Lecture #4: 6 April 2011
Topics: A Weak Upper Bound on Lower Envelopes of Line Segments
Scribe: Nan Hu

Lower Envelope of Segments

Let $\mathcal{L} = \{s_1, s_2, \cdots, s_n\}$ be a set of n nonvertical line segments in the plane, in general position. By general position we mean that any two segments intersect in at most a single point, and that no three segments share a single point. Let $\mathcal{A}(\mathcal{L})$ denote the arrangement of $\mathcal{L}$. An intersection point p between the interiors of two segments of $\mathcal{L}$ is said to be a *k-level inner vertex* of $\mathcal{A}(\mathcal{L})$ if there are exactly k segments in $\mathcal{L}$ passing strictly below p. (We refer to such vertices as inner vertices, to distinguish them from the endpoints of the segments.) Let $C_k(\mathcal{L})$ denote the number of k-level inner vertices in $\mathcal{A}(\mathcal{L})$, and let $C_k(n)$ denote the maximum of $C_k(\mathcal{L})$ over all sets of n segments in the plane, in general position. We want to bound $C_0(n)$, the maximum number of inner vertices on the lower envelope of $\mathcal{L}$. It is well known that $C_0(n) = \Theta(n\alpha(n))$; here we only prove a weaker upper bound of $O(n \log n)$. For a similar proof of the theorem from a probabilistic point of view, interested readers are referred to [1].

Theorem 1. *For n line segments in the plane, the complexity of the lower envelope is $O(n \log n)$.*

Proof:

1. Let p be a 0-level inner vertex of $\mathcal{A}(\mathcal{L})$. We sweep a vertical line from p to the right, and stop the sweeping as soon as we encounter one of the following two types of events (see Fig. 1):

(i) We reach an endpoint of one of the segments.
(ii) We reach a 1-level vertex.

An important observation is that the sweeping line can never reach a 0-level inner vertex before any of these events are encountered, so we can reach every such event only from (at most) one 0-level vertex. Let $D(\mathcal{L})$ denote the number of events of type (i); we clearly have $D(\mathcal{L}) < 2n$, which leads to the inequality

$$C_0(\mathcal{L}) \leq C_1(\mathcal{L}) + D(\mathcal{L}) < C_1(\mathcal{L}) + 2n. \tag{1}$$

2. We now consider removing each of the line segments in turn. Let $\mathcal{L}_i$ be the set of line segments with s_i removed, namely, $\mathcal{L}_i = \{s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n\}$, $c_i = C_0(\mathcal{L}_i)$, and $C = \sum_i c_i$. Each 0-level inner vertex p of $\mathcal{A}(\mathcal{L})$ will appear as a 0-level vertex of

Figure 1: Sweeping from a 0-level inner vertex: terminal events of type (i) and (ii).

$\mathcal{A}(\mathcal{L}_i)$ if and only if the two intersecting segments containing p are in $\mathcal{L}_i$. This happens $(n-2)$ times in all $\mathcal{L}_i$'s. Similarly, each 1-level inner vertex p' of $\mathcal{A}(\mathcal{L})$ will appear as a 0-level vertex of $\mathcal{A}(\mathcal{L}_i)$ if and only if the segment in $\mathcal{L}$ below p' is not chosen in $\mathcal{L}_i$. This will happen once for each p'. Besides that, no other inner vertex of $\mathcal{A}(\mathcal{L})$ can appear as a 0-level vertex of $\mathcal{A}(\mathcal{L}_i)$. Above all, we will have

$$C = (n-2)C_0(\mathcal{L}) + C_1(\mathcal{L}). \tag{2}$$

3. From (1) and (2), we have

$$\begin{aligned}(n-1)C_0(\mathcal{L}) &= (n-2)C_0(\mathcal{L}) + C_0(\mathcal{L})\\ &\le (n-2)C_0(\mathcal{L}) + C_1(\mathcal{L}) + 2n\\ &= C + 2n.\end{aligned}$$

While $C_0(n) = \max_{\mathcal{L}} C_0(\mathcal{L})$, clearly we will have

$$(n-1)C_0(n) = \max_{\mathcal{L}} C_0(\mathcal{L}) \le C + 2n \le nC_0(n-1) + 2n. \tag{3}$$

Therefore, we would have the following recursive relationship (by simply dividing $n(n-1)$ on both sides of (3))

$$\frac{C_0(n)}{n} - \frac{C_0(n-1)}{n-1} \le \frac{2}{n-1}.$$

Summing over all n, and note that $C_0(2) = 1$, we have

$$\begin{aligned}C_0(n) &\le n\left(C_0(2) + 2\sum_{i=1}^{n-1}\frac{1}{i}\right)\\ &= n\left(1 + 2H_{n-1}\right) = O(n\log n)\end{aligned}$$

An event on the lower envelope will be to encounter either i) an endpoint or ii) a 0-level inner vertex. As the total number of endpoints on the lower envelope is bounded by $2n$, the overall complexity of the lower envelope will be $O(n\log n)$. Q.E.D.

References

[1] Boaz Tagansky. *A New Technique for Analyzing Substructures in Arrangements of Piecewise Linear Surfaces*. Discrete and Computational Geometry, 16(5), pp. 455–479, 1996.