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Topics: Representing rotations with quaternions
Scribe: from your lecturers

1 Representing rotations with quaternions

We begin by considering linear transformations of Euclidean 3-space, which have 3-by-3 matrices. A 3-by-3 matrix is called *orthogonal* when its three columns, considered as vectors, are each of unit length and are orthogonal to each other. When the three columns of a matrix are unit-length and orthogonal, the three rows are also unit-length and orthogonal; so it doesn't matter, when we define 'orthogonal matrix', whether we talk about columns or rows. The determinant of an orthogonal matrix is either $+1$ or -1. The matrices of the former type are called *rotations*, while those of the latter type are called *reflections*.

Note that every linear transformation of 3-space fixes the origin, since that is part of what we mean by 'linear'. Thus, the rotations that we are discussing here are rotations about the origin; similarly, our reflections are reflections in planes passing through the origin. It would take us too far afield to study the isometries of Euclidean 3-space in general. Suffice it to say that they consist of translations, rotations about arbitrary points, screw motions, reflections in arbitrary planes, glide reflections, and rotary reflections. In this handout, we restrict ourselves to the rotations that fix the origin.

Rotations are confusing to work with because they compose in surprising ways. For example, what is the result of taking a cube centered at the origin, rotating it first by 90 degrees about one axis, and then rotating it by 90 degrees about a second axis? Answer: The composite rotation rotates through 120 degrees about one of the cube's main diagonals. Try it and see.

One way to deal with such surprises is to write out the 3-by-3 matrices in full and multiply them, being careful to get the factors in the right order. The subject of this handout is a neat alternative, involving quaternions. The quaternion technique lets us represent a rotation with four numbers subject to one constraint, instead of — as is the case with matrices — nine numbers subject to six constraints.

Recall that the complex numbers are a way to turn $\mathbf{R}^2$ into an algebra; the quaternions are a way to turn $\mathbf{R}^4$ into an algebra, but one in which the multiplication doesn't commute. In particular, a *quaternion* is an expression of the form $a+bI+cJ+dK$, where the coefficients a, b, c, and d are real numbers and the symbols I, J, and K multiply according to the rules $I^2=J^2=K^2=IJK=-1$. (So $\pm I$, $\pm J$, and $\pm K$ are all different square roots of -1.) It is easy to check that those rules imply

$$\begin{array}{lll} IJ=K & JK=I & KI=J \\ JI=-K & KJ=-I & IK=-J. \end{array}$$

The quaternions form a non-commutative division algebra — also called a skew field — of dimension 4 over the real numbers. That is, the quaternions have all of the algebraic properties that we expect numbers to have, except that, for quaternions p and q, the products pq and qp are usually different.

The *norm* of a quaternion is the non-negative real number defined by

$$|a+bI+cJ+dK| := \sqrt{a^2+b^2+c^2+d^2}.$$

Note that the norm $|q|$ of a quaternion $q = a+bI+cJ+dK$ is zero only when all four coordinates of q are zero, that is, when $q = 0+0I+0J+0K = 0$. If $q = a+bI+cJ+dK$ is any quaternion, the symbol $\bar{q}$ denotes the *conjugate* quaternion given by $\bar{q} := a-bI-cJ-dK$. If we multiply a quaternion by its conjugate, in either order, the result is always a real number; in fact, it is easy to check that

$$q\bar{q} = (a+bI+cJ+dK)(a-bI-cJ-dK) = a^2+b^2+c^2+d^2 = |q|^2,$$

and also $\bar{q}q = |q|^2$. So, if q is any nonzero quaternion, we have

$$\frac{q\bar{q}}{|q|^2} = 1,$$

from which it follows that q^{-1}, the multiplicative inverse of q, is given by $q^{-1} = \bar{q}/|q|^2$. Now, the multiplicative inverse of a product — of quaternions or of anything else — is the product of the inverses, but in the opposite order:

$$(pq)^{-1} = q^{-1}p^{-1}.$$

Since the conjugate quaternion $\bar{q}$ differs from the multiplicative inverse q^{-1} just by the scalar factor $|q|^2$ and scalars commute with everything, it follows that the conjugate of a product is the product of the conjugates, in the opposite order:

$$\overline{pq} = \overline{q}\,\overline{p}.$$

If p is any fixed quaternion, let C_p denote the map from quaternions to quaternions given by $C_p(r) := pr\bar{p}$. The map C_p differs only by a constant factor from the map $r \mapsto prp^{-1}$, which is called *conjugation by p*, so we shall refer to the map C_p as *scaled conjugation by p*. Conjugating by two quaternions in succession is the same as conjugating by their product, as long as we get the order of the factors correct, and the same holds for scaled conjugation: For any two quaternions p and q, we have

$$C_p(C_q(r)) = C_p(qr\bar{q}) = pqr\bar{q}\bar{p} = pq\,r\,\overline{pq} = C_{pq}(r).$$

