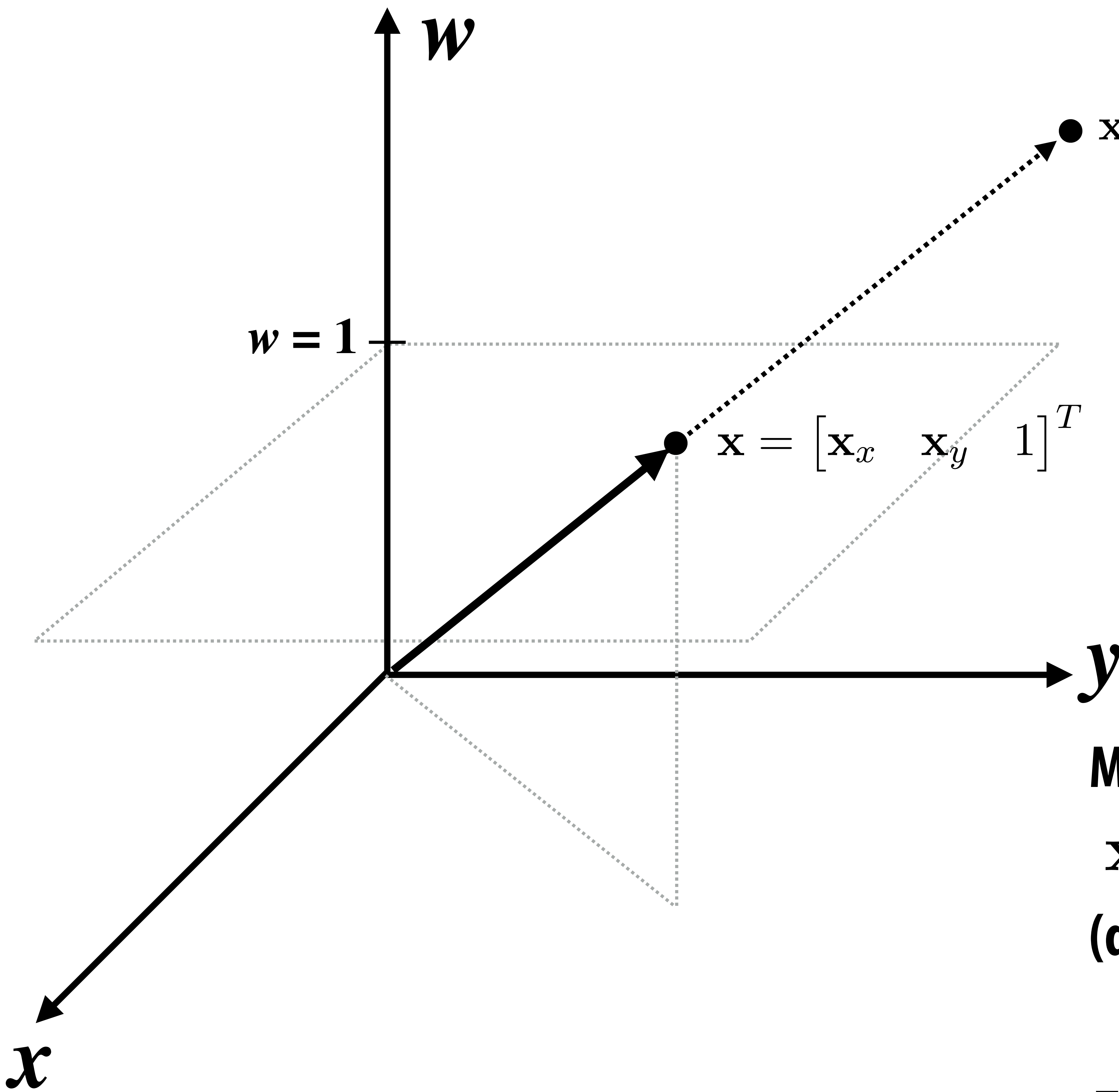
Translation expressed as 3x3 matrix multiplication:

$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T_b x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_x \\ \mathbf{x}_y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_x + \mathbf{b}_x \\ \mathbf{x}_y + \mathbf{b}_y \\ 1 \end{bmatrix}$$ **(remember: linear combination of columns!)**

Cool: homogeneous coordinates let us encode translations as *linear* transformations!

Homogeneous coordinates: some intuition



Many points in 2D-H correspond to same point in 2D

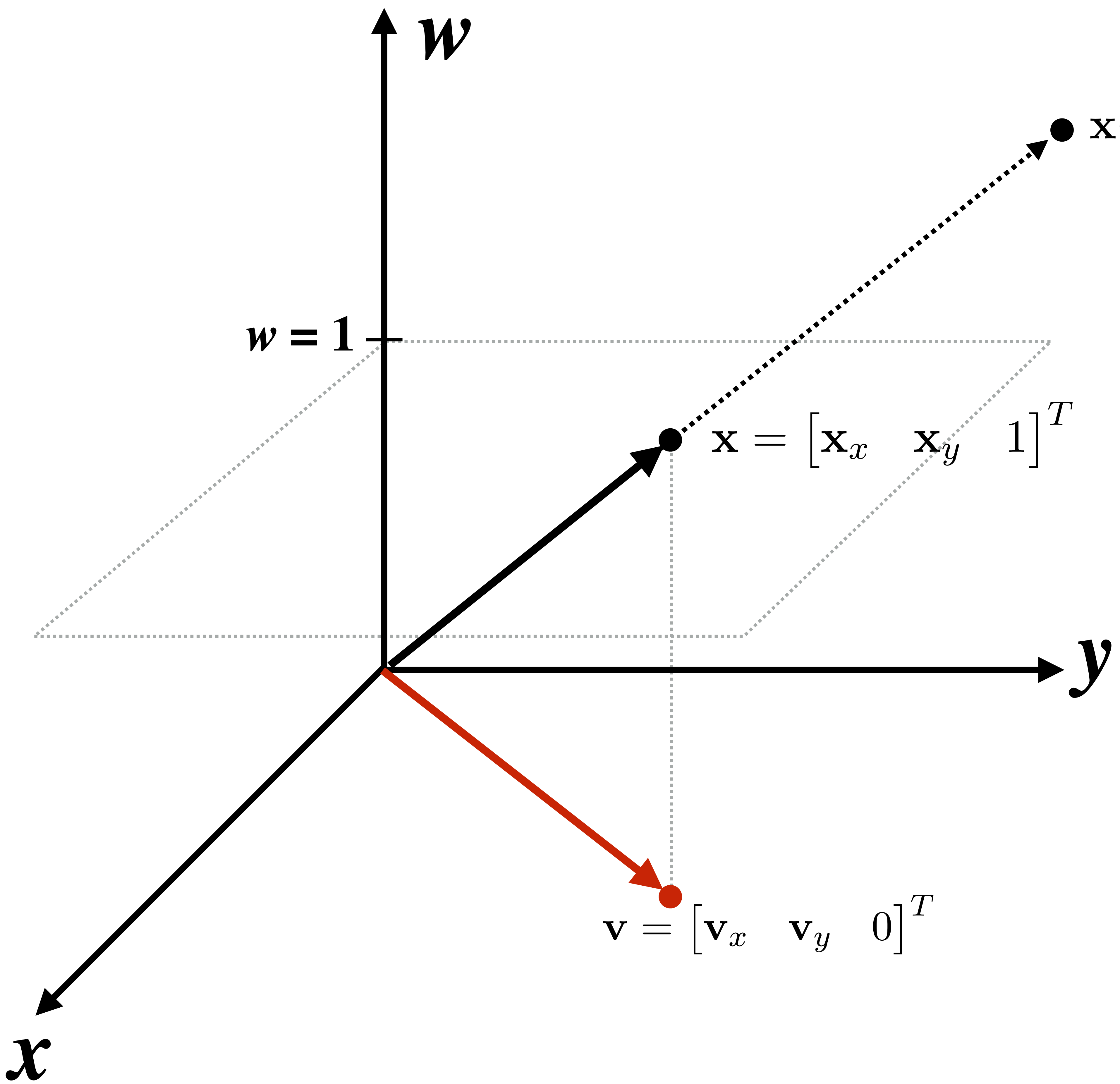
$\mathbf{x}$ and $w\mathbf{x}$ correspond to the same 2D point

(divide by w to convert 2D-H back to 2D)

Translation is a shear in x and y in 2D-H space

$$\mathbf{T_b x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w\mathbf{x}_x \\ w\mathbf{x}_y \\ w \end{bmatrix} = \begin{bmatrix} w\mathbf{x}_x + w\mathbf{b}_x \\ w\mathbf{x}_y + w\mathbf{b}_y \\ w \end{bmatrix}$$

