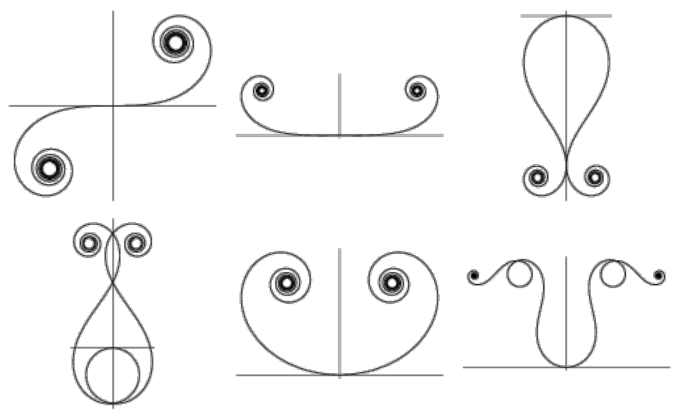
Differential Geometry of Curves

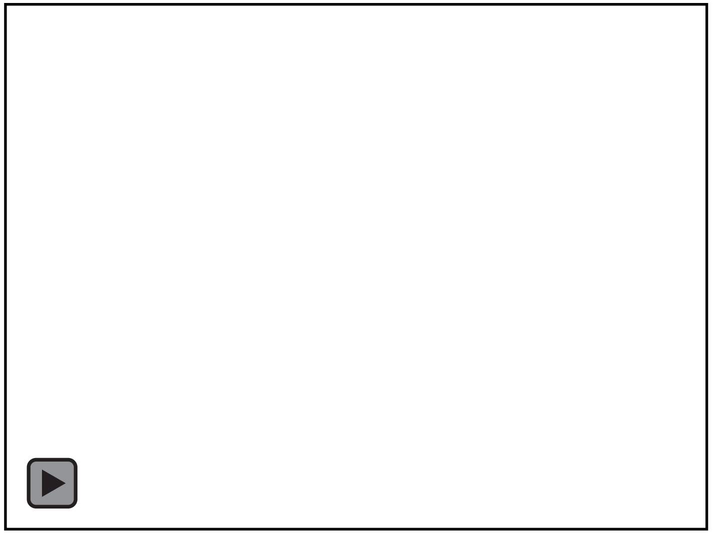


Thanks to Mirela Ben-Chen

Planar and Space Curves

- Good intro to differential geometry on surfaces
- Nice theorems
- Applications

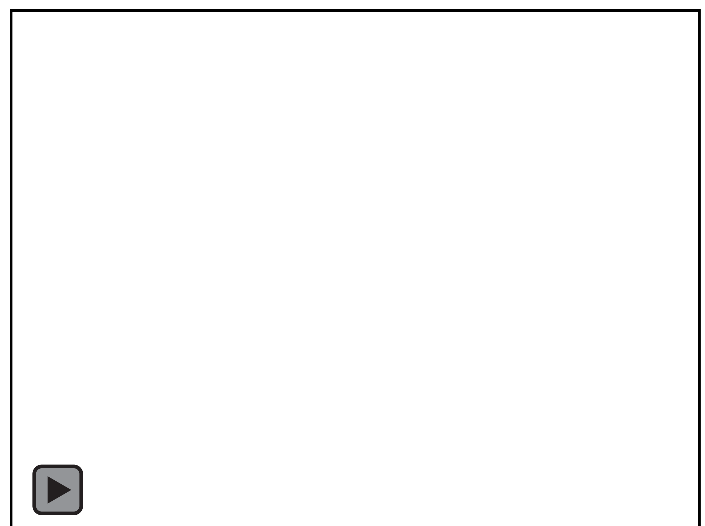
From “Discrete Elastic Rods” by Bergou et al.



Planar and Space Curves

- Good intro to differential geometry on surfaces
- Nice theorems
- Applications

From Geometric Computer Vision by Ron Kimmel

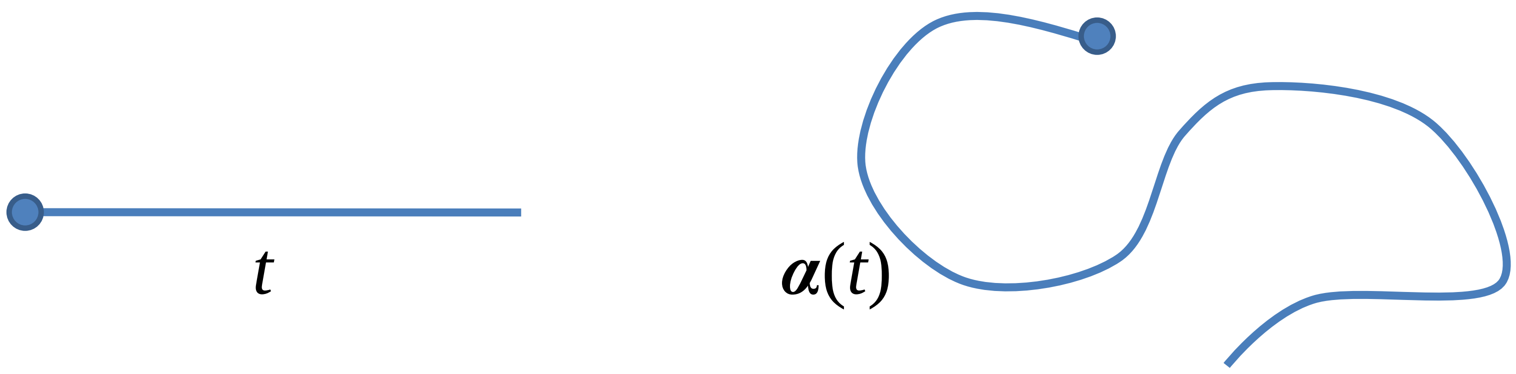


Parameterized Curves
Intuition

A particle is moving in space (E^2, E^3)