Furthermore, each scaled conjugation map C_q is a linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$, so it has a matrix, which we will denote $M(q)$. A straightforward, but somewhat tedious, calculation shows that, if $q = a + bI + cJ + dK$, then

$$M(q) = M(a + bI + cJ + dK) =$$

$$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

Since we checked above that $C_p \circ C_q = C_{pq}$, it follows that $M(p)M(q) = M(pq)$, where the multiplication on the left is matrix multiplication and that on the right is quaternion multiplication.

The map M is interesting because it has the following properties, which you might want to verify for yourself (perhaps with the help of Maple or Mathematica):

- The dot product of any row of $M(q)$ with itself is $|q|^4 = (a^2 + b^2 + c^2 + d^2)^2$. Similarly for the dot product of any column with itself.
- The dot product of any row of $M(q)$ with any other row is zero. Similarly for any column with any other column.
- We have $\det(M(q)) = |q|^8 = (a^2 + b^2 + c^2 + d^2)^4$.
- As we noted above, if p and q are any two quaternions, we have $M(pq) = M(p)M(q)$. That is, M is a homomorphism from the multiplicative group of nonzero quaternions into the group of 4-by-4 invertible matrices.

Suppose that we restrict ourselves to quaternions q of unit norm. The first row of the matrix $M(q)$ is then $(1, 0, 0, 0)$, so we can interpret $M(q)$ as an affine transformation of 3-space. Since the first column is also $(1, 0, 0, 0)$, this affine transformation takes the origin to the origin. Furthermore, the bottom-right 3-by-3 submatrix of $M(q)$ is orthogonal and has determinant $+1$. So the entire matrix $M(q)$ represents a rotation of 3-space that fixes the origin. We are going to represent rotations by using the unit-norm quaternions that correspond to them under the map M.

The correspondence M has one tricky property: It's easy to see that it can't be one-to-one. Let R be any rotation matrix and let $q = a + bI + cJ + dK$ be some unit-norm quaternion that satisfies $M(q) = R$. We then also have $M(-q) = R$, since reversing the signs of all four of the coordinates a, b, c, and d leaves every entry of the matrix $M(q)$ unchanged. Thus, every rotation actually has two corresponding unit-norm quaternions, one the negative of the other.

This two-to-one character of the correspondence between unit-norm quaternions and rotations is confusing at first. We can clarify matters a bit by introducing the concept of a *super-rotation*. Note that adding any multiple of 360 degrees to the angle of a rotation leaves that rotation unchanged. Super-rotations are like rotations, except that two super-rotations around the same axis are equal only when their angles differ by some multiple of 720 degrees. To specify a super-rotation, we need both a rotation matrix and an extra bit. The extra bit says whether the relevant multiple of 360 degrees is even or odd. For example, there are two super-rotations that correspond to the identity matrix. One of them involves rotating by 0 degrees, or ± 720 degrees, or ± 1440 degrees, or the like around any axis. The other involves rotating by ± 360, or ± 1080, or the like around any axis.

Let's go back to rotations for a bit. The relationship between the coordinates of the quaternion $q = a + bI + cJ + dK$ and the geometric structure of the rotation $M(q)$ is pretty simple. Here — without proof — is how it works: The axis of the rotation $M(q)$ goes through the origin and is parallel to the vector (b, c, d). That is, the axis is the line joining the origin $[1; 0, 0, 0]$ to the point at infinity $[0; b, c, d]$. The angle by which $M(q)$ rotates around that axis is θ, where

$$a = \cos(\theta/2) \qquad \text{and} \qquad b^2 + c^2 + d^2 = \sin^2(\theta/2).$$

