The Zone Theorem Revisited

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Abstract

We give a simple proof for the tight(!) upper bound of $\lfloor 9.5n \rfloor - 3$ for the complexity of a zone of a line in an arrangement of $n+1$ lines in the affine plane.

1 Introduction

Let $\mathcal{L}$ be an arrangement of $n+1$ lines in general position in the affine plane, that is, no two lines are parallel and no three pass through a common point. The *zone* of a line ℓ in $\mathcal{L}$ is the collection of all faces in $\mathcal{L}$ supported by ℓ. The complexity of the zone of ℓ is the sum of the sizes of all faces in the zone of ℓ, where the size of a face is the number of its edges (those edge may be bounded or unbounded in case of an unbounded face).

The notion of a zone of a line was defined in [CGL85] where an upper bound of $10n+2$ is shown for the complexity of the zone of a line in an arrangement of $n+1$ lines. This bound was further improved to $\lfloor 9.5n \rfloor - 1$ in [BEY91] and shown to be best possible up to small additive factor.

In this paper we give a simpler proof of the latter upper bound and even improve it by two units to be optimal. Our proof in fact follows the approach in [CGL85] more closely than the one in [BEY91].

Theorem 1. *The complexity of a zone of a line in an arrangement of $n+1$ lines in general position in the plane is at most $\lfloor 9.5n \rfloor - 3$.*

We note that a construction with zone complexity $\lfloor 9.5n \rfloor - 3$ is introduced in [BEY91], implying that Theorem 1 is tight. Yet, we consider the merit of this paper also for the simplified proof of the upper bound and not only for the small improvement in the bound. Having said this we note that perhaps one third of the proof is devoted to getting the optimal additive factor in the bound. The proof can be shortened significantly if one just wants to obtain a bound of the form $9.5n + O(1)$, and the reader will notice this easily.

2 The Proof

We follow the proof in [CGL85]. We consider ℓ to be horizontal line. Let F be a face supported by ℓ such that F lies above ℓ. If F is bounded we call the two edges of F incident to the highest vertex

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of F *top edges* (of F). If F is unbounded, then the unbounded edge(s) of F are top edges. The other edges of F are divided into left edges and right edges (of F) according to whether they are in the left or right component of the boundary of F minus ℓ and the top edges of F. (see Figure 1). If F lies below ℓ we analogously define two *bottom edges* as well as left and right edges (of F).

The basic observation in [CGL85] is that every line in the arrangement may contain at most one left edge and at most one right edge of some faces in the zone of ℓ that lie above ℓ and similarly at most one left edge and at most one right edge of some faces in the zone of ℓ below ℓ. This count gives the upper bound of $10n + 2$ in [CGL85] ($2n$ top edges, $2n$ bottom edge, $2n$ right edges, $2n$ left edges, and $2(n + 1)$ edges on ℓ).

To prove the upper bound in Theorem 1 we will show that for every line in $\mathcal{L}$ that contains two left edges and two right edges there corresponds a unique "missing" right or left edge on one of the other lines in $\mathcal{L}$.

For every line $t \in \mathcal{L} \setminus \{\ell\}$ we denote the first bounded (if exists) edge on t above the line ℓ by $e_1(t)$. We denote by $e_2(t)$ the first bounded (if exists) edge on t below the line ℓ. Observe that each of $e_1(t)$ and $e_2(t)$ serves as either a left or a right edge in some face in the zone of ℓ (see Figure 1).