Homogeneous coordinates: points vs. vectors



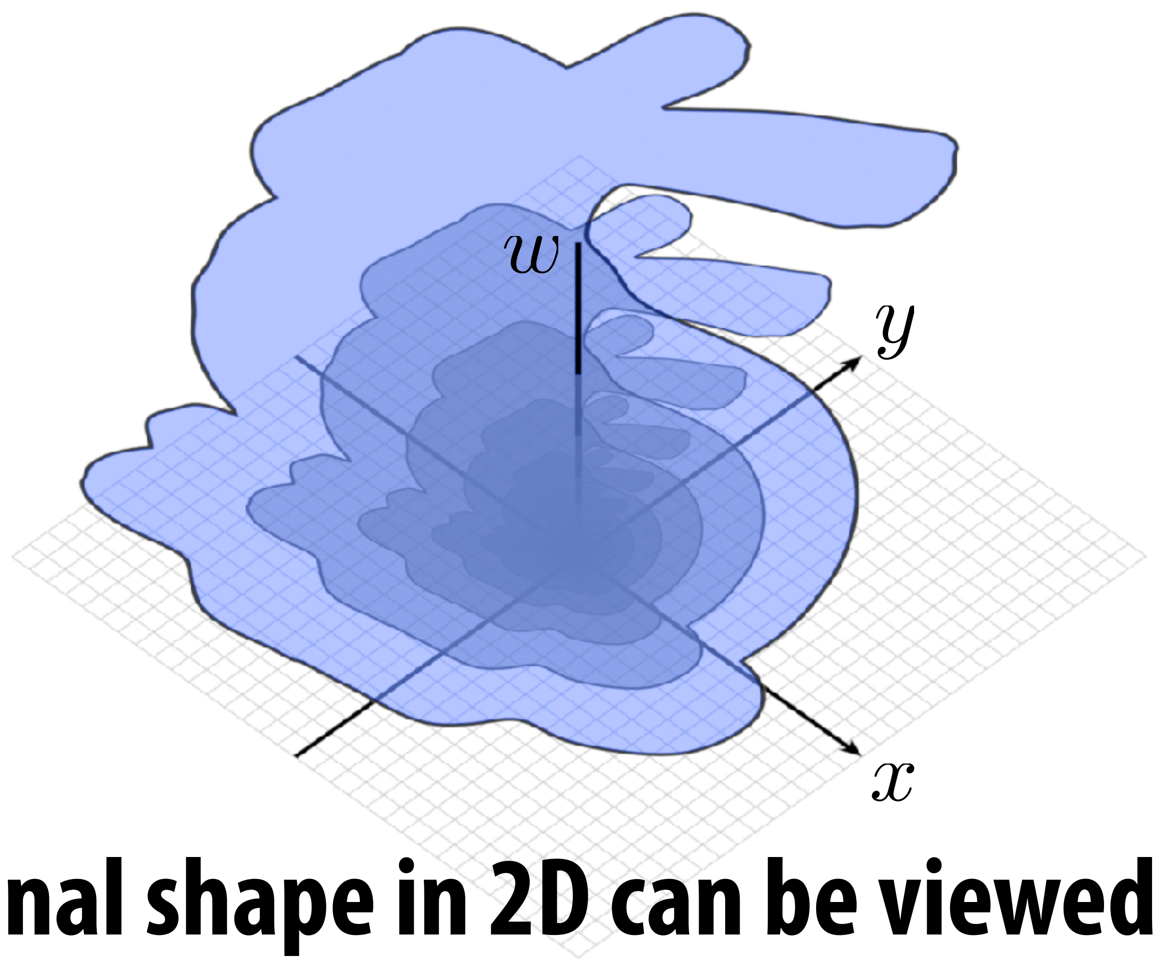
2D-H points with w=0 represent 2D vectors (think: directions are points at infinity)

Unlike 2D, points and directions are distinguishable by their representation in 2D-H

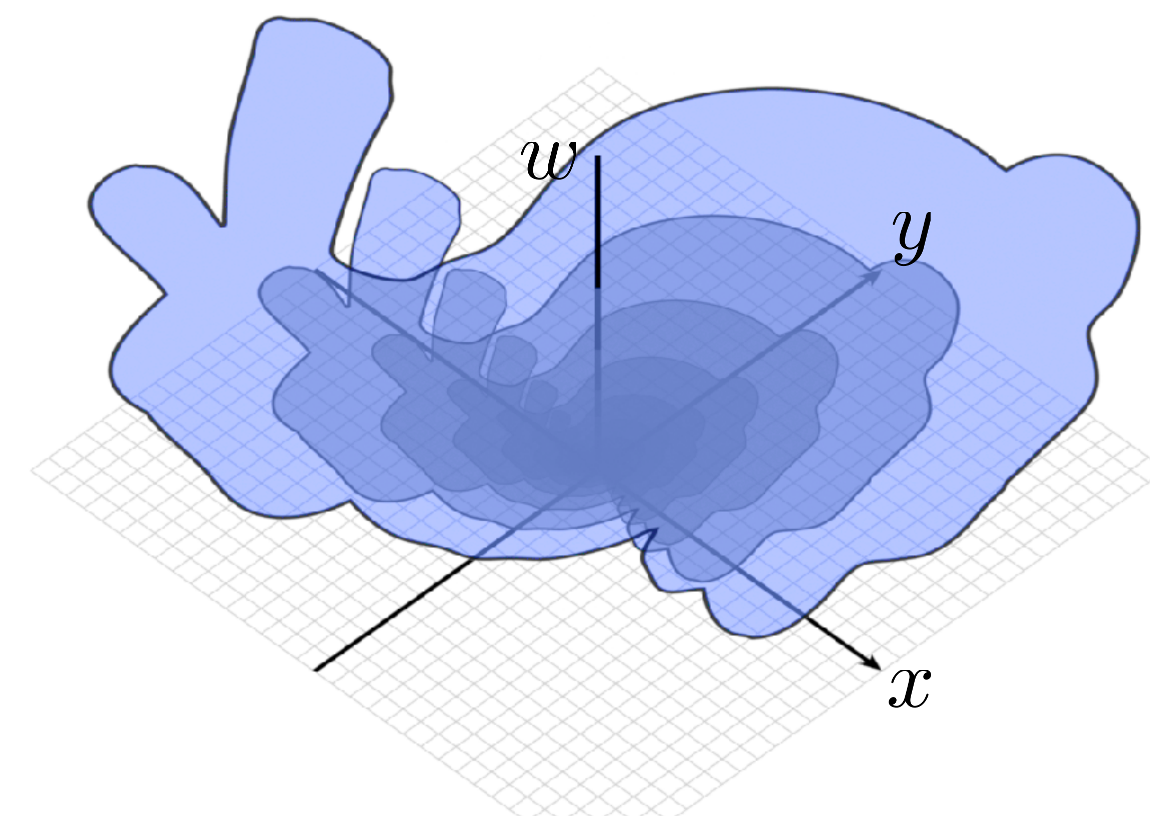
Note: translation does not modify directions:

$$\mathbf{T_b}\mathbf{v} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

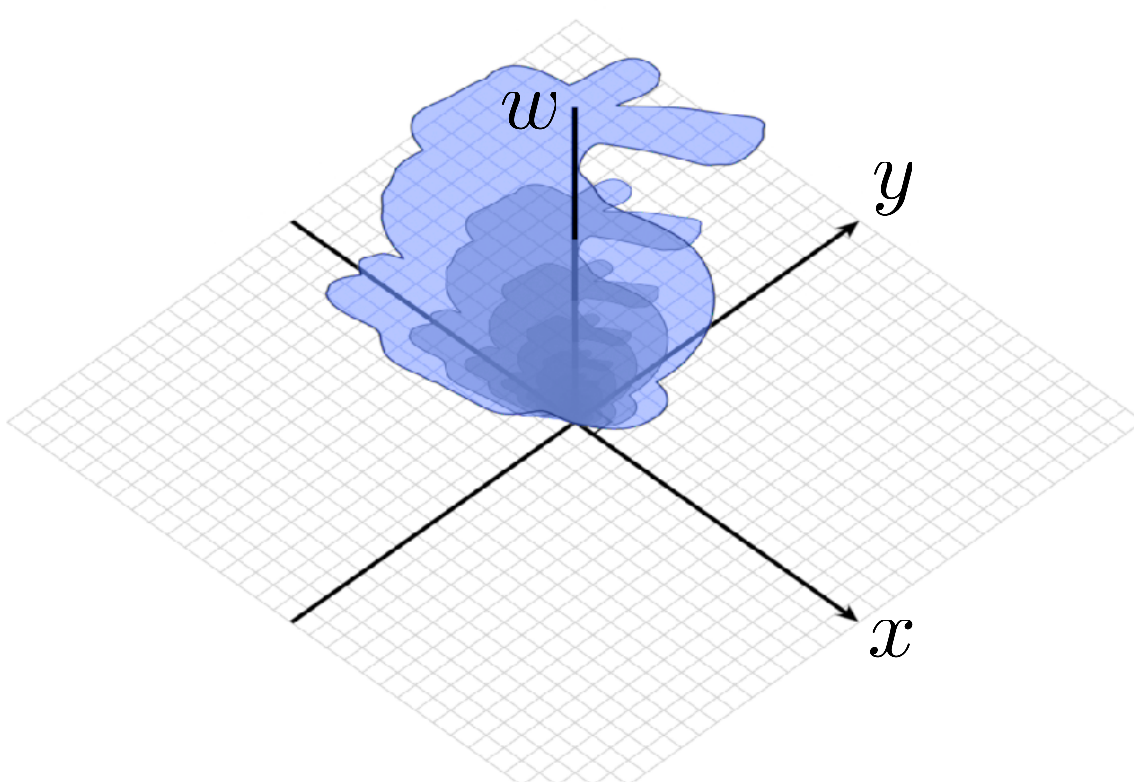
Visualizing 2D transformations in 2D-H



Original shape in 2D can be viewed as many copies, uniformly scaled by w.