Note that these two conditions are consistent with each other, since

$$|q|^2 = a^2 + b^2 + c^2 + d^2 = \cos^2(\theta/2) + \sin^2(\theta/2) = 1.$$

These formulas let us go backwards from a rotation R to the associated unit-norm quaternion $q = a + bI + cJ + dK$, at least up to some questions of signs. Given a rotation R, we first figure out the angle θ through which R rotates. We have several choices here. First, the sign of θ isn't clear: Do positive angles correspond to clockwise or to counterclockwise rotations? That issue has to be settled by adopting some sort of right-hand rule. Second, we can add any multiple of 360 to θ without affecting the rotation R. (That assumes that R is a regular rotation. If R were a super-rotation, adding an even multiple of 360 to θ wouldn't change R, but adding an odd multiple of 360 to θ would toggle the extra bit.) Ignoring these ambiguities for the moment, we next find a vector (b, c, d) that is parallel to the axis of R and is scaled in length so that $b^2 + c^2 + d^2 = \sin^2(\theta/2)$. Note that we have one more choice here, since $(-b, -c, -d)$ works just as well as (b, c, d). Finally, we set a by the formula $a = \cos(\theta/2)$.

What about the various ambiguities? If $M(a + bI + cJ + dK)$ is the rotation R, then $M(-a - bI - cJ - dK) = R$ also, while $M(a - bI - cJ - dK)$ and $M(-a + bI + cJ + dK)$ are both R^{-1}. That's the whole story if we think of the values of M as rotations. If we think of the values of M as super-rotations, then $M(q) = M(a + bI + cJ + dK)$ and $M(-q) = M(-a - bI - cJ - dK)$ are the two distinct super-rotations whose rotation component is R, while $M(\bar{q}) = M(a - bI - cJ - dK)$ and $M(-\bar{q}) = M(-a + bI + cJ + dK)$ are the two distinct super-rotations whose rotation component is R^{-1}.

Note that the appearance of the angle $\theta/2$ in the formula $a = \cos(\theta/2)$ is tied up with the concept of a super-rotation. For example, the quaternion 1 has $a = 1$, so the corresponding rotation $M(1)$ is a rotation through some angle θ that satisfies $1 = \cos(\theta/2)$. This means that $\theta/2$ must be a multiple of 360 degrees, so θ must be a multiple of 720 degrees. Thus, we can think of $M(1)$ as the super-rotation that involves rotating through any even multiple of 360 around any axis. Similarly, $M(-1)$ is the super-rotation that involves rotating through any odd multiple of 360 around any axis.

Here's the story about rotations and super-rotations more abstractly. The map M puts the set of all super-rotations in one-to-one correspondence with the set of unit-norm quaternions, which is just the unit 3-sphere $a^2+b^2+c^2+d^2=1$, sitting in 4-space. When we move from super-rotations to rotations, we ignore the extra bit, which means that we identity q with $-q$. Just as identifying antipodal points on a 2-sphere produces projective 2-space, identifying antipodal points on a 3-sphere in this way produces projective 3-space. So the set of all rotations is isomorphic to real, projective 3-space.

We have seen that we can represent rotations — more properly, super-rotations — by using quaternions of unit norm. It's time for an example of how this quaternion representation works in practice.

Let p denote the quaternion $p := (1+I)/\sqrt{2}$, so $a=b=1/\sqrt{2}$ and $c=d=0$. The matrix associated with p is

$$M(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which represents a rotation through 90 degrees around the X axis. There are two such rotations: One of them carries the positive Y axis to the positive Z axis, the other does the reverse. Under the CS348a convention of prefix functions — points are column vectors written on the right, transformations are matrices written on the left — $M(p)$ carries the positive Y axis to the positive Z axis.

In a similar way, let q denote the quaternion $q := (1+J)/\sqrt{2}$. The associated matrix

$$M(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

represents a rotation through 90 degrees around the Y axis, carrying the positive Z axis to the positive X axis.

Suppose that we first perform the rotation $M(q)$ and then perform the rotation $M(p)$. What will the composite rotation be? Multiplying the matrices, we have

$$M(p)M(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(Note that $M(q)$ is performed first, so it goes on the right in the product.) Multiplying the quaternions, we find that

$$pq = \frac{1+I}{\sqrt{2}}\,\frac{1+J}{\sqrt{2}} = \frac{1+I+J+K}{2}.$$

Thus, the axis of the composite rotation is the main diagonal of XYZ space, the line $X=Y=Z$. The angle of rotation is θ where $a=1/2=\cos(\theta/2)$, so $\theta/2$ is 60 degrees and θ is 120 degrees.

Representing rotations with unit-norm quaternions in this way is an old trick in physics. Ken Shoemake pioneered the use of this trick in computer graphics. For a fuller explanation of quaternions and a description of how to draw curves in rotation space, see Shoemake's articles [S85, S89].