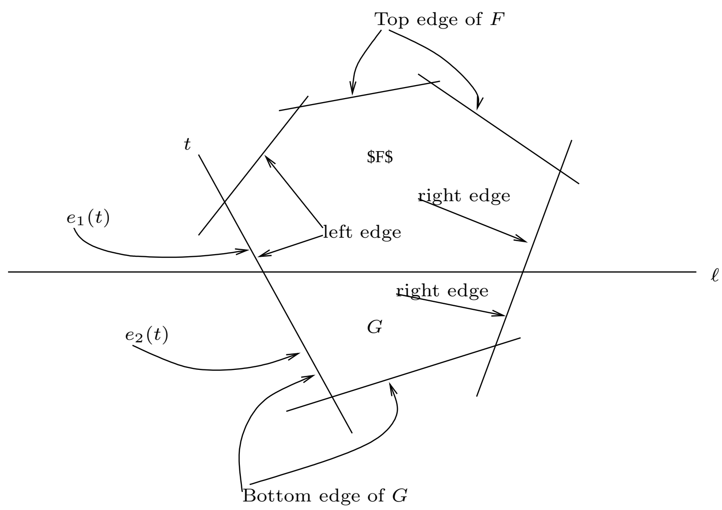

Figure 1: Some terminology in the proof

Let $m \in \mathcal{L} \setminus \{\ell\}$ be a line that contains two left edges and two right edges of some faces in the zone of ℓ. Let z denote the intersection point of ℓ and m. Denote by x_1 and x_2 the vertices of $e_1(m)$ and $e_2(m)$, respectively, different from z. Without loss of generality assume that $e_1(m)$ is a left edge of some face. We claim that $e_2(m)$ must be a right edge of some face. Indeed, assume to the contrary that $e_2(m)$ is also a left edge, then it is easy to see that m may contain at most one right edge, contradicting our assumption (see Figure 2).

Therefore, we assume that $e_2(m)$ is a right edge. Let f_1 and f_2 be the other right and left edges, respectively, on m. Hence f_1 lies above ℓ and f_2 lies below ℓ. Let y_1 and w_1 denote the vertices of f_1, where w_1 is closer to ℓ than y_1, and let y_2 and w_2 denote the vertices of f_2, where w_2 is closer to ℓ than y_2 (see Figure 3).

For a line t we denote by $arg(t)$ the angle that the half line of t above the x-axis creates with the positive part of the x-axis. Let s_1 be the line crossing m at a point on the line segment $[x_1, w_1]$ and such that $arg(s_1)$ is minimum. Let s_2 be the line crossing m at a point on the line segment $[x_2, w_2]$

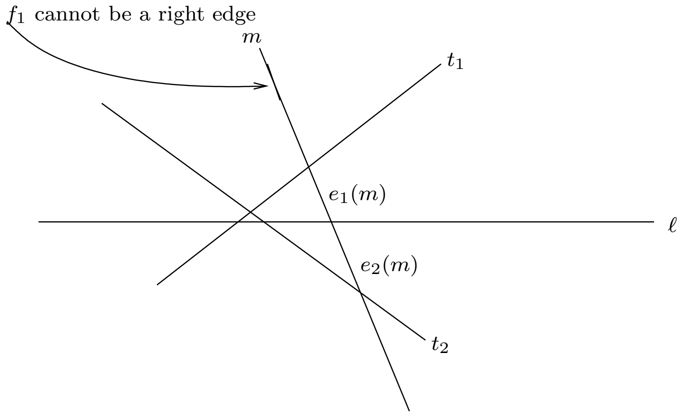

Figure 2: An impossible case.

such that $arg(s_2)$ is minimum (if, to start with, $e_1(m)$ is a right edge, then replace minimum by maximum when defining s_1 and s_2).

Without loss of generality assume that $arg(s_1) > arg(s_2)$. We will show that s_1 does not contain a left edge above ℓ (if $arg(s_1) < arg(s_2)$ one can just rotate the plane at 180 degrees or equivalently show that s_2 does not contain a right edge below ℓ). Observe that $e_1(s_1)$ is a right edge or else we get a contradiction to the minimality of s_1. Because f_1 is a right edge there is a line $k \in \mathcal{L}$ that passes through y_1 and crosses ℓ at a point z' to the right of z. z' must also be to the right of the intersection point of s_2 and ℓ, or else f_2 is not a left edge of a face supported by ℓ. If s_1 contained a left edge g, then g must be contained in the interval between the crossing points of s_1 with ℓ and k. However, in this case the line that crosses s_1 at the higher vertex of g contradicts the minimality of s_1 (see the dashed line in Figure 3).