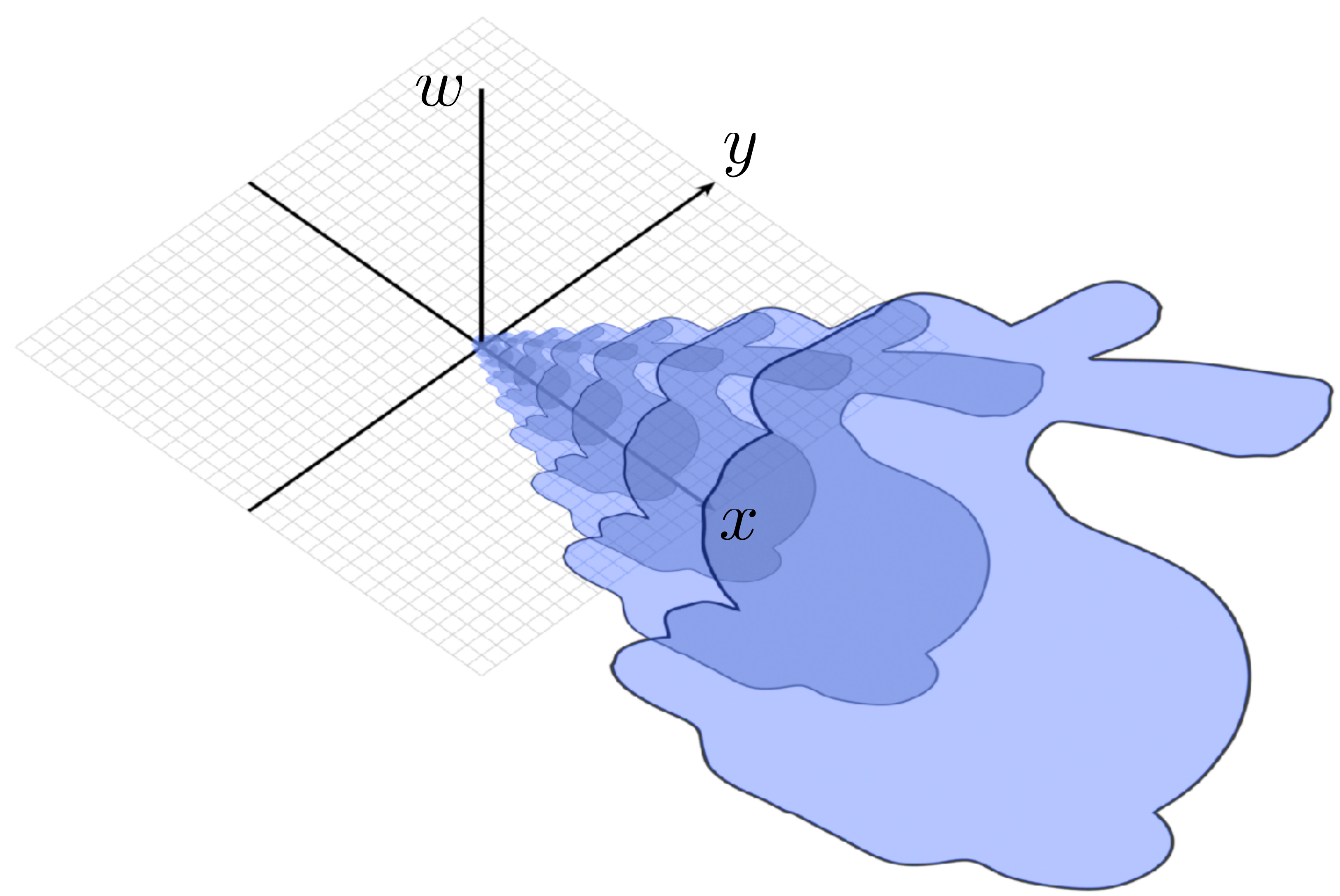


2D rotation ↔ rotate around w



2D scale ↔ scale x and y; preserve w

(Question: what happens to 2D shape if you scale x, y, and w uniformly?)



2D translate ↔ shear in 2D-H

(LINEAR!)

Moving to 3D (and 3D-H)

Represent 3D transformations as 3x3 matrices and 3D-H transformations as 4x4 matrices

Scale:

3D

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & \mathbf{S}_z \end{bmatrix}$$

3D-H

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 & 0 \\ 0 & \mathbf{S}_y & 0 & 0 \\ 0 & 0 & \mathbf{S}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear (in x, based on y,z position):

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

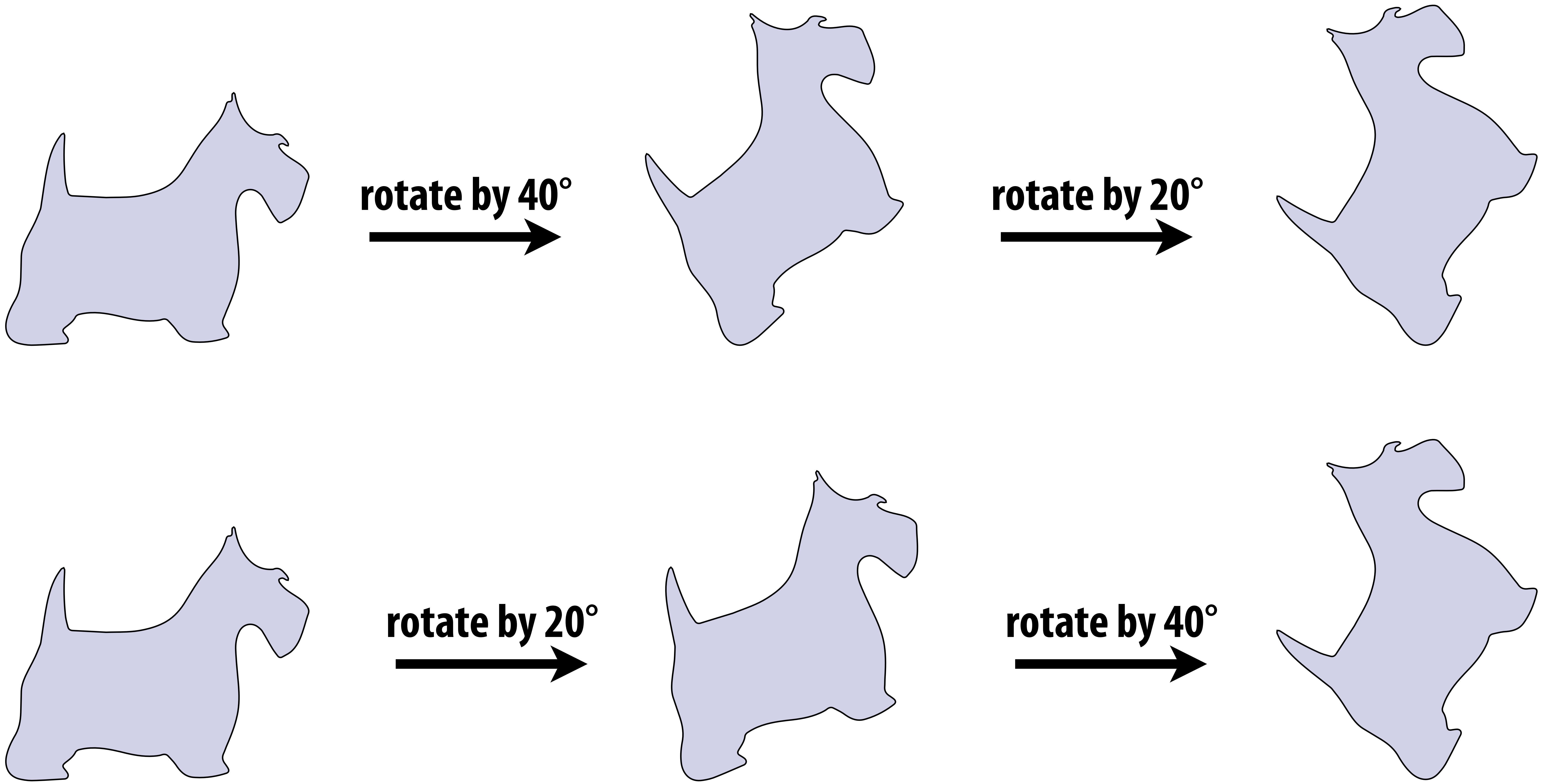
Translate:

3D-H

$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{b}_x \\ 0 & 1 & 0 & \mathbf{b}_y \\ 0 & 0 & 1 & \mathbf{b}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Commutativity of rotations—2D

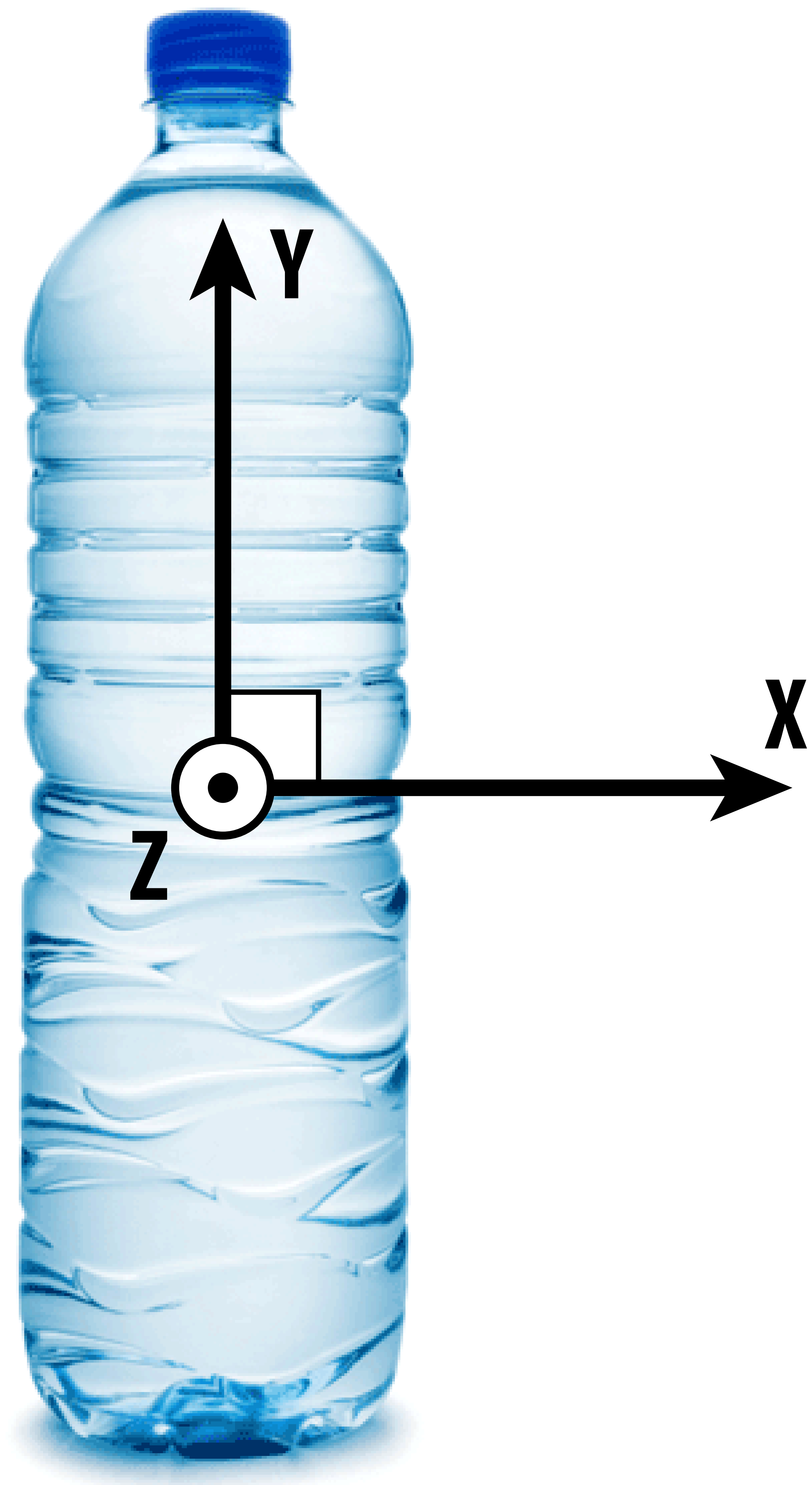
- **In 2D, order of rotations doesn't matter:**