At time t its position is given by

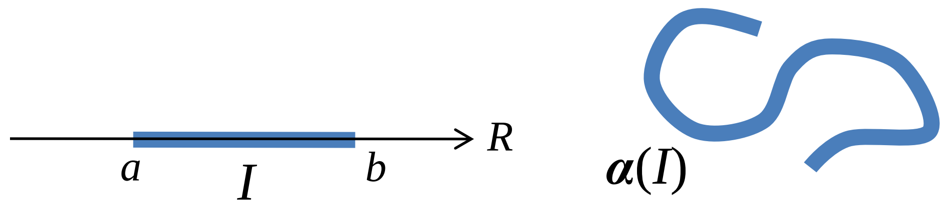
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$$



Parameterized Curves

Definition

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{\alpha}: I \to R^3$ of an interval $I = (\mathrm{a,b})$ of the real line R into R^3

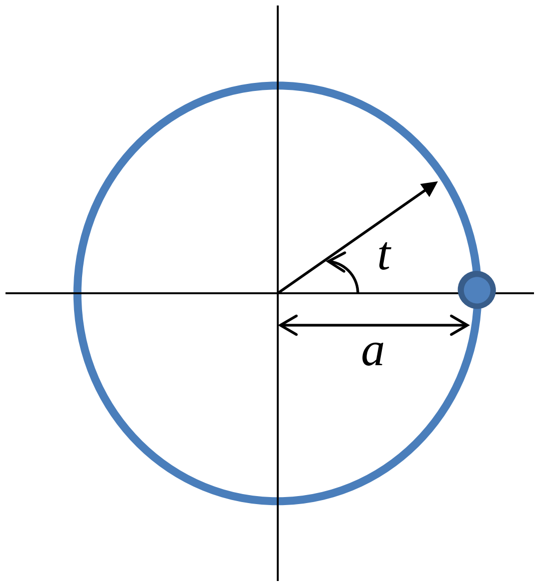


$\boldsymbol{\alpha}$ maps $t \in I$ into a point $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$ such that $x(t)$, $y(t)$, $z(t)$ are *differentiable*

A function is *differentiable* if it has, at all points, derivatives of all orders

Parameterized Curves

A Simple Example



$\boldsymbol{\alpha}_1(t) = (a\cos(t), a\sin(t))$

$t \in [0,2\pi] = I$

$\boldsymbol{\alpha}_2(t) = (a\cos(2t), a\sin(2t))$

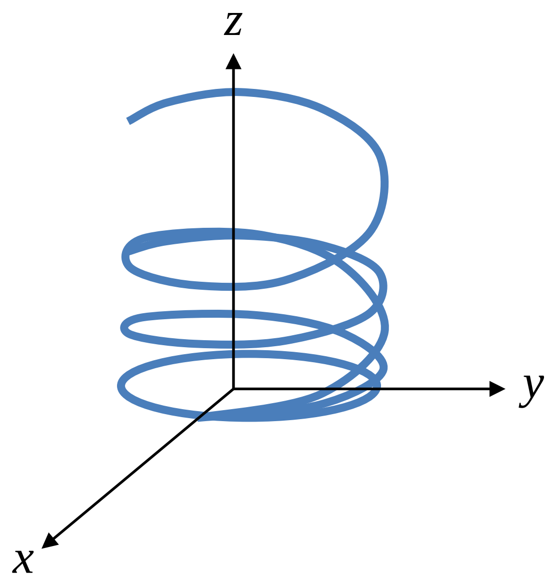
$t \in [0,\pi] = I$

$\boldsymbol{\alpha}(I) \subset R^3$ is the *trace* of $\boldsymbol{\alpha}$ (the tire tracks …)

→ Different curves can have same trace

More Examples

$$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t), bt),\ \ t \in R$$



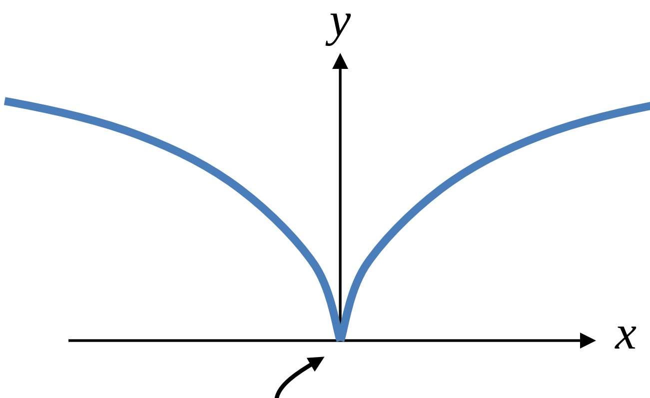
$b = 0$

$b = 1$

$b = 2$

More Examples

$$\boldsymbol{\alpha}(t) = (t^3, t^2), \;\; t \in R$$



Is this "OK"?

The Tangent Vector

Let

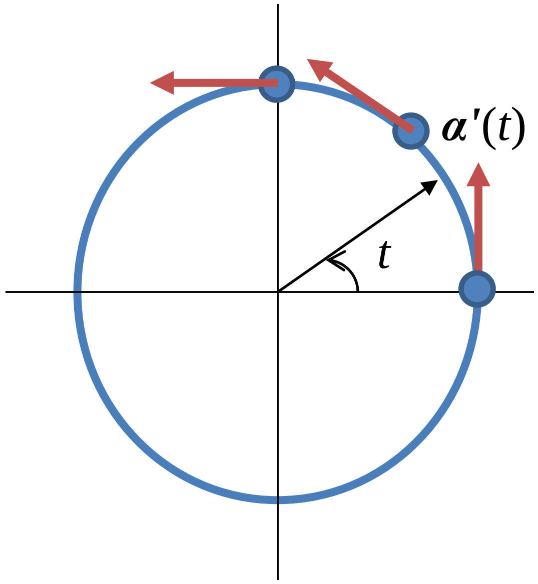
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$$

Then

$$\boldsymbol{\alpha}'(t) = (x'(t), y'(t), z'(t)) \in R^3$$

is called the *tangent vector* (or *velocity vector*) of the curve $\boldsymbol{\alpha}$ at t

