IV.15 Error Measure

The surface simplification algorithm measures the error of an edge contraction as the sum of square distances of a point from a collection of planes. This section develops the details of this error measure.

Signed distance. A plane with unit normal vector v_i and offset δ_i contains all points p whose orthogonal projection to the line defined by v_i is $-\delta_i \cdot v_i$,

$$h_i = \{p \in \mathbb{R}^3 \mid p^T \cdot v_i = -\delta_i\},$$

as illustrated in Figure IV.1. The signed distance of a point $x \in \mathbb{R}^3$ from the plane h_i is

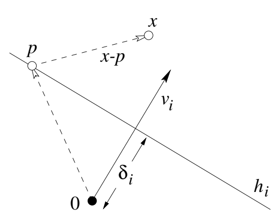

Figure IV.1: We use the unit normal vector to define the signed distance from h_i such that v_i points from the negative to the positive side.

$$\begin{aligned} d(x, h_i) &= (x-p)^T \cdot v_i \\ &= x^T \cdot v_i + \delta_i \\ &= \mathbf{x}^T \cdot \mathbf{v}_i, \end{aligned}$$

where $\mathbf{x}^T = (x^T, 1)$ and $\mathbf{v}_i^T = (v_i^T, \delta_i)$. In words, the signed distance in $\mathbb{R}^3$ can be expressed as a scalar product in $\mathbb{R}^4$ as illustrated in Figure IV.2.

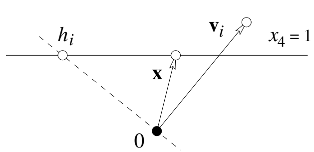

Figure IV.2: The 3-dimensional space $x_4 = 1$ is represented by the horizontal line. It contains point x and plane h_i, which in the 1-dimensional representation are both points.

Fundamental quadric. The sum of square distances of a point x from a collection of planes H is

$$\begin{aligned} E_H(x) &= \sum_{h_i \in H} d^2(x, h_i) \\ &= \sum_{h_i \in H} (\mathbf{x}^T \cdot \mathbf{v}_i) \cdot (\mathbf{v}_i^T \cdot \mathbf{x}) \\ &= \mathbf{x}^T \cdot \Big(\sum_{h_i \in H} \mathbf{v}_i \cdot \mathbf{v}_i^T \Big) \cdot \mathbf{x}, \end{aligned}$$

where

$$\mathbf{Q} = \sum \mathbf{v}_i \cdot \mathbf{v}_i^T = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}.$$

is a symmetric 4-by-4 matrix referred to as the *fundamental quadric* of the map $E_H : \mathbb{R}^3 \to \mathbb{R}$. The sum of square distances is non-negative, so $\mathbf{Q}$ is positive semi-definite. The error of an edge contraction is obtained from an error function like $E = E_H$. Let $\mathbf{x}^T = (x_1, x_2, x_3, 1)$ and note that

$$\begin{aligned} E(x) &= \mathbf{x}^T \cdot \mathbf{Q} \cdot \mathbf{x} \\ &= Ax_1^2 + Ex_2^2 + Hx_3^2 \\ &\quad + 2(Bx_1x_2 + Cx_1x_3 + Fx_2x_3) \\ &\quad + 2(Dx_1 + Gx_2 + Ix_3) \\ &\quad + J. \end{aligned}$$

We see that E is a quadratic map that is non-negative and unbounded. Its graph can only be an elliptic paraboloid as illustrated in Figure IV.3. In other words, the preimage of a constant error value ϵ, $E^{-1}(\epsilon)$, is an ellipsoid. Degenerate ellipsoids are possible, such as cylinders with elliptic cross-sections and pairs of planes.

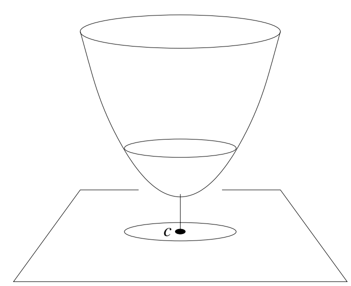

Figure IV.3: Illustration of $E = E_H$ in one lower dimension. The cross-section at a fixed height ϵ is an ellipse.