Same result! ("2D rotations commute")

Commutativity of rotations—3D

- What about in 3D?
- IN-CLASS ACTIVITY:
  - Rotate 90° around Y, then 90° around Z, then 90° around X
  - Rotate 90° around Z, then 90° around Y, then 90° around X
  - (Was there any difference?)

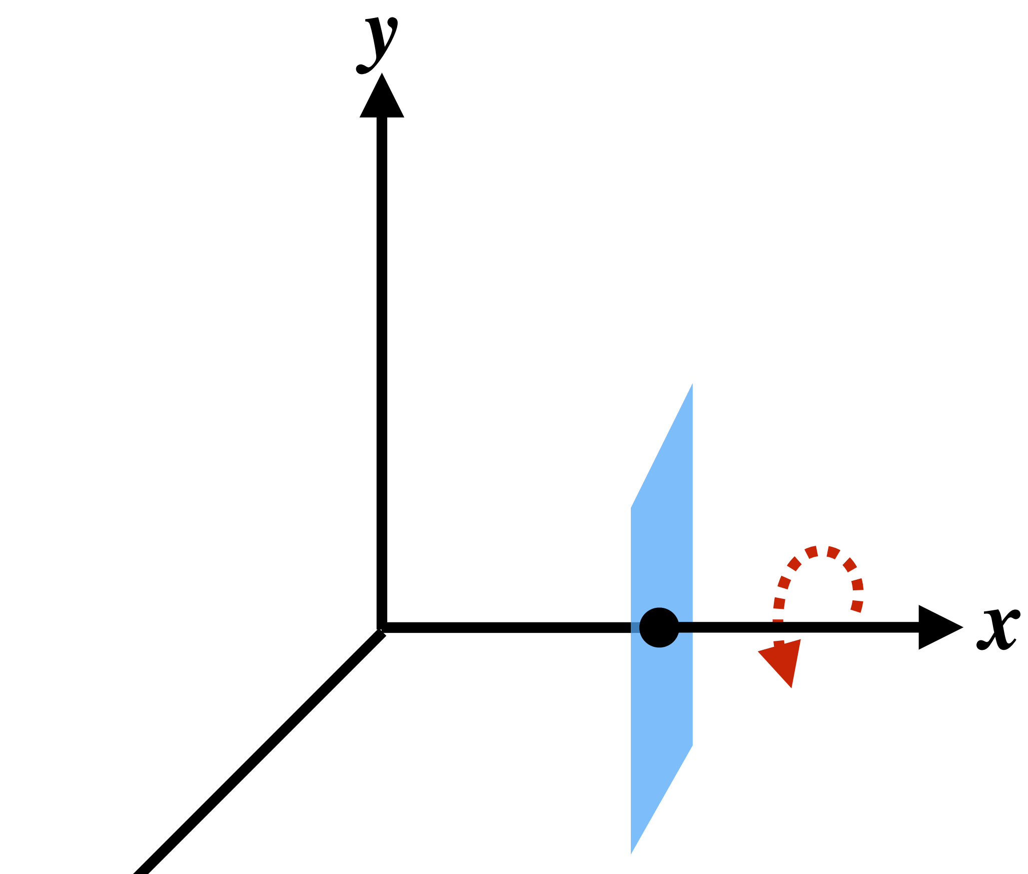


CONCLUSION: bad things can happen if we're not careful about the order in which we apply rotations!

Rotations in 3D

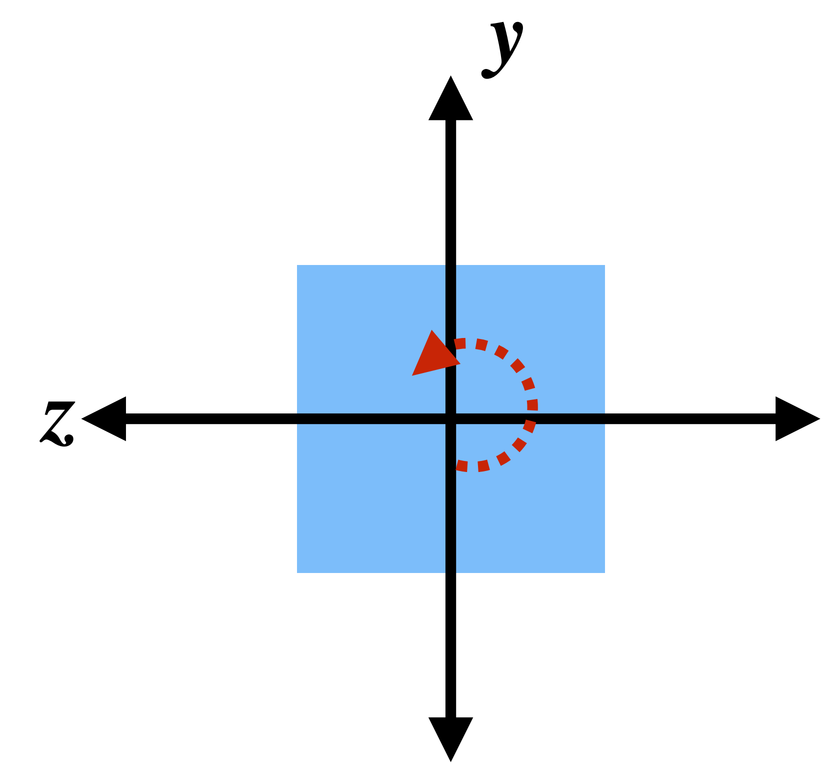
Rotation about x axis:

$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



x coordinate is unchanged by rotation about x

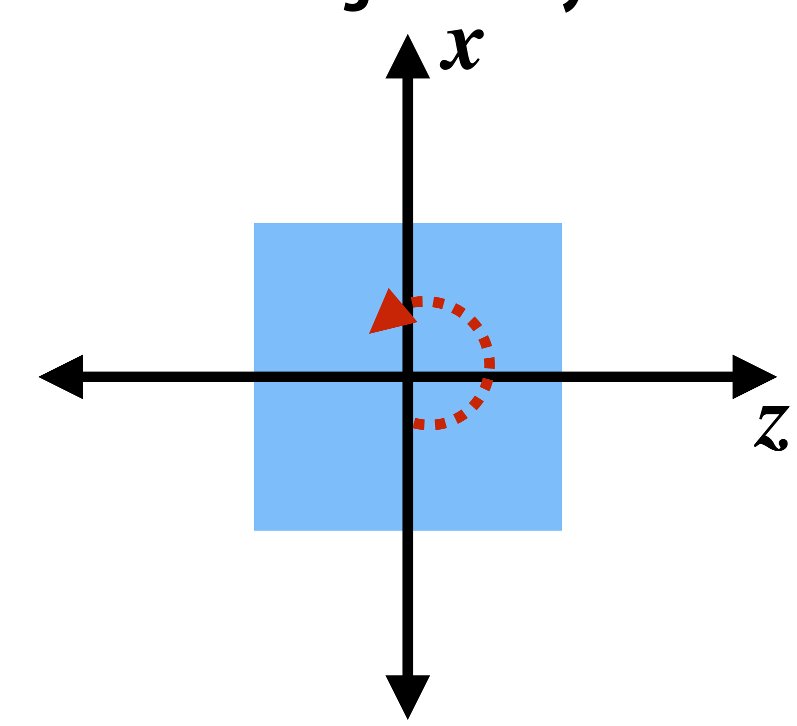
View looking down -x axis:



Rotation about y axis:

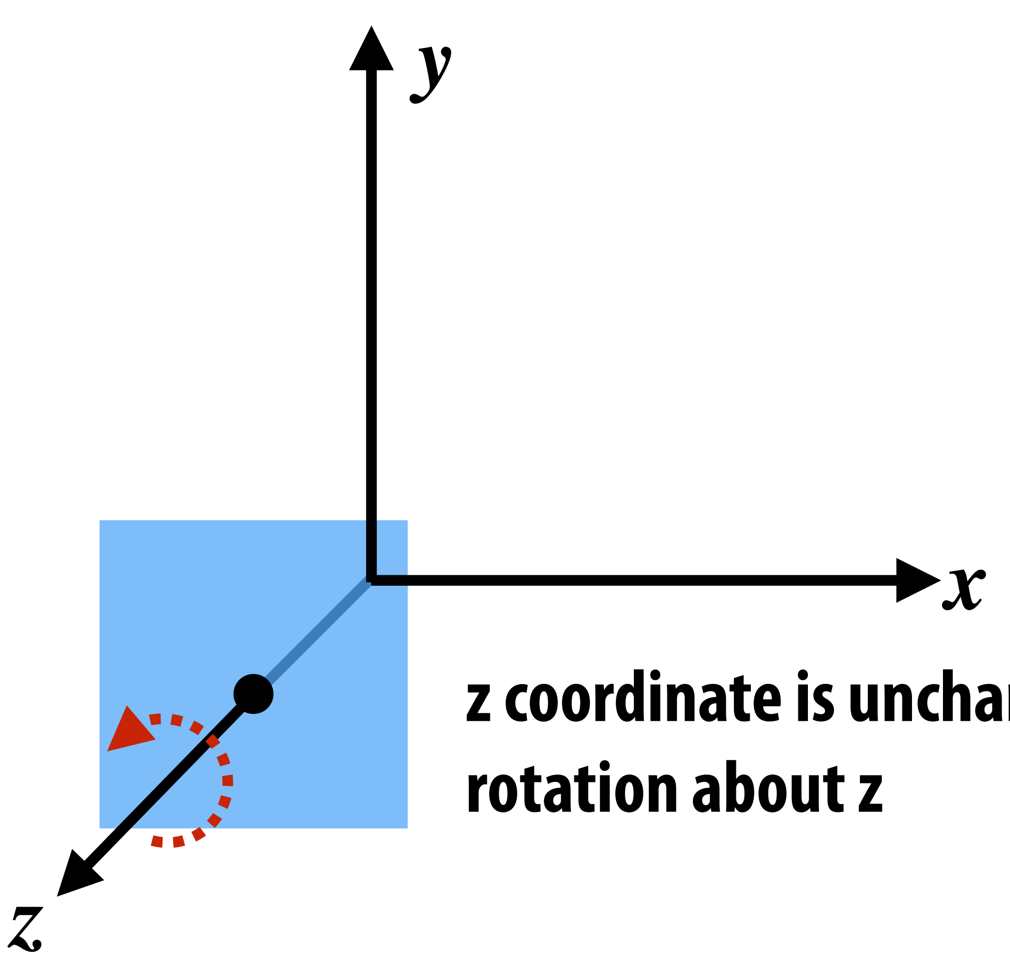
$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

View looking down -y axis:



Rotation about z axis:

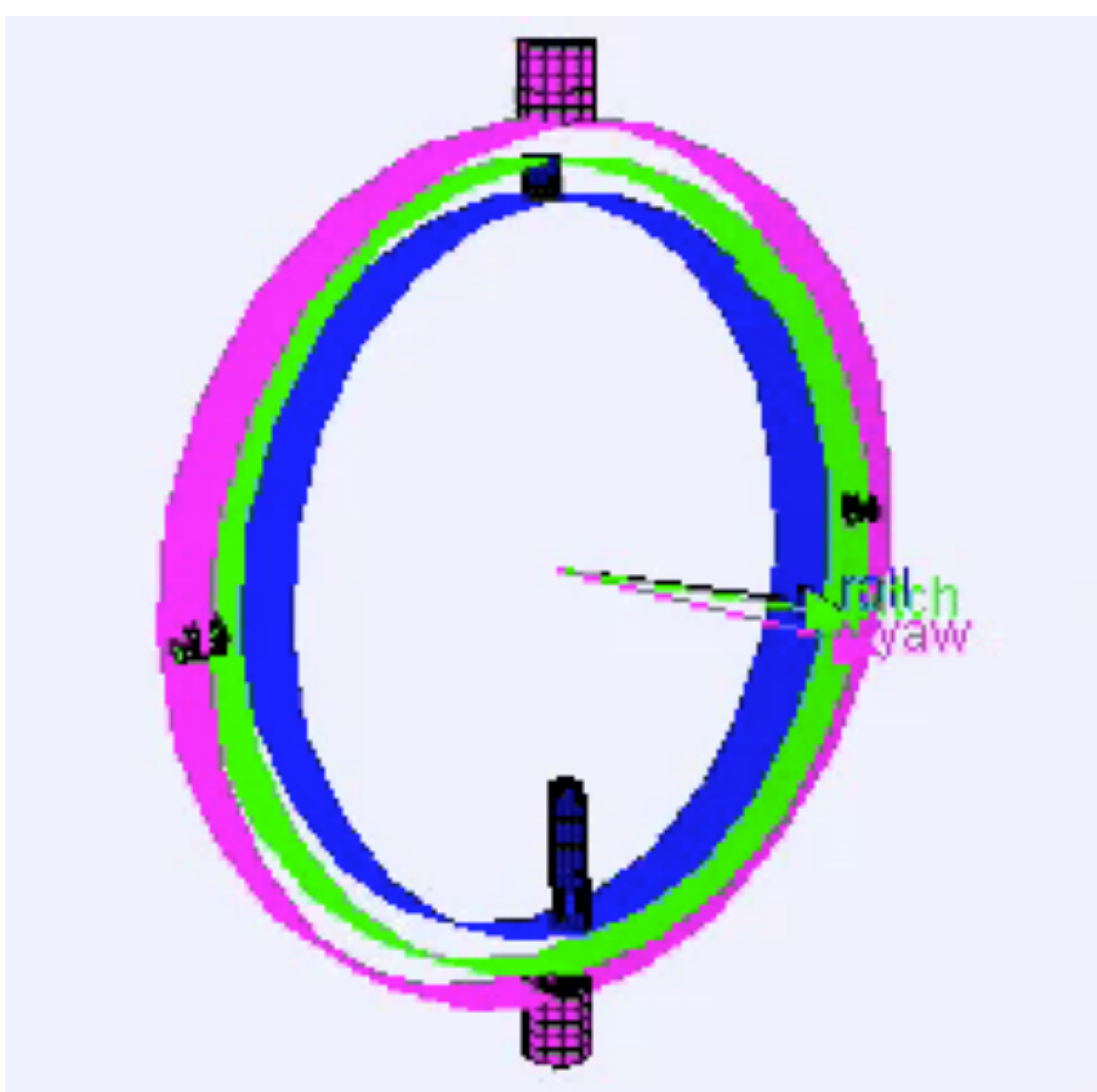
$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

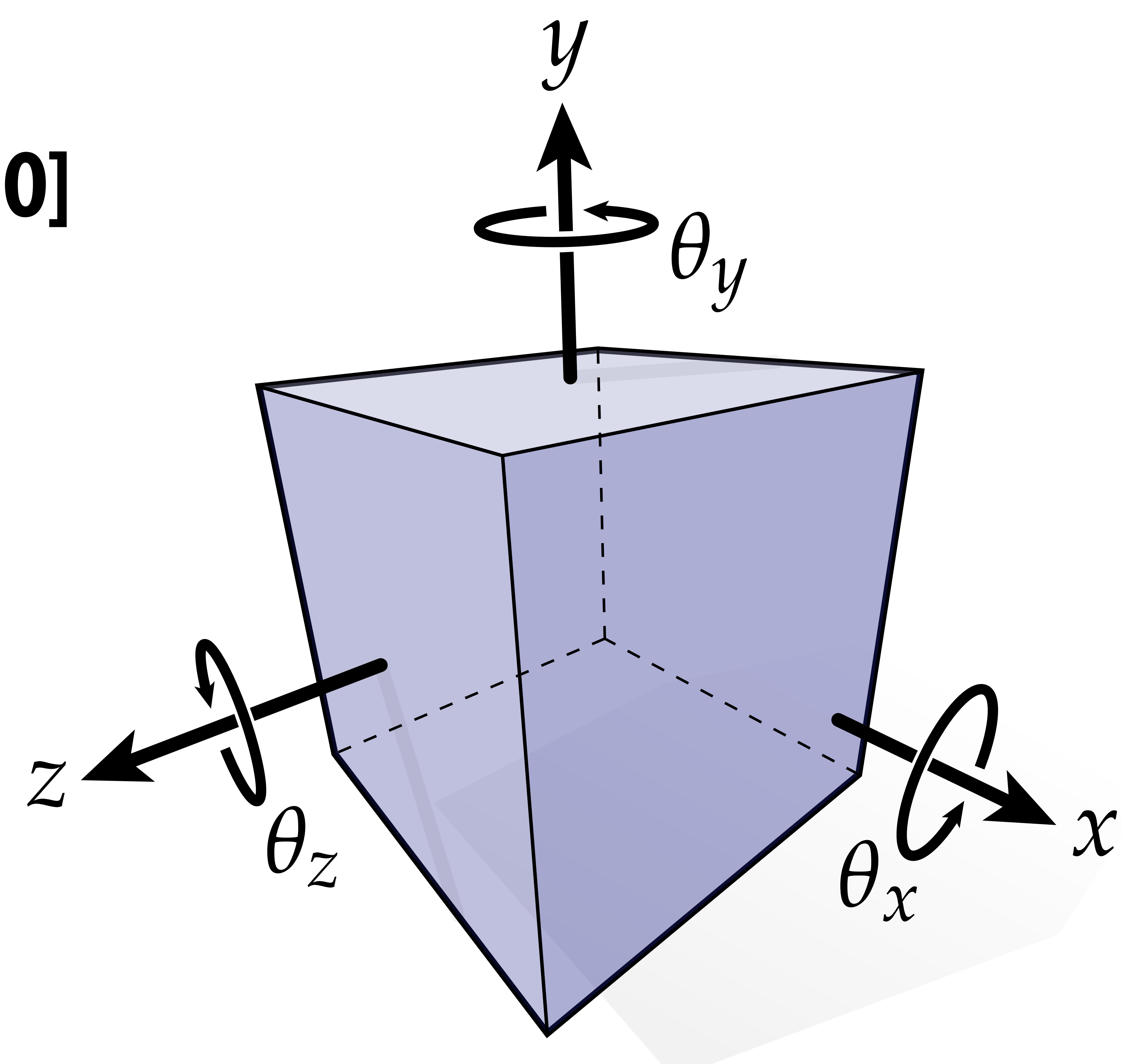


z coordinate is unchanged by rotation about z

Representing rotations in 3D—euler angles

- How do we express rotations in 3D?
- One idea: we know how to do 2D rotations
- Why not simply apply rotations around the three axes? (X,Y,Z)
- Scheme is called *Euler angles*
- PROBLEM: "Gimbal Lock" [DEMO]