Back to the Circle



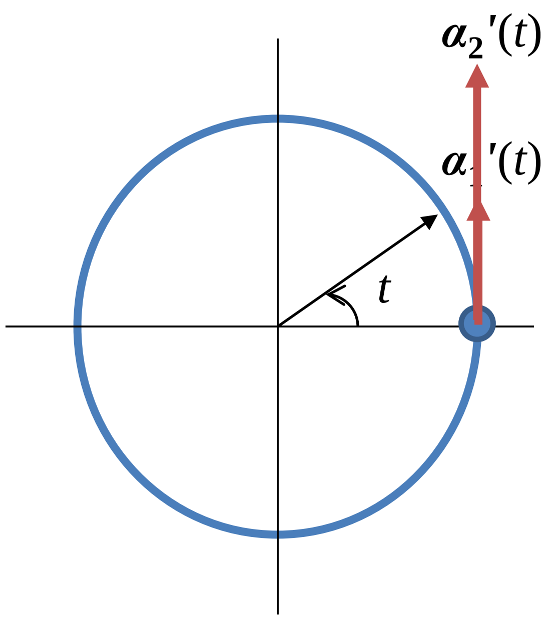
$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$$

$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t))$$

$\boldsymbol{\alpha}'(t)$ - direction of movement

$|\boldsymbol{\alpha}'(t)|$ - speed of movement

Back to the Circle

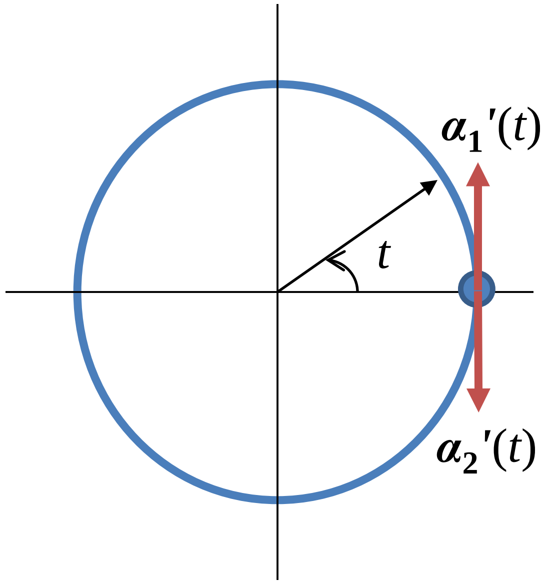


$$\alpha_1(t) = (\cos(t), \sin(t))$$

$$\alpha_2(t) = (\cos(2t), \sin(2t))$$

Same direction, different speed

Back to the Circle



$$\alpha_1(t) = (\cos(t), \sin(t))$$

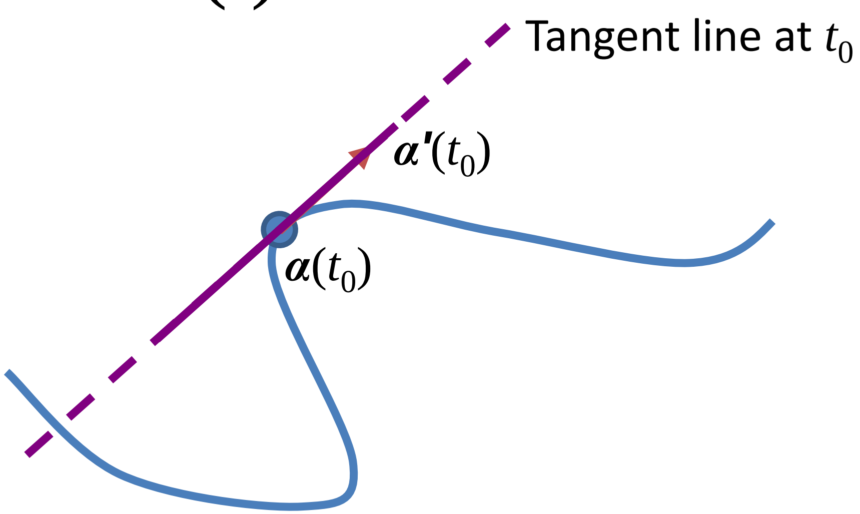
$$\alpha_2(t) = (\cos(-t), \sin(-t))$$

Same speed, different direction

The Tangent Line

Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a parameterized differentiable curve.

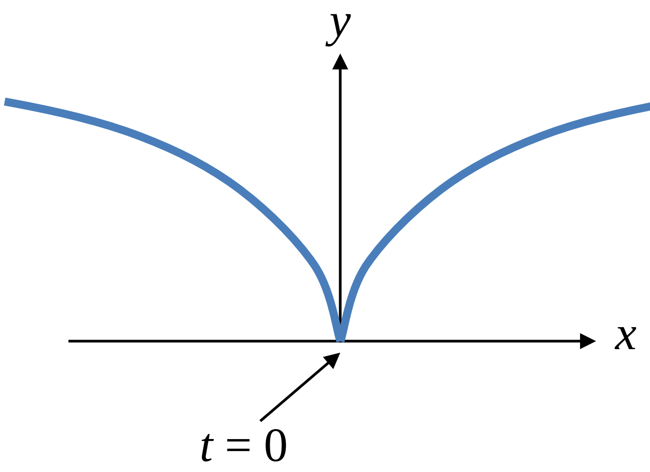
For each $t \in I$ s.t. $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ the *tangent line* to $\boldsymbol{\alpha}$ at t is the line which contains the point $\boldsymbol{\alpha}(t)$ and the vector $\boldsymbol{\alpha}'(t)$



Regular Curves

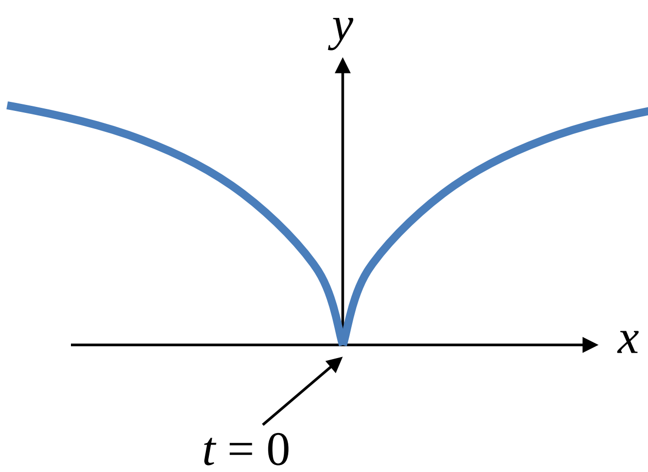
If $\boldsymbol{\alpha}'(t) = \mathbf{0}$, then t is a *singular point* of $\boldsymbol{\alpha}$.

