CS348a: Computer Graphics Handout #16
Geometric Modeling Original Handout #6
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Topics: Different Kinds of Geometries
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1 Overview of Four Kinds of Geometries

Each flavor of geometry has certain transformations that it treats as primitive and certain properties of geometric figures that it studies. The properties that are studied must be preserved by those transformations that are taken as primitive, so the more transformations that one allows, the fewer properties are available for study. Here are four examples of geometries of the plane:

- In *congruence geometry*, the primitive transformations are rigid motions of the plane, that is, translations, rotations, and reflections. (These maps are also called *isometries*). It makes sense to talk about lengths, angles, and areas, all of which are preserved by any rigid motion.

- In *similarity geometry*, the primitive transformations include the rigid motions, as above, and also uniform scalings. Since lengths and areas are not preserved by uniform scalings, they no longer make sense. But angles still make sense. In addition, it makes sense to talk about ratios of lengths and ratios of areas.

- In *affine geometry*, the primitive transformations include all of the above and also nonuniform scalings and shearings. As a result, even ratios of lengths are not preserved. But ratios of lengths of parallel segments are preserved, as are ratios of areas.

- In *projective geometry*, an even broader class of transformations is considered primitive, as we will discuss later in the course. As a result, not even ratios of parallel lengths or ratios of areas are preserved. (Indeed, there is no longer any notion of parallelism.) All that is preserved is a more complicated thing called *cross ratio* and incidence properties: which triples of points are collinear and which triples of lines are concurrent.

2 Line Equations in Affine Geometry

There are many ways to express a line equation in the xy-plane. The slope-intercept form $y = mx + b$ is not suitable for vertical lines. In CS 348, we adopt the general form

$$a + bx + cy = 0$$

and denote the line corresponding to that equation by the triple of coefficients $[a,b,c]$. This notation extends to plane equation

$$a+bx+cy+dz=0$$

easily by adding another coefficient to get $[a,b,c,d]$.

Multiplying the line equation $a+bx+cy=0$ by a nonzero scalar doesn't change its meaning, so we have

$$[\lambda a,\lambda b,\lambda c]=[a,b,c], \qquad \lambda \neq 0.$$

As a result, the three numbers a, b, c are called *homogeneous coefficients* of the line $[a,b,c]$. In CS348, we use square brackets to indicate homogeneity.

It is also easy to see the following:

$[a,b,c]$ is not a line equation (for now, that is, in affine geometry) if $b=c=0$.

$[0,b,c]$ goes through the origin;

$[a,0,c]$, and $[a,b,0]$ are parallel to the x and y axis, respectively;

$[0,0,1]$ and $[0,1,0]$ are the x-axis and y-axis.

3 Mappings and Composition

The application of a function

$$F : X \longrightarrow Y$$

to a point x in X can be expressed in two ways: the prefix form $F(x)$ and postfix form xF. In CS 348, we will adopt the prefix notation.

In prefix form, for every element x in X, $F(x)$ in Y is the image under the mapping. Prefix form reverses the order of composition. If

$$G : Y \longrightarrow Z$$

is another mapping, then the composed mapping that applies first F and then G is called

$$G \circ F : X \longrightarrow Z,$$

since we want $(G \circ F)(x) = G(F(x))$.

In postfix form, the image of x under F is denoted xF. The benefit of this approach is that compositions work in the obvious way. If we want to apply first F and then G, the composition is called $F \circ G$, since we have $x(F \circ G) = (xF)G$. However, postfix form will not be used in this course.

If X and Y are linear spaces and $F : X \longrightarrow Y$ is a linear map, then we compute an application $F(x)$ as a matrix multiplication where the matrix corresponding to F is on the left and the column vector corresponding to x is on the right. Note that a point x is represented as a column vector, because we are using the prefix convention. (If we were using the postfix convention, the application xF would be computed as a matrix product with a row vector corresponding to x on the left. Composition of mappings corresponds to matrix multiplications.

Figure 1: Projecting a line onto another line

4 Weight Coordinates

Points and vectors are both expressed with a pair of numbers. A *weight coordinate* w with values 1 or 0 is therefore introduced to distinguish between points and vectors. In the form

$$\begin{pmatrix} w \\ x \\ y \end{pmatrix}$$

the weight $w = 1$ specifies a point, and $w = 0$ means a vector. (Other sources may choose to put the weight coordinate w last, instead of first.) If a point P and a vector v have the following coordinate values,

$$P = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

then, $P + v$ is another point and $2v$ is another vector, with

$$P + v = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} \text{ and } 2v = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix}$$

But $2P$ is illegal, because its weight coordinate is neither 0 nor 1.

It is easy to see that a point is on the line $[a,b,c]$ or a vector v is parallel to the line $[a,b,c]$ if and only if

$$[a,b,c]\begin{pmatrix} w \\ x \\ y \end{pmatrix} = 0.$$

5 Projective Geometry

Projecting a line onto another line with rays coming from an origin E (See Figure 1) requires that every ray that intersects the first line also intersects the second or, more generally, that any two distinct lines intersect at one and only one point.

In affine geometry, this is not the case: Parallel lines do not intersect. This causes a problem when projecting. If the projecting ray is parallel to the second line, what do we do? The cure for this is to move from affine geometry to projective geometry by introducing *points at infinity* and a *line at infinity*.

For each slope s, finite or infinite, we add precisely one new point to the plane, called the *point at infinity* with the slope s. This new point lies on all lines that have the slope s. Indeed, given any line with slope s, this new point can be thought of as infinitely far to the right along that line and also as infinitely far to the left. Note that there is only one point at infinity on each line, not two. When you go out along a line to the right, the point at infinity that you get to is the same point as the one that you get to by going out along that line to the left. Two distinct lines that are parallel have the same slope, and hence intersect at the point at infinity corresponding to their common slope.

We also introduce one new line, called the *line at infinity*, whose points are precisely all of the points at infinity. Now we find that:

- Every two distinct lines (one of which might be the line at infinity) intersect at a unique point (possibly infinite);
- Every two distinct points (finite or infinite) determine a unique line (possibly the line at infinity).

These facts begin to show how projective geometry is simpler than affine geometry; there are no special cases associated with parallelism.

What are the coordinates of points at infinity and the coefficients of the line at infinity? The line is easier. The triple $[1,0,0]$ of homogeneous coefficients, corresponding to the equation

$$1 + 0x + 0y = 0,$$

is taken, in projective geometry, as representing the line at infinity. Note that this equation can hold, intuitively speaking, only when either x or y are infinitely large, so the left hand side involves the indeterminate form 0∞.

What about points at infinity? For example, what coordinates shall we give to the point at infinity with slope 1, which lies on lines such as $[2,-1,1]$ (whose equation is $y = x - 2$)? One possibility is the vector $(0,1,1)^T$ (where the superscript T denotes transpose), which we have agreed should lie on that line. But every other vector $(0,t,t)^T$ is equally good. So we switch to treating the coordinates of points homogeneously, just like we treat the coefficients of lines homogeneously. With this convention, all of the vectors $(0,x,x)^T$ become identified, and that common point, with *homogeneous coordinates* $[0,1,1]^T$ is the point at infinity with slope 1.

We remark that $[0,0,0]$ is not a line in projective geometry, just as $[0,0,0]^T$ is not a point.