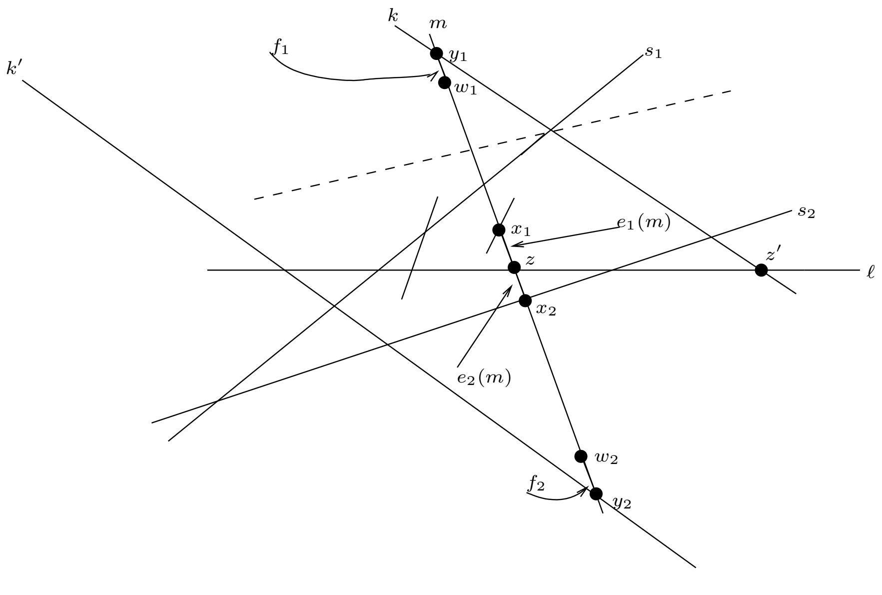

Figure 3: The missing edge.

Before continuing to showing that s_1 is uniquely assigned to the line m we make two simple observations about s_1 that will help up to obtain a tight bound up to the additive factor as well. The first observation is that s_1 does not contain a top infinite edge of a face supported by ℓ (see Figure 3). In particular s_1 is never the line such that $arg(s_1)$ is maximum possible or minimum possible. The second observation is that the crossing point of s_1 with ℓ is never the left most or the right most on ℓ. We refer the reader to Figure 3 where it is evident that the crossing point of s_1 with ℓ is to the left of z and z'. To see that this crossing point is not the left most observe that because f_2 is a left edge there is a line $k' \in \mathcal{L}$ through y_2 that crosses ℓ to the left of the crossing point of ℓ and s_1 for otherwise f_1 cannot be a right edge of a face supported by ℓ (see Figure 3 again).

We will now show that s_1 does not correspond in this way to any other line m' that contains two right edges and two left edges. Suppose to the contrary that it does. We must have that $e_1(m')$ is a left edge and $e_2(m')$ is a right edge (as is the case with the line m, because the missing edge is a *LEFT* edge *ABOVE* ℓ). We may assume without loss of generality that m' crosses ℓ at a point z'' to the right of z. Then z'' must be to the left of the intersection point of s_2 and ℓ or else s_2 will contradict the minimality of s_1 with respect to m' (recall that $arg(s_2) < arg(s_1)$). It is important to observe that m' must cross m at a point above ℓ. For if m' crosses m below ℓ, then m' cannot contain a right edge of a face in the zone of ℓ that lies above ℓ. However, now it is evident that m cannot contain (f_2 as) a left edge below ℓ, contradicting our assumption (see Figure 4).