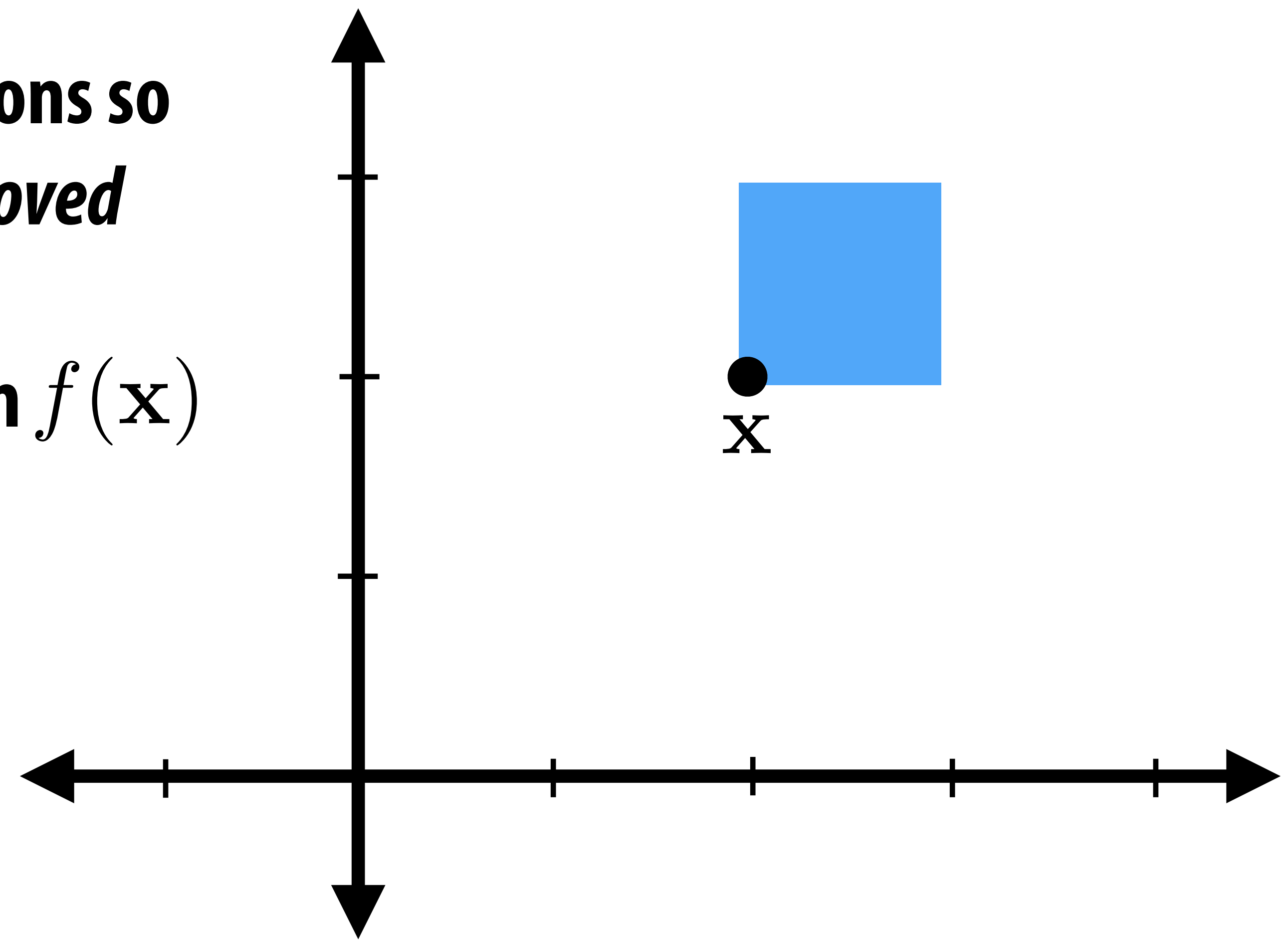
Alternative representations of 3D rotations

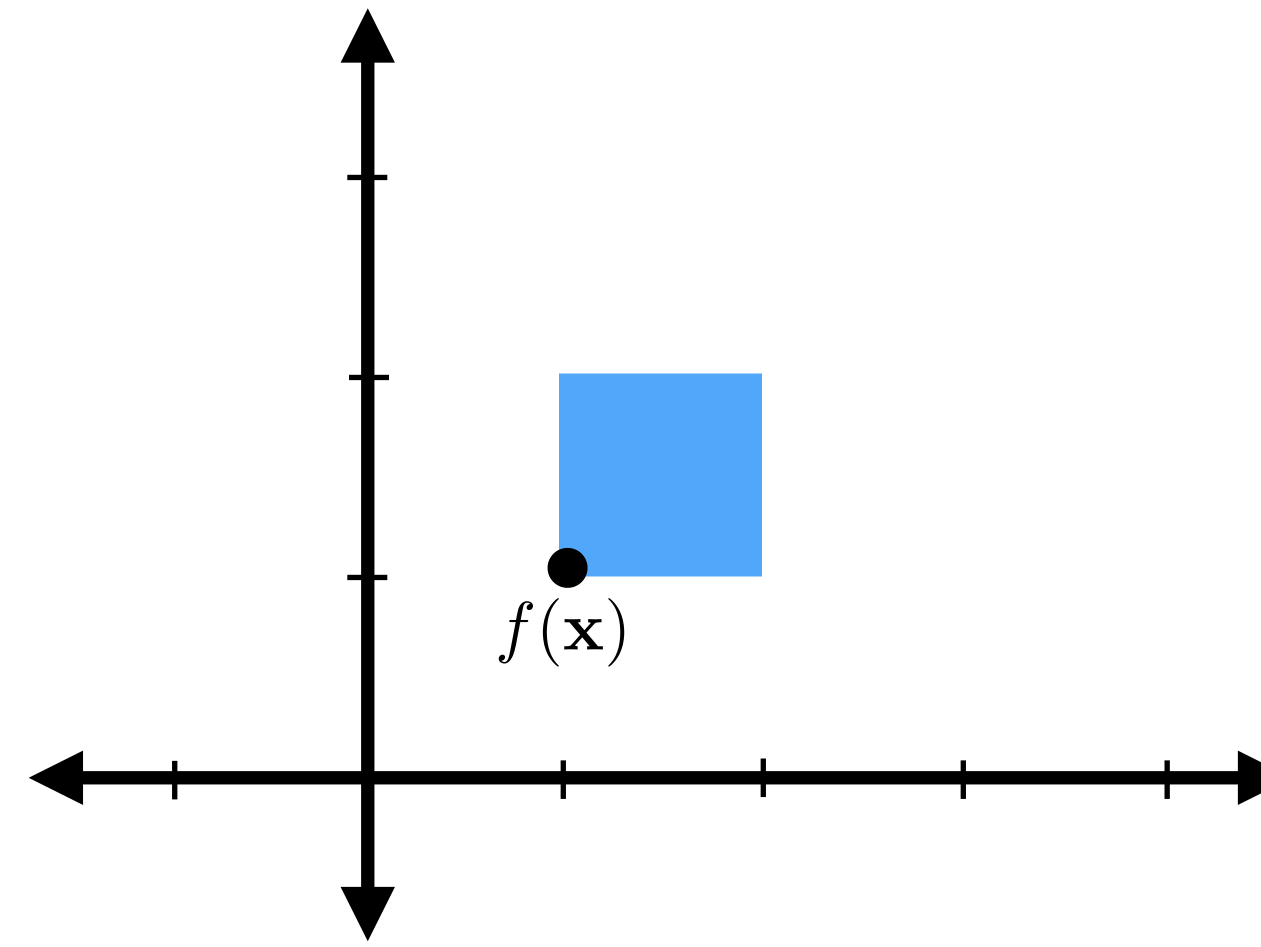
- **Axis-angle rotations**
- **Quaternions**

