Lecture #4:	Notes from the 2011 offering
Topics:	A Weak Upper Bound on Lower Envelopes of Line Segments
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Lower Envelope of Segments

Let $\mathcal{L} = \{s_1, s_2, \dots, s_n\}$ be a set of *n* nonvertical line segments in the plane, in general position. By general position we mean that any two segments intersect in at most a single point, and that no three segments share a single point. Let $\mathcal{A}(\mathcal{L})$ denote the arrangement of \mathcal{L} . An intersection point *p* between the interiors of two segments of \mathcal{L} is said to be a *k*-level inner vertex of $\mathcal{A}(\mathcal{L})$ if there are exactly *k* segments in \mathcal{L} passing strictly below *p*. (We refer to such vertices as inner vertices, to distinguish them from the endpoints of the segments.) Let $C_k(\mathcal{L})$ denote the number of *k*-level inner vertices in $\mathcal{A}(\mathcal{L})$, and let $C_k(n)$ denote the maximum of $C_k(\mathcal{L})$ over all sets of *n* segments in the plane, in general position. We want to bound $C_0(n)$, the maximum number of inner vertices on the lower envelope of \mathcal{L} . It is well known that $C_0(n) = \Theta(n\alpha(n))$; here we only prove a weaker upper bound of $O(n \log n)$. For a similar proof of the theorem from a probabilistic point of view, interested readers are referred to [1].

Theorem 1. For *n* line segments in the plane, the complexity of the lower envelope is $O(n \log n)$.

Proof:

1. Let *p* be a 0-level inner vertex of $\mathcal{A}(\mathcal{L})$. We sweep a vertical line from *p* to the right, and stop the sweeping as soon as we encounter one of the following two types of events (see Fig. 1):

- (i) We reach an endpoint of one of the segments.
- (ii) We reach a 1-level vertex.

An important observation is that the sweeping line can never reach a 0-level inner vertex before any of these events are encountered, so we can reach every such event only from (at most) one 0-level vertex. Let $D(\mathcal{L})$ denote the number of events of type (i); we clearly have $D(\mathcal{L}) < 2n$, which leads to the inequality

$$C_0(\mathcal{L}) \le C_1(\mathcal{L}) + D(\mathcal{L}) < C_1(\mathcal{L}) + 2n.$$
(1)

2. We now consider removing each of the line segments in turn. Let \mathcal{L}_i be the set of line segments with s_i removed, namely, $\mathcal{L}_i = \{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n\}$, $c_i = C_0(\mathcal{L}_i)$, and $C = \sum_i c_i$. Each 0-level inner vertex p of $\mathcal{A}(\mathcal{L})$ will appear as a 0-level vertex of

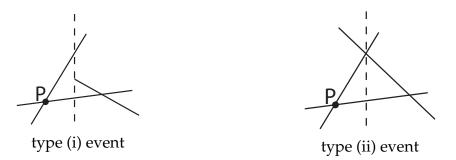


Figure 1: Sweeping from a 0-level inner vertex: terminal events of type (i) and (ii).

 $\mathcal{A}(\mathcal{L}_i)$ if and only if the two intersecting segments containing p are in \mathcal{L}_i . This happens (n-2) times in all \mathcal{L}_i 's. Similarly, each 1-level inner vertex p' of $\mathcal{A}(\mathcal{L})$ will appear as a 0-level vertex of $\mathcal{A}(\mathcal{L}_i)$ if and only if the segment in \mathcal{L} below p' is not chosen in \mathcal{L}_i . This will happen once for each p'. Besides that, no other inner vertex of $\mathcal{A}(\mathcal{L})$ can appear as a 0-level vertex of $\mathcal{A}(\mathcal{L}_i)$. Above all, we will have

$$C = (n-2)C_0(\mathcal{L}) + C_1(\mathcal{L}).$$
 (2)

3. From (1) and (2), we have

$$(n-1)C_0(\mathcal{L}) = (n-2)C_0(\mathcal{L}) + C_0(\mathcal{L})$$

$$\leq (n-2)C_0(\mathcal{L}) + C_1(\mathcal{L}) + 2n$$

$$= C + 2n.$$

While $C_0(n) = \max_{\mathcal{L}} C_0(\mathcal{L})$, clearly we will have

$$(n-1)C_0(n) = \max_{\mathcal{L}} C_0(\mathcal{L}) \le C + 2n \le nC_0(n-1) + 2n.$$
(3)

Therefore, we would have the following recursive relationship (by simply dividing n(n-1) on both sides of (3))

$$\frac{C_0(n)}{n} - \frac{C_0(n-1)}{n-1} \le \frac{2}{n-1}.$$

Summing over all *n*, and note that $C_0(2) = 1$, we have

$$C_0(n) \le n \left(C_0(2) + 2 \sum_{i=1}^{n-1} \frac{1}{i} \right)$$

= $n (1 + 2H_{n-1}) = O(n \log n)$

An event on the lower envelope will be to encounter either i) an endpoint or ii) a 0level inner vertex. As the total number of endpoints on the lower envelope is bounded by 2n, the overall complexity of the lower envelope will be $O(n \log n)$. Q.E.D.

References

 Boaz Tagansky. A New Technique for Analyzing Substructures in Arrangements of Piecewise Linear Surfaces. Discrete and Computational Geometry, 16(5), pp. 455– 479, 1996.