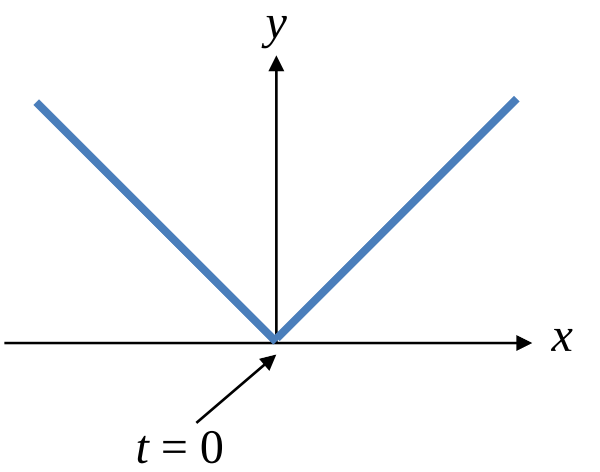
$\boldsymbol{\alpha}(t) = (t^3, t^2),\ \ t \in R$



A parameterized differentiable curve $\boldsymbol{\alpha}: I \longrightarrow R^3$ is *regular* if $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ for all $t \in I$

Spot the Difference





$\boldsymbol{\alpha}_{\mathbf{1}}(t) = (t^3, t^2)$

Differentiable

Not regular

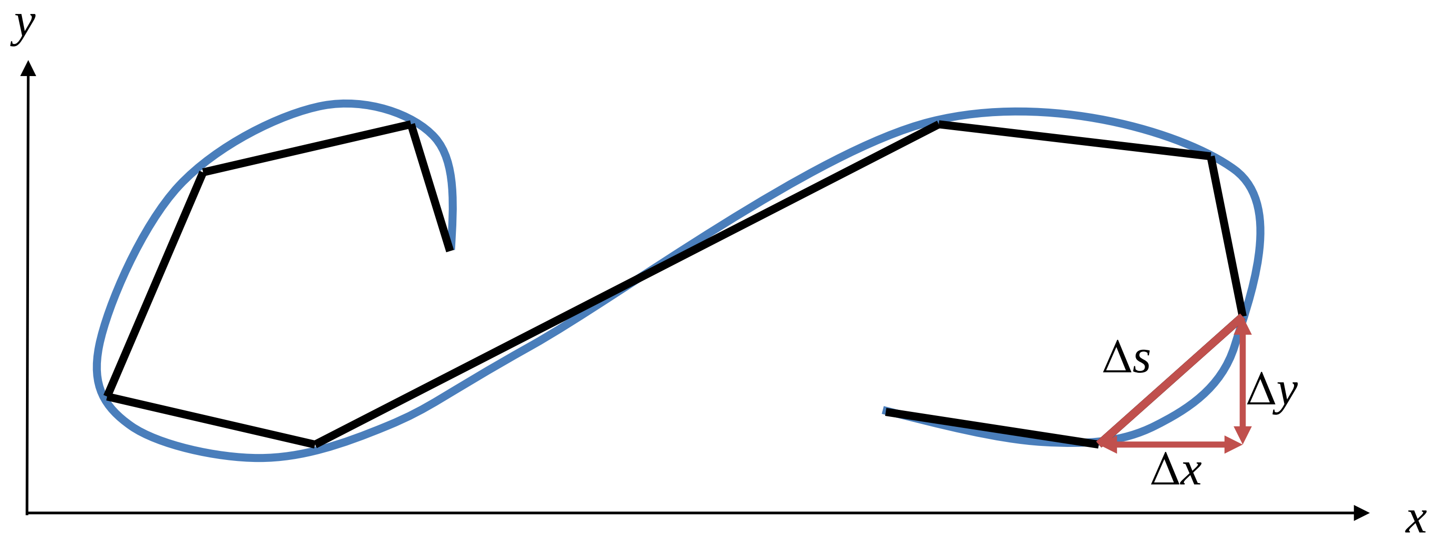
$\boldsymbol{\alpha}_{\mathbf{2}}(t) = (t, |t|)$

Not differentiable

Which differentiable curve has the same trace as $\boldsymbol{\alpha}_2$?

Arc Length of a Curve

How long is this curve?



Approximate with straight lines

Sum lengths of lines: $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Arc Length

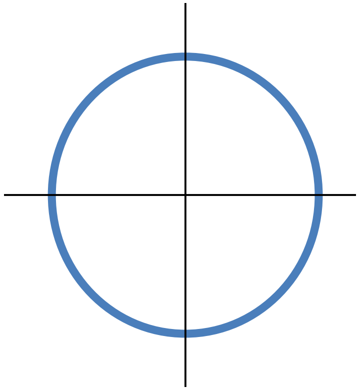
Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a parameterized differentiable curve. The *arc length* of $\boldsymbol{\alpha}$ from the point t_1 is:

$$s(t) = \int_{t_1}^{t} \left|\alpha'(p)\right| dp$$

$$= \int_{t_1}^{t} \sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2} \, dp$$

The arc length is an *intrinsic* property of the curve – does not depend on choice of parameterization

Examples



$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t)),\ t \in [0,2\pi]$

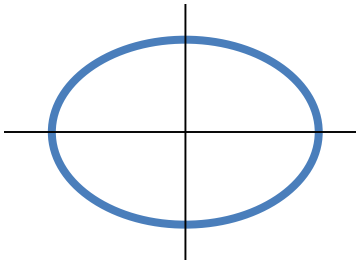
$\boldsymbol{\alpha}'(t) = (-a\sin(t), a\cos(t))$