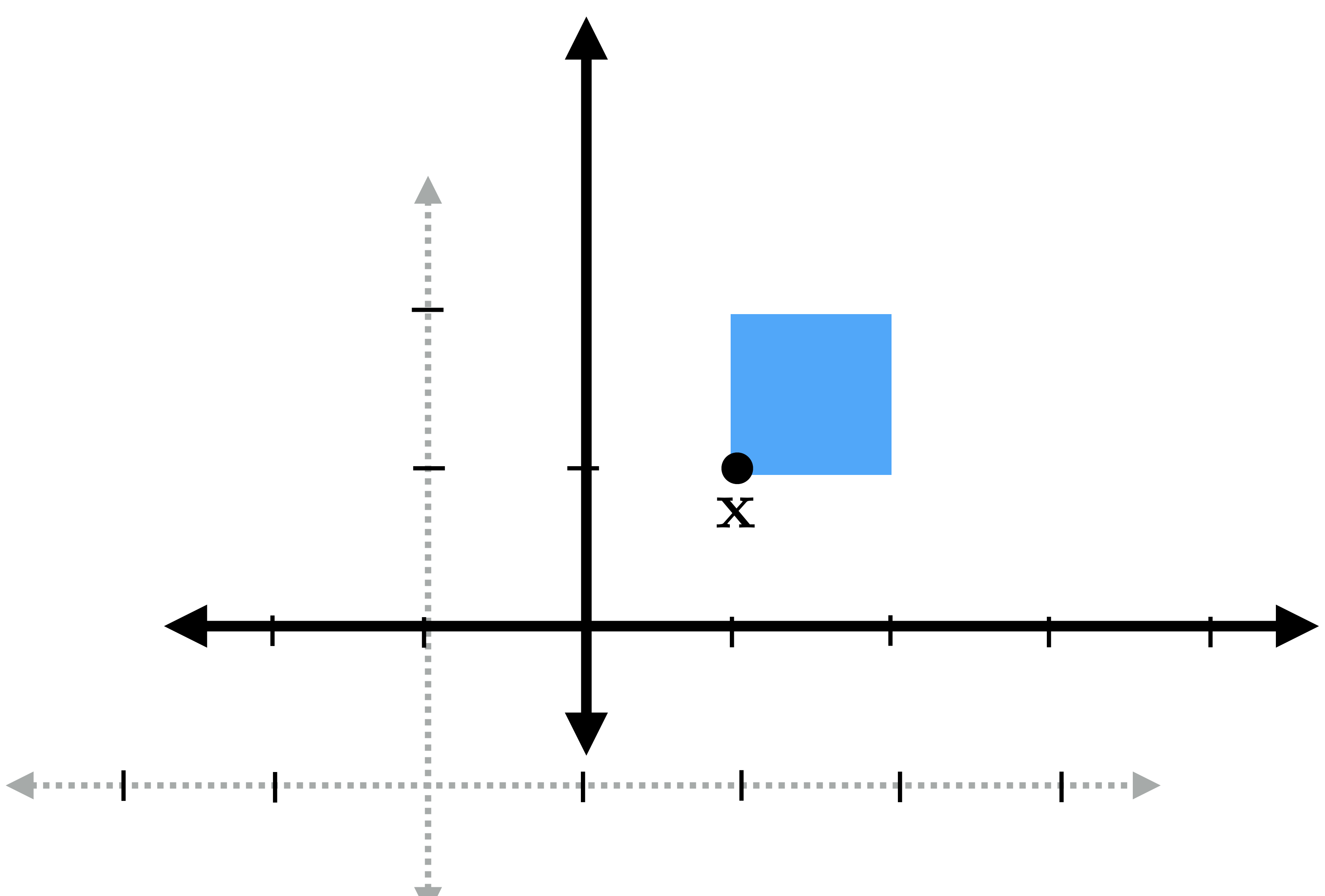
Another way to think about transformations: change of coordinates

Interpretation of transformations so far in this lecture: ***points get moved***

Point $\mathbf{x}$ moved to new position $f(\mathbf{x})$







Alternative interpretation:

Transformations induce of change of coordinate frame: Representation of $\mathbf{x}$ changes since point is now expressed in new coordinates

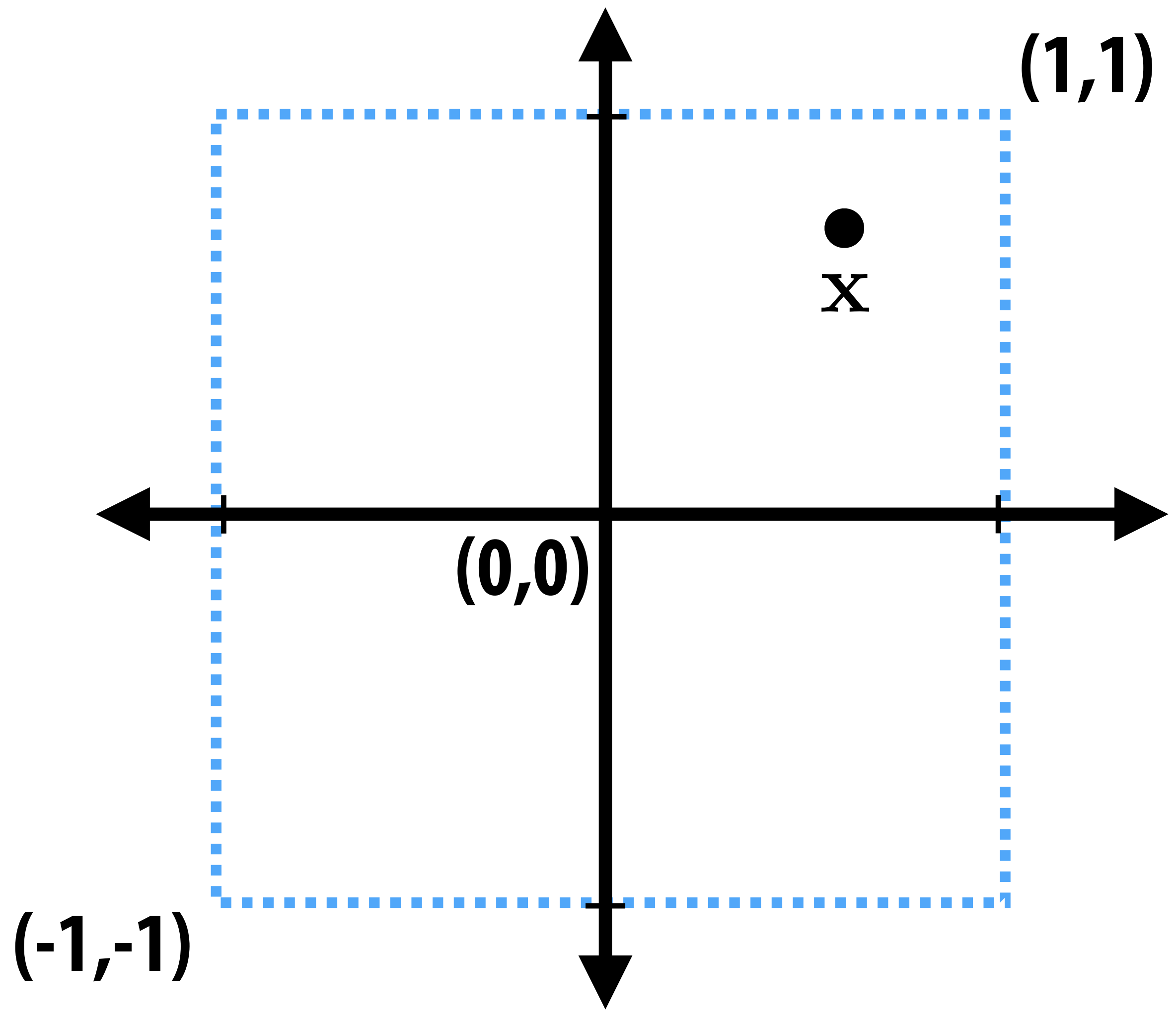
Screen transformation *

Convert points in normalized coordinate space to screen pixel coordinates

Example: all points within (-1,1) to (1,1) region are on screen

- **(1,1) in normalized space maps to (W,0) in screen space**
- **(-1,-1) in normalized space maps to (0,H) in screen space**

Normalized coordinate space:

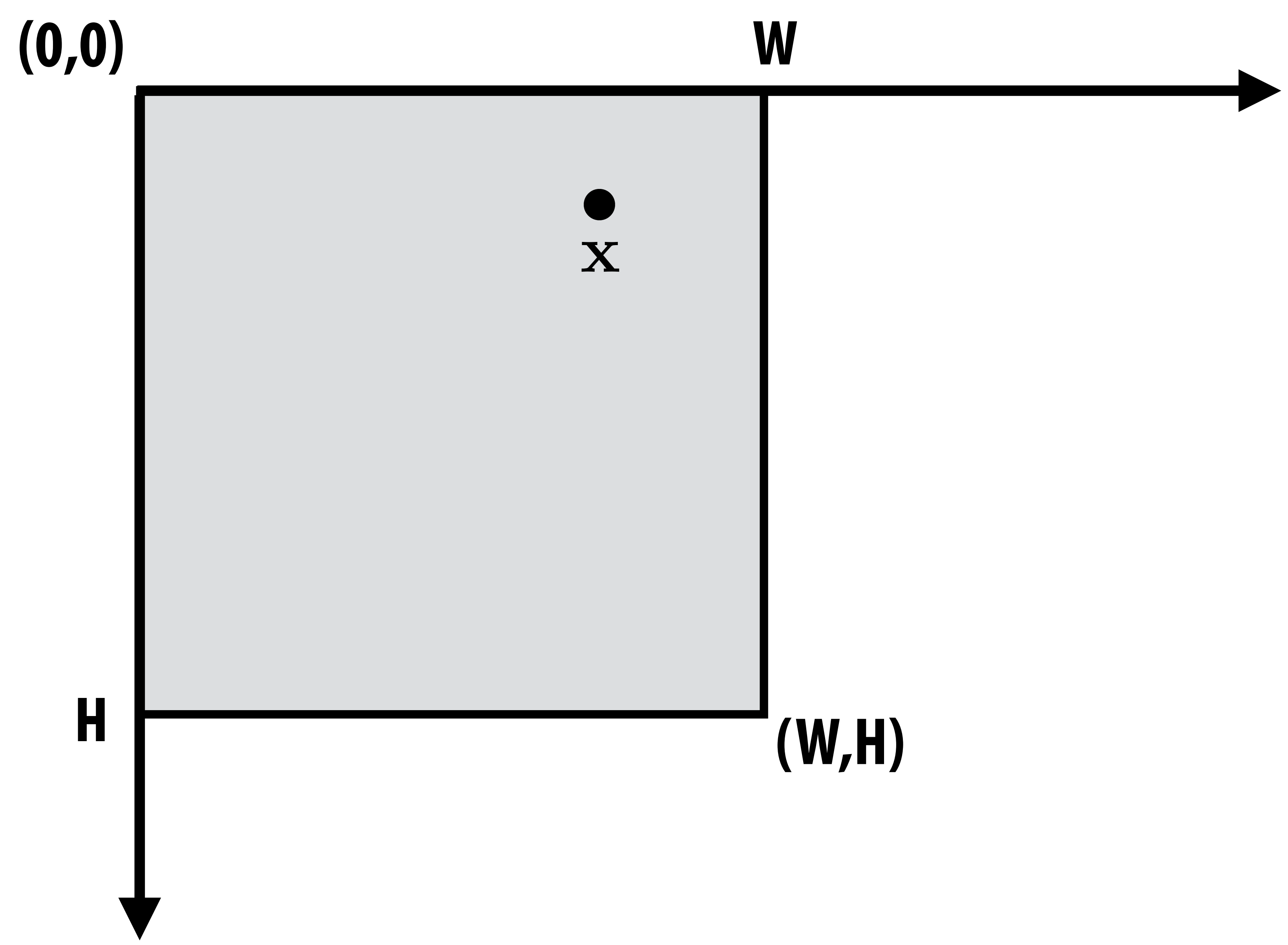


Step 1: reflect about x

Step 2: translate by (1,1)