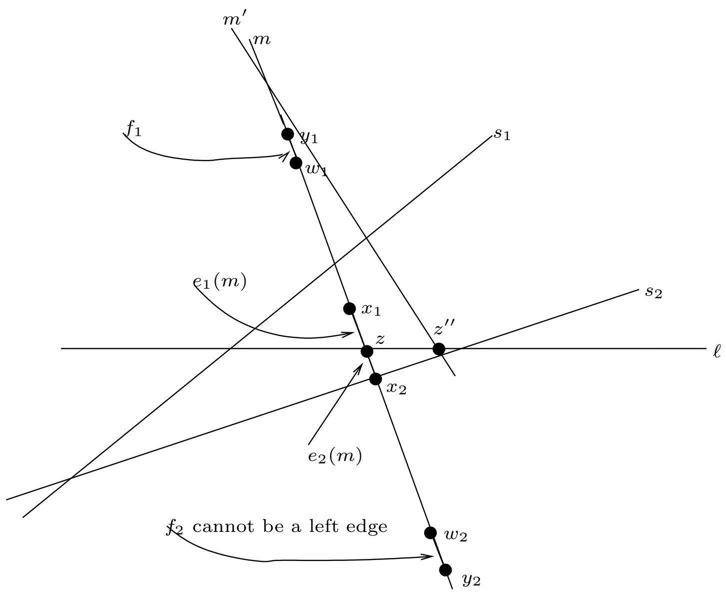

Figure 4: The missing edge is uniquely assigned to m.

We can now turn to counting part of the paper. We have $2n$ top edges and $2n$ bottom edges. We also have $2(n+1)$ edges on ℓ. It remains to estimate the number of right and left edges of faces supported by ℓ. Every line may contain at most two left edges and two right edges of such faces. We saw that for every line that contains the complete set of two left edges and two right edges there is a unique missing (left or right) edge on one of the other lines.

Let r_1 be the line such that $arg(r_1)$ is minimum and let r_2 be the line such that $arg(r_2)$ is maximum. Observe that r_1 and r_2 contain at most one right edge and one left edge each. Moreover by applying a suitable projective transformation taking to the line at infinity a line that is very close to a line (to become r_1) containing a complete set of two left and two right edges as in Figure 5 (it may take a moment to get convinced that the complexity of the zone of ℓ remains unchanged), we may assume that r_1 contains only one left edge and no right edges, as we may assume that r_1 creates together with ℓ an unbounded face with just two edges. Denote by r_1' the line such that $arg(r_1')$ is the second smallest (see Figure 5). r_1' contains an infinite top edge and no right edge above the line ℓ. As we already observed, s_1 in the proof is never one of r_1, r_1', or r_2. r_1, r_1', and r_2 contribute altogether at most 6 left and/or right edges to the count as follows: r_1 contributes exactly one left edge below ℓ. r_2 contributes one right edge below ℓ and one left edge above ℓ. r_1' cannot contribute a left edge above ℓ and therefore contributes at most 3 to the count.

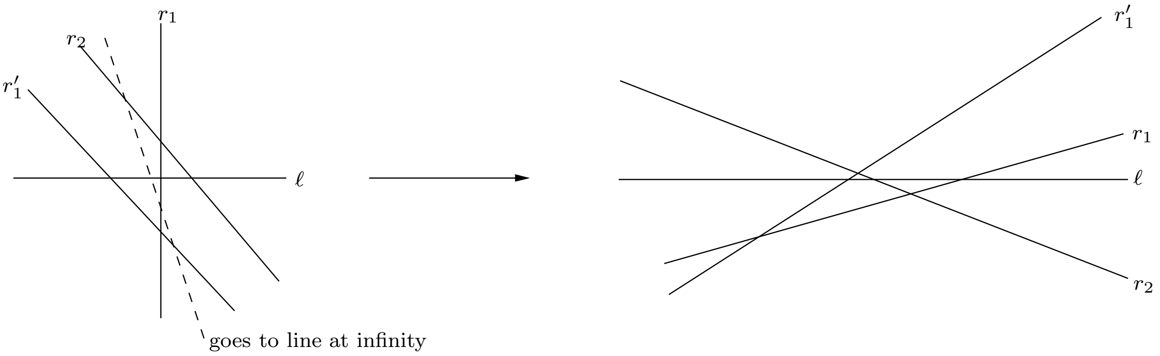

Figure 5: The lines r_1, r_1', and r_2.