$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}\, dt$$

$$= a\int_0^{2\pi} dt = 2\pi a$$

Examples



$\boldsymbol{\alpha}(t) = (a\cos(t), b\sin(t)),\ t \in [0,2\pi]$

$\boldsymbol{\alpha}'(t) = (-a\sin(t), b\cos(t))$

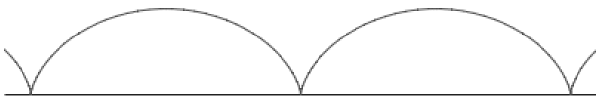
$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt$$

$$= ??$$

No closed form expression for an ellipse

Closed-Form Arc Length Gallery



Cycloid

$$\boldsymbol{\alpha}(t) = (at - a\sin(t),\ a - a\cos(t))$$

$$L(\boldsymbol{\alpha}) = 8a$$

Logarithmic Spiral

$$\boldsymbol{\alpha}(t) = (ae^{bt}\cos(t),\ ae^{bt}\sin(t))$$

Catenary

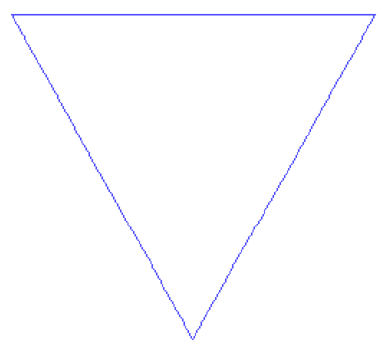
$$\boldsymbol{\alpha}(t) = (t,\ a/2\ (e^{t/a} + e^{-t/a}))$$

$$= (t,\ a\cosh\left(\frac{t}{a}\right))$$

Curves with Infinite Length

The integral $s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt$ does not always converge

$\rightarrow$ Some curves have infinite length



Koch Snowflake

Arc Length Parameterization

A curve $\alpha: I \rightarrow R^3$ is *parameterized by arc length* if $|\alpha'(t)| = 1$, for all t

For such curves we have

$$s(t) = \int_{t_0}^{t} dt' = t - t_0$$

Arc Length *Re*-Parameterization

Let $\boldsymbol{\alpha}: I \to R^3$ be a regular parameterized curve, and $s(t)$ its arc length.

Then the inverse function $t(s)$ exists, and

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s))$$

is parameterized by arc length.

Proof:

$\boldsymbol{\alpha}$ is regular → $s'(t) = |\boldsymbol{\alpha}'(t)| > 0$

→ $s(t)$ is a monotonic increasing function

→ the inverse function $t(s)$ *exists*

→ $\beta'(s) =$

→ $|\boldsymbol{\beta}'(s)| = 1$

The Local Theory of Curves

Defines local properties of curves

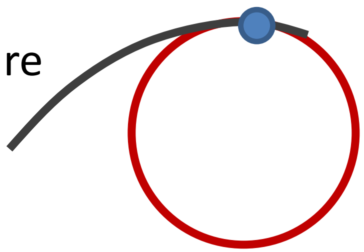
Local = properties which depend only on behavior in neighborhood of point

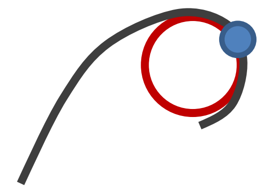
We will consider only curves parameterized by arc length

Curvature

Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a curve parameterized by arc length s. The *curvature* of $\boldsymbol{\alpha}$ at s is defined by:

$$\left|\boldsymbol{\alpha}''(s)\right| = \kappa(s)$$





$\boldsymbol{\alpha}'(s)$ – the tangent vector at s

$\boldsymbol{\alpha}''(s)$ – the *change in* the tangent vector at s

$R(s) = 1/\kappa(s)$ is called the *radius of curvature* at s.

Examples

<u>Straight line</u>

$\boldsymbol{\alpha}(s) = us + v,\ \ u,v \in R^2$

$\boldsymbol{\alpha}'(s) = u$

$\boldsymbol{\alpha}''(s) = \mathbf{0} \quad \rightarrow \kappa(s) = |\boldsymbol{\alpha}''(s)| = 0$

<u>Circle</u>

$\boldsymbol{\alpha}(s) = (a \cos(s/a), a \sin(s/a)),\ s \in [0, 2\pi a]$

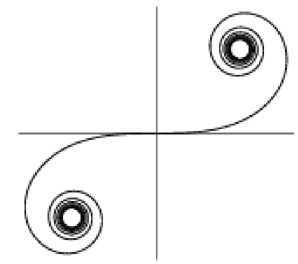
$\boldsymbol{\alpha}'(s) = (-\sin(s/a), \cos(s/a))$

$\boldsymbol{\alpha}''(s) = (-\cos(s/a)/a, -\sin(s/a)/a) \rightarrow \kappa(s) = |\boldsymbol{\alpha}''(s)| = 1/a$

Examples

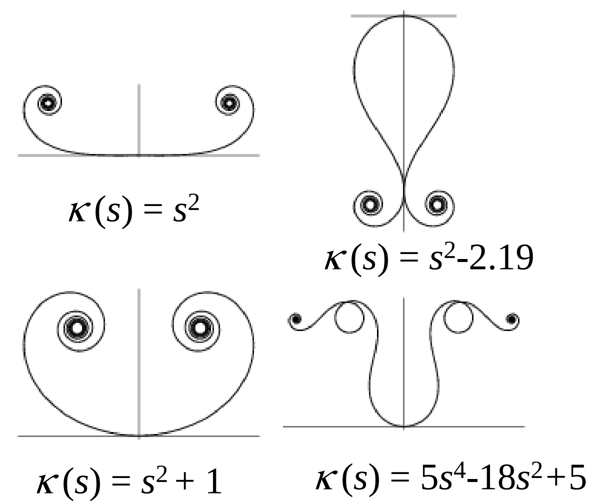
<u>Cornu Spiral</u>

A curve for which $\kappa(s) = s$



<u>Generalized Cornu Spiral</u>

A curve for which $\kappa(s)$ is a polynomial function of s



The Normal Vector

$|\boldsymbol{\alpha}'(s)|$ is the speed (derivative of arc length)

$\boldsymbol{\alpha}'(s)$ is the tangent vector

$|\boldsymbol{\alpha}''(s)|$ is the curvature

$\boldsymbol{\alpha}''(s)$ is ?