Step 3: scale by (W/2,H/2)

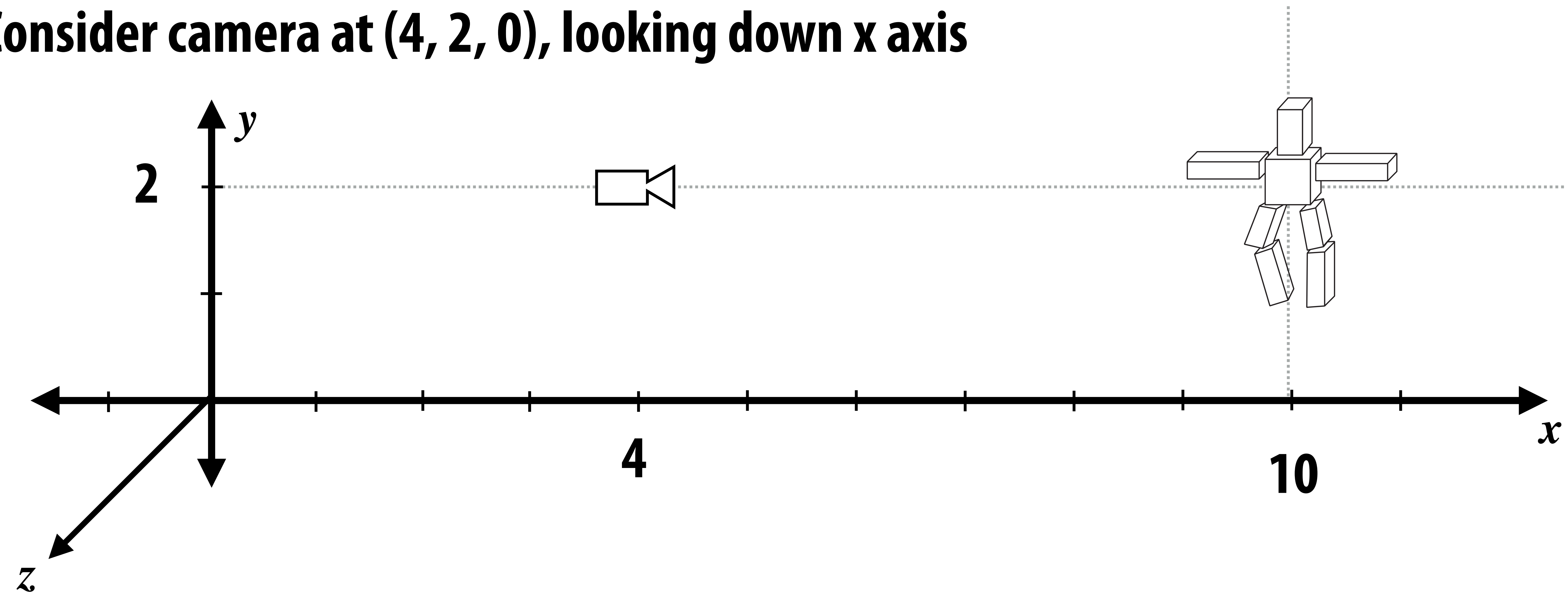
Screen (W x H output image) coordinate space:



* This slide adopts convention that top-left of screen is (0,0) to match SVG convention in Assignment 1.
Many 3D graphics systems like OpenGL place (0,0) in bottom-left. In this case what would the transform be?

Example: simple camera transform

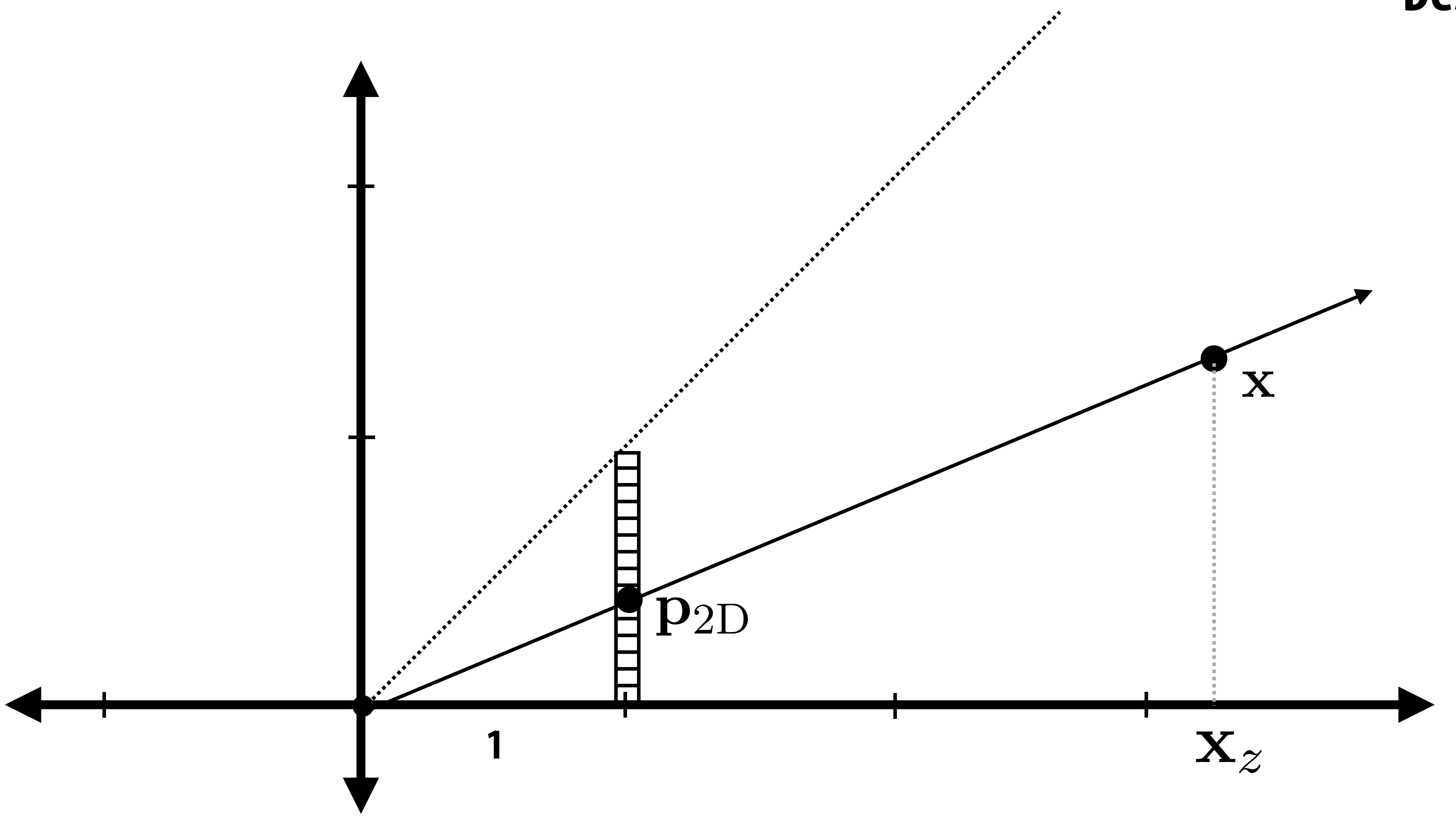
- **Consider object in world at (10, 2, 0)**
- **Consider camera at (4, 2, 0), looking down x axis**



- **Translating object vertex positions by (-4, -2, 0) yields position relative to camera.**
- **Rotation about *y* by $-\pi/2$ gives position of object in coordinate system where camera's view direction is aligned with the *z* axis ***

*** The convenience of such a coordinate system will become clear on the next slide!**

Basic perspective projection



Desired perspective projected result (2D point):

$$\mathbf{p}_{2\text{D}} = \begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z \end{bmatrix}^T$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Input: point in 3D-H $\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & 1 \end{bmatrix}$

After applying $\mathbf{P}$: point in 3D-H $\mathbf{P}\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & \mathbf{x}_z \end{bmatrix}^T$

After homogeneous divide: $\begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z & 1 \end{bmatrix}^T$

(throw out third component)

Assumption:
Pinhole camera at (0,0) looking down z

More about perspective in later lecture!

Transformations summary

- Transformations can be interpreted as operations that move points in space
  - e.g., for modeling, animation
- Or as a change of coordinate system
  - e.g., screen and view transforms
- Construct complex transformations as compositions of basic transforms
- Homogeneous coordinate representation allows for expression of non-linear transforms (e.g., affine, perspective projection) as matrix operations (linear transforms) in higher-dimensional space
  - Matrix representation affords simple implementation and efficient composition