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# 1 Comment on Perspective Transformations

The previous class meeting focused upon perspective transformations. The process requires the following steps:

1. Translate the origin of the given *xyz* coordinate system to match the viewing coordinate system *pqr*.

2. Rotate to align the the *xyz* with the *pqr* axes.

3. Apply the perspective transformation matrix.

In handout 11 the rotation matrix was derived using a computation of the coordinates of the $p$, $q$, and $r$ unit vectors. An alternate way to express this rotation is as a product of three rotations, each about one of the principal axes. Specifically, we can (say) first rotate around the $y$ axis to bring the $r$ axis onto the $yz$ plane. Then rotate around the $x$ axis to align the $r$ axis with the $z$ axis. And then rotate about the $z$ axis to align $p$ and $q$ with the $x$ and $y$ axes respectively.

# 2 Quaternions

The discussion of the rotation matrix for perspective above and in the previous lecture applies to an arbitrary rotation. The techniques already mentioned allow us to derive the corresponding transformation matrix quite easily. Such a rotation matrix is always an orthogonal matrix: its rows (or columns) are mutually perpendicular unit vectors. Any orthogonal matrix has determinant $\pm 1$. But a rotation matrix must have determinant $+1$, because rotations preserve the orientation of space and conversely—any orthogonal matrix of determinant $+1$ represents a rotation. An orthogonal matrix of determinant $-1$ reverses orientation. Such matrices correspond to reflections around a plane through the origin.

A transformation matrix is, unfortunately, not a very intuitive way of understanding a rotation. One cannot determine by inspection of the matrix what the geometric significance of the rotation is. Below we introduce another representation of rotations based on *quaternions.* Quaternions, as we will see, can be used to represent rotations in a more perspicuous manner.

Before discussing quaternions, one should note some characteristics of rotations. The composition of rotations is order specific. Rotation $\theta$ followed by rotation $\phi$ gives a different result from rotation $\phi$ followed by rotation $\theta$. The composition of rotations is not intuitively obvious. For instance, a 90 degree rotation of a cube about the $x$-axis followed by a 90 degree rotation about the $y$-axis is equivalent to a 120 degree rotation about one of the cube's major diagonals. The product of the rotations in the opposite order is a 120 degree rotation about another major diagonals

Quaternions are a mathematical formalism which are somewhat like complex numbers. Numbers are expressed as an expression combing four terms, three of them the product of a scalar and the three special numbers designated by $I$, $J$, and $K$. Thus the representation has the form $a+bI+cJ+dK$.

The resulting quaternions may be added together, or multiplied with other quaternions. Quaternion multiplication follows the following rules:

$$\begin{aligned} I^2 &= J^2 = K^2 = IJK = -1, \\ IJ &= K, \\ JK &= I, \\ KI &= J, \\ JI &= -K, \\ KJ &= -I, \\ IK &= -J. \end{aligned}$$

Obviously, quaternion multiplication is not commutative.

Consider a quaternion $q$, $q = a+bI+cJ+dK$. The norm of $q$ is defined as $\| q \| = \sqrt{a^2+b^2+c^2+d^2}$. The conjugate of $q$ is defined as $\bar{q} := a-bI-cJ-dK$. Multiplying by the conjugate gives us: $q\bar{q} = a^2+b^2+c^2+d^2 = \| q \|^2 = \| \bar{q} \|^2$ .

So, as long as $\| q \|$ is non-zero, $\bar{q}/|q|^2$ is the multiplicative inverse of $q$. If we know $pq$, we can quickly get the inverse $(pq)^{-1}$ by the following trivial identity: $(pq)^{-1} = q^{-1}p^{-1}$.

Quaternion conjugation by $q$, denoted by $C_q$, maps quaternion $p$ to $qp\bar{q}$. This can also be expressed as: $C_q(p) = qp\bar{q} = \| q \|^2 qpq^{-1}$. If we conjugate twice, first by $q$ then by $r$, we get: $C_rC_q(p) = rqp\bar{q}\bar{r} = (rq)p(\bar{r}\bar{q})$. Conjugating a sum is the same as summing the conjugates. Conjugating a scalar product is the same as scaling the conjugate.

Thus conjugation on quaternions acts as a linear map on the corresponding four-tuplets of coefficients. If we consider the corresponding matrix $M$ for this transformation $C_q$, we can start to see the connection between quaternions and rotations.

$$M(q) = M(aI+bJ+cK+d) = \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

If we take the sum of the squares of the entries in any row or column of the matrix, we get $(a^2+b^2+c^2+d^2)^2$. Furthermore, we can check that any two rows, or any two columns, are orthogonal to each other (their dot product is zero). Further, the determinant of $M$ is also $(a^2+b^2+c^2+d^2)^4$.

If $q$ is a unit norm quaternion ($\| q \|= 1$), then the all rows and columns of $M$ become unit vectors, and its determinant is $+1$, so it looks like a rotation matrix. It takes three parameters to represent rotations. The matrix has 9 variables, but 6 constraints, which gives us the requisite 3 degrees of freedom.

The question remains "How is this an easier way to represent rotations?" Consider a unit quaternion $q$, $q = a+bI+cJ+dK$. There is an intuitive description of what this quaternion represents. It is the rotation whose axis is the line connecting $(1;0,0,0)$ to $(0;b,c,d)$, (the origin to the point at infinity in direction $(b,c,d)$. The angle of rotation is given by the (equivalent) relations $a = cos\theta/2$, or $\sqrt{b^2+c^2+d^2} = sin\theta/2$.

It would be nice if this was a one-to-one to mapping between unit quaternions and rotations but, unfortunately it is not: both $q$ and $-q$ represent the same rotation. So this mapping is exactly two-to-one. We can get a one-to-one relationship by using the concept of super-rotations. Super-rotations are just like rotations, except we only equate rotations up to multiples of 720 degrees (not 360). So, 180 degrees equals 900 degrees, but not 540 degrees.

If $q$ represents a rotation, $q^{-1}$ represents the inverse rotation.

For more information on complex quaternions, consult handout 12.

# 3 Curves and Surfaces

The next major section of the course will explore curves and surfaces.

There are a number of strategies for representing curves. The first one is to use a polynomial expression, such as

$$p(x) = a_0x^n + b_1x^{n-1} + ... + a_{n-1}x + a_n.$$

This method can be very useful for modeling. One need only store the coefficients in order to store the whole polynomial.

Another representation is to provide the roots, such as in the following:

$$P(x) = s(x-r_1)(x-r_2)...(x-r_n),$$

where $s$ is the scaling factor.

The third way is to be give some specific values of the polynomial:

$$\begin{aligned} P(x_0) &= y_0, \\ P(x_1) &= y_1, \\ &\cdots, \\ P(x_n) &= y_n. \end{aligned}$$

Converting among these three representations is a classical problem. To convert from coefficients to points, of course, just substitute the appropriate values of $x$. To convert from coefficients to roots is an old problem in algebra. Finally, to convert from points to coefficients, one must use *interpolation*—a technique to be covered in the next lecture.