2 Complex quaternions and Lorentz transformations

The function M above is a 2-to-1 homomorphism from the group of unit-norm quaternions onto the group of origin-fixing rotations of 3-space. There is a generalization $\mathscr{M}$ of M that is useful in physics: a 2-to-1 homomorphism from the group of unit-norm complex quaternions onto the proper Lorentz group. This generalization is too delicious to ignore, but it has little to do with CS348; so keep reading only if you are curious.

To a good approximation, we live in a *spacetime*, a semi-Riemannian manifold whose tangent spaces are isomorphic to Minkowski 4-space. Minkowski 4-space is a 4-dimensional linear space over the real numbers, equipped with the *Minkowski dot product*:

$$(t_1, x_1, y_1, z_1) \cdot (t_2, x_2, y_2, z_2) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2.$$

The Minkowski dot product is a non-degenerate, symmetric bilinear form, but it is not positive definite; hence, it isn't a inner product in the usual, mathematical sense — it's a *metric tensor* instead. A *Lorentz transformation* is the analog of an orthogonal matrix, but using the Minkowski dot product. That is, a Lorentz transformation is a 4-by-4 matrix with the following properties:

- The Minkowski dot product of any row with any other row is 0.
- The Minkowski dot product of the first row with itself is 1.
- The Minkowski dot product of any of the last three rows with itself is -1.
- All of the same goes for columns.

These row properties involve ten equations, which turn out to be independent; but once the row properties are satisfied, the column properties are automatic. So there are $4 \cdot 4 - 10 = 6$ degrees of freedom in a Lorentz transformation: three dimensions' worth of spatial rotations that leave the observer at rest, and three dimensions' worth of *boosts*, changes to the observer's velocity.

Making sure that a Lorentz transformation doesn't reverse orientation is slightly tricky. In Euclidean n-space, with its positive definite dot product, the orthogonal matrices split up into two classes: the rotations, with determinant $+1$, and the reflections, with determinant -1. But the Lorentz transformations split up into four classes, depending upon whether they preserve or reverse the orientations of space and time separately. A Lorentz transformation is called *proper* when it preserves both orientations. In matrix terms, a Lorentz transformation is proper when both its determinant is $+1$ and its top-left element is positive. The top-left element being positive (in fact, it will be at least 1) means that the orientation of time is preserved. The overall determinant being $+1$ means that the joint orientation of spacetime is preserved, from which it follows that the orientation of space is also preserved.

Now, what about complex quaternions? The idea is to replace each of the four real coefficients of a regular quaternion with a complex coefficient. That is, a *complex quaternion* is an expression of the form $Q = A + BI + CJ + DK$, where A, B, C, and D are complex numbers. Writing $A = a + \alpha i$ and the like, we have

$$Q = (a + \alpha i) + (b + \beta i)I + (c + \gamma i)J + (d + \delta i)K.$$

To multiply two complex quaternions, we handle the symbols I, J, and K via the rules $I^2 = J^2 = K^2 = IJK = -1$ and we handle their scalar coefficients by the usual rules for complex numbers. The complex quaternions form an associative algebra of dimension 8 over the real numbers, but that algebra is not a division algebra, as we will see shortly.

We define the *squared norm* $\sigma(Q)$ of a complex quaternion Q by the rule

$$\sigma(A + BI + CJ + DK) := A^2 + B^2 + C^2 + D^2,$$

where A^2 denotes the usual square of the complex number A. That is, $A^2 = (a + \alpha i)^2 = (a^2 - \alpha^2) + 2a\alpha i$. Thus, we have

$$\begin{aligned}\sigma((a + \alpha i) &+ (b + \beta i)I + (c + \gamma i)J + (d + \delta i)K) = \\ &(a^2 - \alpha^2 + b^2 - \beta^2 + c^2 - \gamma^2 + d^2 - \delta^2) + 2(a\alpha + b\beta + c\gamma + d\delta)i.\end{aligned}$$

We don't define 'the norm' of a complex quaternion Q, since the squared norm $\sigma(Q)$ is complex and it isn't clear which square root of that complex number to choose. The squared norm is multiplicative, as you would expect from the name: $\sigma(PQ) = \sigma(P)\sigma(Q)$. Thus, the complex quaternions Q that satisfy $\sigma(Q) = 1$ form a multiplicative subgroup. Let's refer to those complex quaternions as being *of unit norm*, even though we aren't defining 'the norm' of a complex quaternion in general. Note that the equation $\sigma(Q) = 1$ is an equality between complex numbers, and hence represents two real constraints. Thus, there are six degrees of freedom in a complex quaternion of unit norm.