Error. The *error* of the edge contraction $ab \to c$ is the minimum value of $E(x) = E_H(x)$ over all $x \in \mathbb{R}^3$, where H is the set of planes spanned by triangles in the preimage of the star of the new vertex c. The geometric location of c is the point x that minimizes E. In the non-degenerate case, this point is unique and can be computed by setting the gradient $\nabla E = (\partial E/\partial x_1, \partial E/\partial x_2, \partial E/\partial x_3)$ to zero. The derivative with respect to x_i is

$$\begin{aligned}\frac{\partial E}{\partial x_i}(x) &= \frac{\partial \mathbf{x}^T}{\partial x_i} \cdot \mathbf{Q} \cdot \mathbf{x} + \mathbf{x}^T \cdot \mathbf{Q} \cdot \frac{\partial \mathbf{x}}{\partial x_i} \\ &= \mathbf{Q}_i^T \cdot \mathbf{x} + \mathbf{x}^T \cdot \mathbf{Q}_i \\ &= 2\mathbf{Q}_i^T \cdot \mathbf{x},\end{aligned}$$

where $\mathbf{Q}_i^T$ is the i-th row of $\mathbf{Q}$. The point $c \in \mathbb{R}^3$ that minimizes $E(x)$ is the solution to the system of three linear equations $Q \cdot x + q = 0$, where

$$Q = \begin{pmatrix} A & B & C \\ B & E & F \\ C & F & H \end{pmatrix} \text{ and } q = \begin{pmatrix} D \\ G \\ I \end{pmatrix}.$$

Hence $c = Q^{-1} \cdot (-q)$, and the sum of square distances of c from the planes in H is $E(c)$. The equation for c sheds light on the possible degeneracies. The non-degenerate case corresponds to $\operatorname{rank} Q = 3$, the case of an elliptic cylinder corresponds to $\operatorname{rank} Q = 2$, and the case of two parallel planes corresponds to $\operatorname{rank} Q = 1$. Rank 0 is not possible because Q is the non-empty sum of products of unit vectors.

Eigenvalues and eigenvectors. We may translate the planes by $-c$ such that E attains its minimum at the origin. In this case $D = G = I = 0$ and $J = E(0)$. The shape of the ellipsoid $E^{-1}(\epsilon)$ can be described by the eigenvalues and eigenvectors of Q. By definition, the *eigenvectors* are unit vectors x that satisfy $Q \cdot x = \lambda \cdot x$. The value of λ is the corresponding *eigenvalue*. The eigenvalues are the roots of the *characteristic polynomial* of Q, which is

$$\begin{aligned}P(\lambda) &= \det \begin{pmatrix} A-\lambda & B & C \\ B & E-\lambda & F \\ C & F & H-\lambda \end{pmatrix} \\ &= \det Q - \lambda \cdot \operatorname{dtr} Q + \lambda^2 \cdot \operatorname{tr} Q - \lambda^3,\end{aligned}$$

where $\det Q$ is the determinant, $\operatorname{dtr} Q$ is the sum of cofactors of the three diagonal elements, and $\operatorname{tr} Q$ is the trace of Q. For symmetric positive semi-definite matrices, the characteristic polynomial has three non-negative roots, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Once we have an eigenvalue, we can compute the corresponding eigenvector to span the nullspace of the underconstrained system $(Q - \lambda) \cdot x = 0$.

What is the geometric meaning of eigenvectors and eigenvalues? For symmetric matrices, the eigenvectors are pairwise orthogonal, or if there are multiple eigenvalues the eigenvectors can be chosen pairwise orthogonal. They can thus be viewed as defining another coordinate system for $\mathbb{R}^3$. The three symmetry planes of the ellipsoid $E^{-1}(\epsilon)$ coincide with the coordinate planes of this new system, see Figure IV.4. We can write the error function as

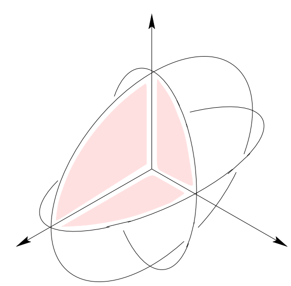

Figure IV.4: The ellipsoid is indicated by drawing the elliptic cross-sections along the three symmetry planes spanned by the eigenvectors.

$$\begin{aligned}E(x) &= \mathbf{x}^T \cdot \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & J \end{pmatrix} \cdot \mathbf{x} \\ &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + J.\end{aligned}$$

Since $E(x) \geq 0$ for every $x \in \mathbb{R}^3$ this proves that the three eigenvalues are indeed real and non-negative. The preimage for a fixed error $\epsilon > J$ is the ellipsoid with axes of half-lengths $\sqrt{(\epsilon - J)/\lambda_i}$ for $i = 1, 2, 3$.

Bibliographic notes. The idea of using the sum of square distances from face planes for surface simplification is due to Garland and Heckbert [1]. Eigenvalues and eigenvectors of matrices are topics in linear algebra. A very readable introductory text is the book by Gilbert Strang [2].

[1] M. Garland and P. S. Heckbert. Surface simplification using quadratic error metrics. *Computer Graphics*, Proc. SIGGRAPH 1997, 209–216.

[2] G. Strang. *Introduction to Linear Algebra.* Wellesley-Cambridge Press, Wellesley, Massachusetts, 1993.