Detour to Vector Calculus

Lemma:

Let $\boldsymbol{f},\boldsymbol{g}: I \to R^3$ be differentiable maps which satisfy $\boldsymbol{f}(t)\cdot\boldsymbol{g}(t) = const$ for all t.

Then:

$$\boldsymbol{f}'(t)\cdot\boldsymbol{g}(t) = -\boldsymbol{f}(t)\cdot\boldsymbol{g}'(t)$$

And in particular by taking $g = f$:

$|\boldsymbol{f}(t)| = const$ if and only if $\boldsymbol{f}(t)\cdot\boldsymbol{f}'(t)=0$ for all t

Detour to Vector Calculus

Proof:

If $\boldsymbol{f} \cdot \boldsymbol{g}$ is constant for all t, then $(\boldsymbol{f} \cdot \boldsymbol{g})' = 0$.

From the product rule we have:

$$(\boldsymbol{f} \cdot \boldsymbol{g})'(t) = \boldsymbol{f}(t)' \cdot \boldsymbol{g}(t) + \boldsymbol{f}(t) \cdot \boldsymbol{g}'(t) = 0$$

$$\rightarrow \quad \boxed{\boldsymbol{f}'(t) \cdot \boldsymbol{g}(t) = -\boldsymbol{f}(t) \cdot \boldsymbol{g}'(t)}$$

Taking $\boldsymbol{f} = \boldsymbol{g}$ we get:

$$\boldsymbol{f}'(t) \cdot \boldsymbol{f}(t) = -\boldsymbol{f}(t) \cdot \boldsymbol{f}'(t)$$

$$\rightarrow \quad \boxed{\boldsymbol{f}'(t) \cdot \boldsymbol{f}(t) = 0}$$

Back to Curves

$\boldsymbol{\alpha}$ is parameterized by arc length

$\rightarrow \quad \boldsymbol{\alpha}'(s) \cdot \boldsymbol{\alpha}'(s) = 1$

Applying the Lemma

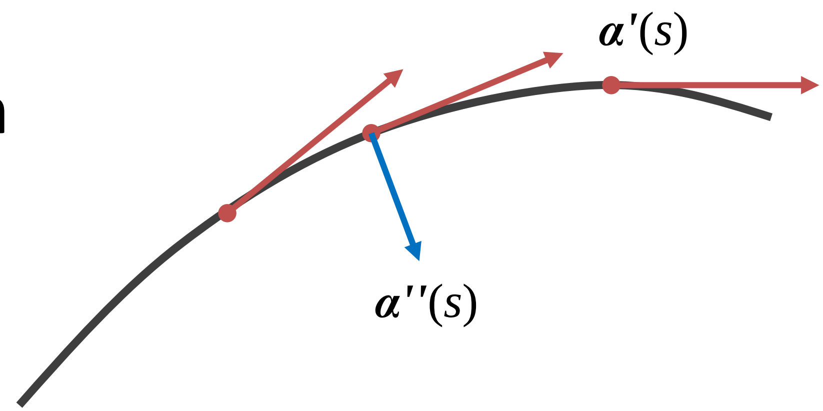
$\rightarrow \quad \boldsymbol{\alpha}''(s) \cdot \boldsymbol{\alpha}'(s) = 0$

$\rightarrow$ $\boldsymbol{\alpha}''(s)$ is *orthogonal* to the tangent vector

The Normal Vector

$\boldsymbol{\alpha}'(s) = \boldsymbol{T}(s)$ - tangent direction

$|\boldsymbol{\alpha}'(s)|$ - speed

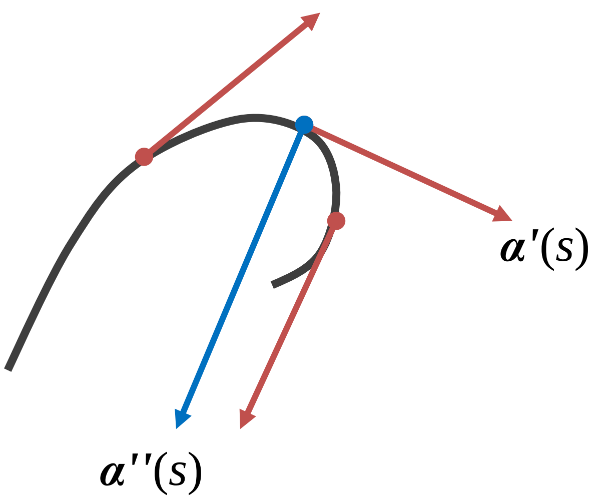


$\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s)$ - normal direction

$|\boldsymbol{\alpha}''(s)|$ - curvature

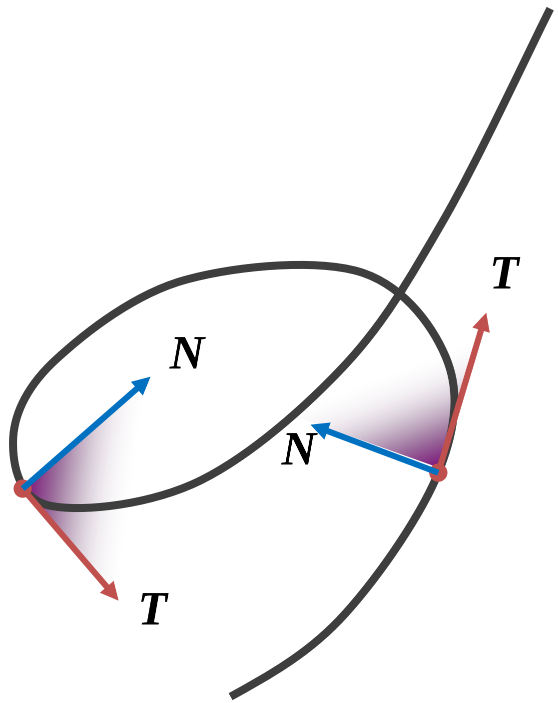
If $|\boldsymbol{\alpha}''(s)| \neq \mathbf{0}$, define $\boldsymbol{N}(s) = \boldsymbol{T}'(s)/|\boldsymbol{T}'(s)|$