The squared-norm function σ is non-norm-like in the sense that there are nonzero complex quaternions, such as $Q = 1 + iI$, that have $\sigma(Q) = 0$. Note that such a complex quaternion can't have a multiplicative inverse. So the complex quaternions do not form a division algebra.

If $A = a + \alpha i$ is a complex number, let A^* denote the usual complex conjugate of A, given by $A^* = a - \alpha i$. If $Q = A + BI + CJ + DK$ is a complex quaternion, let $\bar{Q}$ denote the complex quaternion $\bar{Q} := A^* - B^*I - C^*J - D^*K$. Note that $\bar{Q}$ is conjugated at two different levels: both at the quaternion level and at the complex-number level. For any fixed complex quaternion P, we can define a scaled conjugation map by the rule $\mathscr{C}_P(R) = PR\bar{P}$, and these maps have the homomorphism property that $\mathscr{C}_P \circ \mathscr{C}_Q = \mathscr{C}_{PQ}$. Each scaled conjugation map $\mathscr{C}_Q$ corresponds to a linear transformation of $\mathbf{R}^8$, and so has an 8-by-8 matrix. We want only a 4-by-4 matrix, of course — a Lorentz transform, in fact. Fortunately, every scaled conjugation map $\mathscr{C}_Q$ has the property that it takes arguments of the form $R = w + xiI + yiJ + ziK$ — that is, arguments of the form $R = A + BI + CJ + DK$ where A is pure real and B, C, and D are pure imaginary — to results of the same form. So we can get a 4-by-4 matrix $\mathscr{M}(Q)$ out of the linear map $\mathscr{C}_Q$ by looking just at what $\mathscr{C}_Q$ does to arguments R of this form.

Here is the resulting 4-by-4 matrix $\mathscr{M}(Q)$:

$$\mathscr{M}((a+\alpha i)+(b+\beta i)I+(c+\gamma i)J+(d+\delta i)K) :=$$

$$\begin{pmatrix} \begin{matrix}+a^2+\alpha^2+b^2+\beta^2\\+c^2+\gamma^2+d^2+\delta^2\end{matrix} & \begin{matrix}+2a\beta-2\alpha b\\-2c\delta+2\gamma d\end{matrix} & \begin{matrix}+2a\gamma-2\alpha c\\+2b\delta-2\beta d\end{matrix} & \begin{matrix}+2a\delta-2\alpha d\\-2b\gamma+2\beta c\end{matrix} \\ \\ \begin{matrix}+2a\beta-2\alpha b\\+2c\delta-2\gamma d\end{matrix} & \begin{matrix}+a^2+\alpha^2+b^2+\beta^2\\-c^2-\gamma^2-d^2-\delta^2\end{matrix} & \begin{matrix}+2bc+2\beta\gamma\\-2ad-2\alpha\delta\end{matrix} & \begin{matrix}+2bd+2\beta\delta\\+2ac+2\alpha\gamma\end{matrix} \\ \\ \begin{matrix}+2a\gamma-2\alpha c\\-2b\delta+2\beta d\end{matrix} & \begin{matrix}+2bc+2\beta\gamma\\+2ad+2\alpha\delta\end{matrix} & \begin{matrix}+a^2+\alpha^2-b^2-\beta^2\\+c^2+\gamma^2-d^2-\delta^2\end{matrix} & \begin{matrix}+2cd+2\gamma\delta\\-2ab-2\alpha\beta\end{matrix} \\ \\ \begin{matrix}+2a\delta-2\alpha d\\+2b\gamma-2\beta c\end{matrix} & \begin{matrix}+2bd+2\beta\delta\\-2ac-2\alpha\gamma\end{matrix} & \begin{matrix}+2cd+2\gamma\delta\\+2ab+2\alpha\beta\end{matrix} & \begin{matrix}+a^2+\alpha^2-b^2-\beta^2\\-c^2-\gamma^2+d^2+\delta^2\end{matrix} \end{pmatrix}$$

The map $\mathscr{M}$ from the set of complex quaternions to the set of 4-by-4 matrices defined in this way has the following wonderful properties:

- The Minkowski dot product of any row of $\mathscr{M}(Q)$ with any other row is 0, and similarly for columns.
- The Minkowski dot product of the top row with itself is $|\sigma(Q)|^2$, while the Minkowski dot product of any of the last three rows with itself is $-|\sigma(Q)|^2$; and similarly for columns.
- We have $\det(\mathscr{M}(Q)) = |\sigma(Q)|^4$.
- The top-left element of $\mathscr{M}(Q)$ is nonnegative.
- The map $\mathscr{M}$ is a homomorphism: If P and Q are any complex quaternions, we have $\mathscr{M}(PQ) = \mathscr{M}(P)\mathscr{M}(Q)$.