Let r_3 denote the line that crosses ℓ at the left most vertex on ℓ. Notice that r_3 cannot be equal to r_1 or to r_2, as r_1' crosses ℓ to the left of the crossing points of ℓ with r_1 and r_2. Notice also that r_3 cannot contain a complete set of two left edges and two right edges, as such a line never crosses ℓ at an extreme intersection point on ℓ (see Figure 3). Moreover, as we observed earlier, s_1 in the proof is never r_3, for s_1 never crosses ℓ at an extreme crossing point on ℓ.

Case 1. $r_3 \neq r_1'$. Notice that r_1, r_1', r_2, and r_3 contain altogether at most 9 left and/or right edges.

Denote by x_i $(i = 1, 2, 3, 4)$ the number of lines (other than r_1, r_1', r_2, and r_3) containing exactly i right and/or left edges of faces supported by ℓ. (Observe that every line $m \in \mathcal{L}$ must contain at least one left or right edge, namely $e_1(m)$ and/or $e_2(m)$, as at least one exists.)

We have $x_1 + x_2 + x_3 + x_4 = n - 4$. The correspondence between lines counted by x_4 and "missing" edges implies $x_4 \leq x_3 + 2x_2 + 3x_1$. We are interested in estimating the number of left and right edges on lines different from r_1, r_1', r_2 and r_3, that is, we are interested in the sum $4x_4 + 3x_3 + 2x_2 + x_1$. This is now quite easy:

$$\begin{aligned} 4x_4 + 3x_3 + 2x_2 + x_1 &\leq 4x_4 + 3x_3 + 2.5x_2 + 2x_1 = \\ &= 3.5(x_4 + x_3 + x_2 + x_1) + 0.5(x_4 - x_3 - 2x_2 - 3x_1) \leq \\ &\leq 3.5(n-4) = 3.5n - 14 \end{aligned}$$

We conclude that the complexity of the zone of ℓ is bounded from above by

$$2n + 2n + 2(n+1) + (\lfloor 3.5n - 14 \rfloor) + 9 = \lfloor 9.5n \rfloor - 3.$$

Case 2. $r_3 = r'_1$. In this case r'_1 contain precisely 2 left and/or right edges (in fact right edges only), implying that r_1, r'_1, and r_2 contain altogether at most 5 left and/or right edges.

Therefore, for x_1, x_2, x_3, x_4 as in the previous case, we have $x_1 + x_2 + x_3 + x_4 = n - 3$. The correspondence between lines counted by x_4 and "missing" edges implies $x_4 \le x_3 + 2x_2 + 3x_1$. Now the sum $4x_4 + 3x_3 + 2x_2 + x_1$ can be bounded as follows:

$$\begin{aligned}4x_4 + 3x_3 + 2x_2 + x_1 &\le 4x_4 + 3x_3 + 2.5x_2 + 2x_1 = \\ &= 3.5(x_4 + x_3 + x_2 + x_1) + 0.5(x_4 - x_3 - 2x_2 - 3x_1) \le \\ &\le 3.5(n-3) = 3.5n - 10.5\end{aligned}$$

We conclude that the complexity of the zone of ℓ is bounded from above by

$$2n + 2n + 2(n+1) + (\lfloor 3.5n - 10.5 \rfloor) + 5 = \lfloor 9.5n - 3.5 \rfloor \le \lfloor 9.5n \rfloor - 3.$$

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