Then $\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s)$



The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

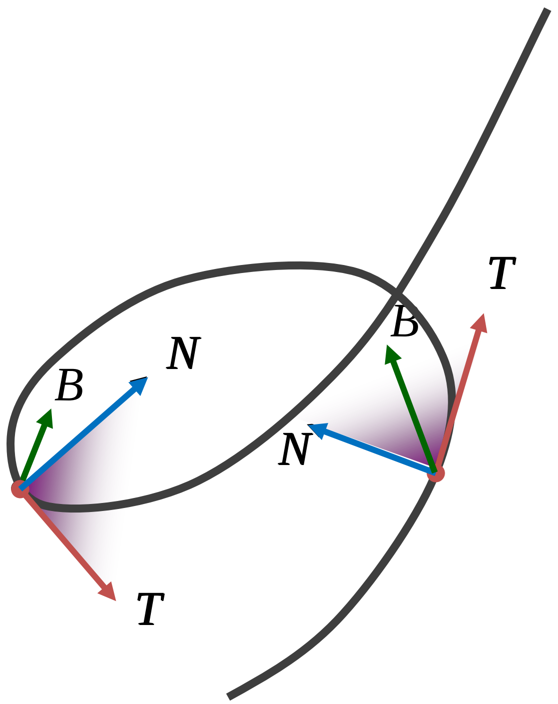


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane

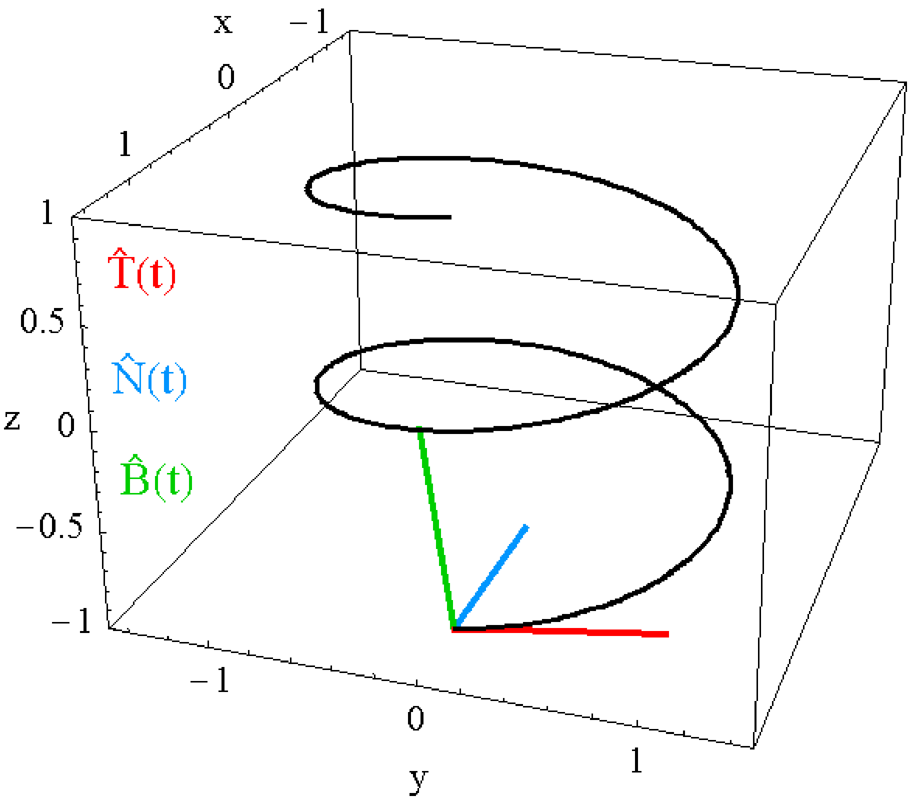


The Frenet Frame

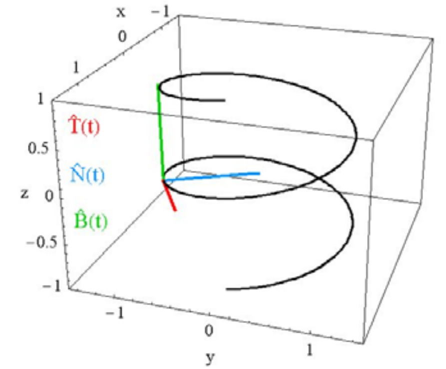
$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*

How does the frame change when the particle moves?

What are $\boldsymbol{T}', \boldsymbol{N}', \boldsymbol{B}'$ in terms of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$?



$\boldsymbol{T}\,'\,(\boldsymbol{s})$

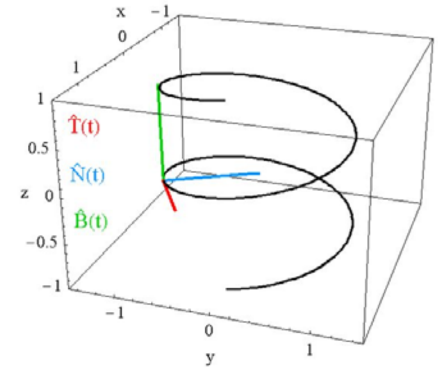


Already used it to define the curvature:

$$\boxed{\boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s)}$$

Since in the direction of the normal, its orthogonal to $\boldsymbol{B}$ and $\boldsymbol{T}$

$N'(s)$



What is $N'(s)$ as a combination of N, T, B ?