The first four properties imply that, if Q is any complex quaternion of unit norm, then $\mathscr{M}(Q)$ is a proper Lorentz transformation. Conversely, given any proper Lorentz transformation L, it turns out that there are precisely two complex quaternions of unit norm, each the negative of the other, say Q and $-Q$, such that $\mathscr{M}(Q) = \mathscr{M}(-Q) = L$.

Note that setting $\alpha = \beta = \gamma = \delta = 0$ in the matrix $\mathscr{M}(Q)$ reduces it to the matrix $M(q)$ above. Thus, the map $\mathscr{M}$ is an extension of M; that is, $\mathscr{M}$ gives the same correspondence between real quaternions and rotations as does M.

The simplest Lorentz transformations that aren't rotations are called *boosts*; they relate the frame of reference of the laboratory to the frame of a rocket ship traveling at a constant velocity in some direction.

Let r be a real number with $|r| < 1$. The boost along the X axis whose velocity is r times the speed of light is the Lorentz transformation

$$\begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $r = \tanh(\theta)$. The number θ is called the *parameter* of the boost, an alternative way to measure its velocity. You can check that the two complex quaternions that correspond to this boost are Q and $-Q$, where

$$Q = \cosh(\theta/2) + \sinh(\theta/2)iI.$$

Once we know the complex quaternions that correspond to boosts along the X axis, we can calculate the rule for composing two such boosts. Suppose that one rocket is moving at r_1 times the speed of light with respect to our laboratory and that a second rocket is moving at r_2 times the speed of light with respect to the first rocket. The transformation that relates the second rocket to the laboratory is the composite boost. Let θ_1 and θ_2 be the parameters of the two boosts; that is, let $r_1 = \tanh(\theta_1)$ and $r_2 = \tanh(\theta_2)$. We then have the complex quaternion product:

$$\begin{aligned} &(\cosh(\theta_1/2) + \sinh(\theta_1/2)iI)(\cosh(\theta_2/2) + \sinh(\theta_2/2)iI) \\ &\quad = \cosh(\theta_1/2)\cosh(\theta_2/2) + \sinh(\theta_1/2)\sinh(\theta_2/2) + \\ &\qquad\qquad (\cosh(\theta_1/2)\sinh(\theta_2/2) + \sinh(\theta_1/2)\cosh(\theta_2/2))iI \\ &\quad = \cosh((\theta_1 + \theta_2)/2) + \sinh((\theta_1 + \theta_2)/2)iI. \end{aligned}$$

Thus, composing two boosts in the same direction corresponds to adding the parameters θ. The formula for the composite velocity r is

$$r = \tanh(\theta_1 + \theta_2) = \frac{r_1 + r_2}{1 + r_1 r_2}.$$

Boosts in arbitrary directions behave something like rotations. Recall that the complex quaternion associated with a rotation has $\alpha = \beta = \gamma = \delta = 0$ (and is hence a real quaternion). The vector (b, c, d) points along the axis of rotation, and we have $b^2 + c^2 + d^2 = \sin^2(\theta/2)$ and $a = \cos(\theta/2)$, where θ is the angle of rotation. In a similar way, the complex quaternion Q associated with a boost of velocity r in some direction has $\alpha = b = c = d = 0$. The vector (β, γ, δ) points in the direction of the boost, and we have $\beta^2 + \gamma^2 + \delta^2 = \sinh^2(\theta/2)$ and $a = \cosh(\theta/2)$, where θ is the parameter of the boost, defined by $r = \tanh(\theta)$.

The analogy between boosts and rotations is not perfect, however. The composition of two rotations about arbitrary axes is always a rotation. But the composition of two boosts in arbitrary directions need not be a boost. If one needs to compose boosts in different directions with each other and with rotations, the homomorphism $\mathscr{M}$ from the complex quaternions provides about the simplest approach known.

References

[S85] Ken Shoemake, “Animating Rotation with Quaternion Curves,” *Computer Graphics*, 19(3): 245–254, SIGGRAPH Conference Proceedings, July 1985.

[S89] Ken Shoemake, “Quaternion Calculus For Animation,” pages B-18 to B-36 of the notes for course C2 at SIGGRAPH, July 1991.