We know: $N(s) \cdot N(s) = 1$

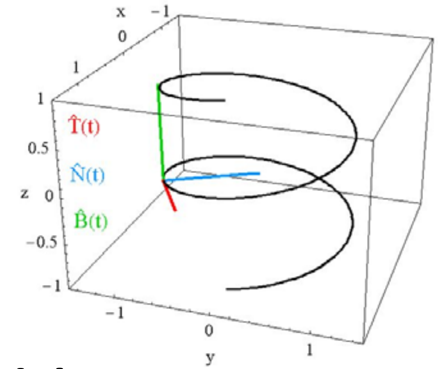
From the lemma $\rightarrow N'(s) \cdot N(s) = 0$

We know: $N(s) \cdot T(s) = 0$

From the lemma $\rightarrow N'(s) \cdot T(s) = -N(s) \cdot T'(s)$

From the definition $\rightarrow \kappa(s) = N(s)\, T'(s)$

$\rightarrow N'(s) \cdot T(s) = -\kappa(s)$



The Torsion

Let $\alpha: I \to R^3$ be a curve parameterized by arc length s. The *torsion* of α at s is defined by:

$$\tau(s) = \boldsymbol{N}'(s) \cdot \boldsymbol{B}(s)$$

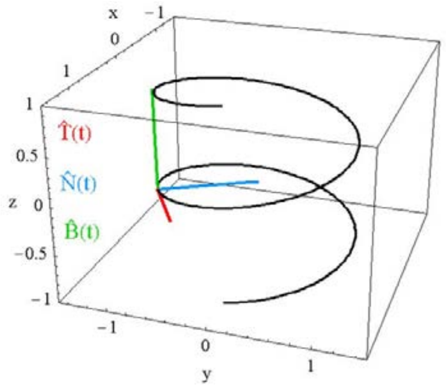
Now we can express $\boldsymbol{N}'(s)$ as:

$$\boxed{\boldsymbol{N}'(s) = -\kappa(s)\,\boldsymbol{T}(s) + \tau(s)\,\boldsymbol{B}(s)}$$

$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$

Curvature vs. Torsion

The *curvature* indicates how much the normal changes, in the direction tangent to the curve

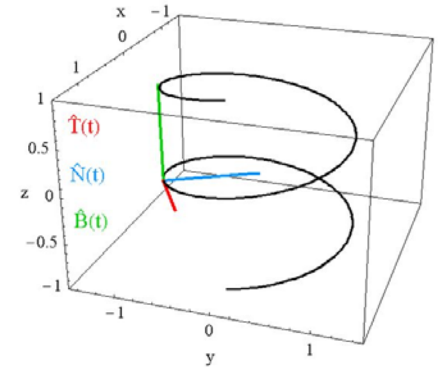


The *torsion* indicates how much the normal changes, in the direction orthogonal to the osculating plane of the curve

The curvature is always positive, the torsion can be negative

Both properties *do not* depend on the choice of parameterization

$\boldsymbol{B}'(s)$



What is $\boldsymbol{B}'(s)$ as a combination of $\boldsymbol{N},\boldsymbol{T},\boldsymbol{B}$?

We know: $\boldsymbol{B}(s) \cdot \boldsymbol{B}(s) = 1$

From the lemma $\rightarrow$ $\boldsymbol{B}'(s) \cdot \boldsymbol{B}(s) = 0$

We know: $\boldsymbol{B}(s) \cdot \boldsymbol{T}(s) = 0, \boldsymbol{B}(s) \cdot \boldsymbol{N}(s) = 0$

From the lemma $\rightarrow$

$$\boldsymbol{B}'(s) \cdot \boldsymbol{T}(s) = -\boldsymbol{B}(s) \cdot \boldsymbol{T}'(s) = -\boldsymbol{B}(s) \cdot \kappa(s)\boldsymbol{N}(s) = 0$$

From the lemma $\rightarrow$

$$\boldsymbol{B}'(s) \cdot \boldsymbol{N}(s) = -\boldsymbol{B}(s) \cdot \boldsymbol{N}'(s) = -\tau(s)$$

Now we can express $\boldsymbol{B}'(s)$ as:

$$\boxed{\boldsymbol{B}'(s) = -\tau(s)\,\boldsymbol{N}(s)}$$

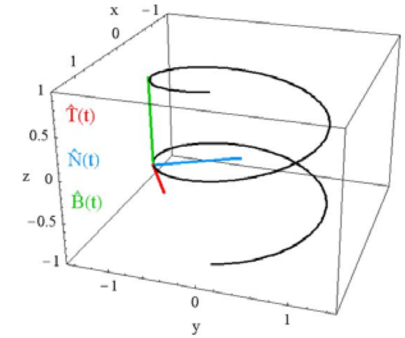
The Frenet Formulas

$$\begin{aligned}
\boldsymbol{T}'(s) &= \kappa(s)\boldsymbol{N}(s) \\
\boldsymbol{N}'(s) &= -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s) \\
\boldsymbol{B}'(s) &= -\tau(s)\boldsymbol{N}(s)
\end{aligned}$$

In matrix form:

$$\begin{bmatrix} | & | & | \\ \boldsymbol{T}'(s) & \boldsymbol{N}'(s) & \boldsymbol{B}'(s) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \boldsymbol{T}(s) & \boldsymbol{N}(s) & \boldsymbol{B}(s) \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}$$

An Example – The Helix



$\boldsymbol{\alpha}(t) = (a \cos(t), a \sin(t), bt)$

In arc length parameterization:

$\boldsymbol{\alpha}(s) = (a \cos(s/c), a \sin(s/c), bs/c)$, where $c = \sqrt{a^2 + b^2}$

Curvature: $\kappa(s) = \dfrac{a}{a^2 + b^2}$ Torsion: $\tau(s) = \dfrac{b}{a^2 + b^2}$

Note that both the curvature and torsion are constants

A Thought Experiment

Take a straight line

Bend it to add curvature

Twist it to add torsion

$\rightarrow$ You got a curve in R^3

Can we define a curve in R^3 by specifying its curvature and torsion at every point?

The Fundamental Theorem of the Local Theory of Curves

Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there **exists a regular parameterized curve** $\boldsymbol{\alpha}: I \to R^3$ such that s is the arc length, $\kappa(s)$ is the **curvature**, and $\tau(s)$ is the **torsion** of $\boldsymbol{\alpha}$.

Moreover, any other curve $\boldsymbol{\beta}$, satisfying the same conditions, differs from $\boldsymbol{\alpha}$ only by a **rigid motion**.