# III.8 Simplicial Complexes

We use simplicial complexes as the fundamental tool to model geometric shapes and spaces. They generalize and formalize the somewhat loose geometric notions of a triangulation. Because of their combinatorial nature, simplicial complexes are perfect data structures for geometric modeling algorithms.

**Simplices.** A finite collection of points is *affinely independent* if no affine space of dimension $i$ contains more than $i+1$ of the points, and this is true for every $i$. A *$k$-simplex* is the convex hull of a collection of $k+1$ affinely independent points, $\sigma = \operatorname{conv} S$. The *dimension* of $\sigma$ is $\dim \sigma = k$. In $\mathbb{R}^d$, the largest number of affinely independent points is $d+1$, and we have simplices of dimension $-1$ through $d$. The $(-1)$-simplex is the empty set. Figure III.1 shows the four types of non-empty simplices in $\mathbb{R}^3$. The convex

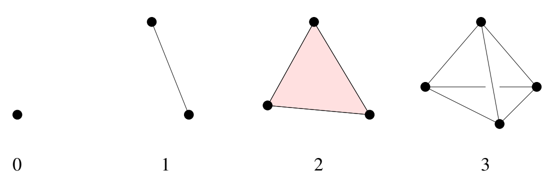


Figure III.1: A 0-simplex is a point or vertex, a 1-simplex is an edge, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

hull of any subset $T \subseteq S$ is again a simplex. It is a subset of $\operatorname{conv} S$ and called a *face* of $\sigma$, which is denoted as $\tau \leq \sigma$. If $\dim \tau = \ell$ then $\tau$ is called an *$\ell$-face*. $\tau = \emptyset$ and $\tau = \sigma$ are *improper* faces and all others are *proper* faces of $\sigma$. The number of $\ell$-faces of $\sigma$ is equal to the number of ways we can choose $\ell + 1$ from $k+1$ points, which is $\binom{k+1}{\ell+1}$. The total number of faces is

$$\sum_{\ell=-1}^{k} \binom{k+1}{\ell+1} \quad = \quad 2^{k+1}.$$

**Simplicial complexes.** A *simplicial complex* is the collection of faces of a finite number of simplices, any two of which are either disjoint or meet in a common face. More formally, it is a collection $K$ such that

(i) $\sigma \in K \wedge \tau \leq \sigma \Longrightarrow \tau \in K$, and

(ii) $\sigma, \upsilon \in K \Longrightarrow \sigma \cap \upsilon \leq \sigma, \upsilon$.

Note that $\emptyset$ is a face of every simplex and thus belongs to $K$ by Condition (i). Condition (ii) therefore allows for the possibility that $\sigma$ and $\upsilon$ be disjoint. Figure III.2 shows three sets of simplices that each violate one of the two conditions and therefore do not form complexes. A *subcomplex* is a subset that is a sim-

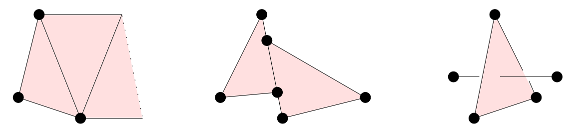

Figure III.2: To the left, we are missing an edge and two vertices. In the middle, the triangles meet along a segment that is not an edge of either triangle. To the right, the edge crosses the triangle at an interior point.

plicial complex itself. Observe that every subset of a simplicial complex satisfies Condition (ii). To enforce Condition (i), we may add faces and simplices to the subset. Formally, the *closure* of a subset $L \subseteq K$ is the smallest subcomplex that contains $L$,

$$\operatorname{Cl} L \quad = \quad \{\tau \in K \mid \tau \leq \sigma \in L\}.$$

A particular subcomplex is the *$i$-skeleton* that consists of all simplices $\sigma \in K$ whose dimension is $i$ or less. The *vertex set* is $\operatorname{Vert} K = \{\sigma \in K \mid \dim \sigma = 0\}$, which is the 0-skeleton minus the $(-1)$-simplex. The *dimension* of $K$ is the largest dimension of any simplex, $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$. If $k = \dim K$ then $K$ is a *$k$-complex*. The *underlying space* is the set of points covered by simplices, $\|K\| = \bigcup K = \bigcup_{\sigma \in K} \sigma$. A *polyhedron* is the underlying space of a simplicial complex.

**Stars and links.** We use special subsets to talk about the local structure of a simplicial complex. These subsets may or may not be closed. The *star*

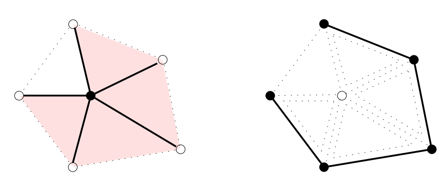

Figure III.3: Star and link of a vertex. To the left, the solid edges and shaded triangles belong to the star of the solid vertex. To the right the solid edges and vertices belong to the link of the hollow vertex.

of a simplex $\tau$ consists of all simplices that contain $\tau$, and the *link* consists of all faces of simplices in the star that do not intersect $\tau$,

$$\begin{aligned} \operatorname{St} \tau &= \{\sigma \in K \mid \tau \leq \sigma\}, \\ \operatorname{Lk} \tau &= \{\sigma \in \operatorname{Cl} \operatorname{St} \tau \mid \sigma \cap \tau = \emptyset\}. \end{aligned}$$

Figure III.3 illustrates this definition by showing the star and the link of a vertex in a 2-complex. The star is generally not closed, but the link is always a simplicial complex.

**Abstract simplicial complexes.** By substituting the *set* of vertices for each simplex, we get a system of subsets of the vertex set. In doing so, we throw away the geometry of the simplices and focus on the combinatorial structure. Formally, a finite system $A$ of finite sets is an *abstract simplicial complex* if $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$. This requirement is similar to Condition (i) for geometric simplicial complexes. A set $\alpha \in A$ is an *(abstract) simplex* and its dimension is $\dim \alpha = \operatorname{card} \alpha - 1$. The *vertex set* of $A$ is $\operatorname{Vert} A = \bigcup A = \bigcup_{\alpha \in A} \alpha$.

Observe that $A$ is a subsystem of the power set of $\operatorname{Vert} A$. We can therefore think of it as a subcomplex of an $n$-simplex, where $n + 1 = \operatorname{card} \operatorname{Vert} A$. This view is expressed in the picture of an abstract simplicial complex shown as Figure III.4. The concepts of

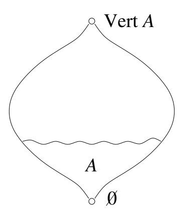


Figure III.4: The onion is the power set of $\operatorname{Vert} A$. The area below the waterline is an abstract simplicial complex.

face, subcomplex, closure, star, link extend straightforwardly from geometric to abstract simplicial complexes.

**Posets.** The set system together with the inclusion relation forms a partially ordered set, or poset, denoted as $(A, \subseteq)$. Posets are commonly drawn using Hasse diagrams, where sets are nodes, smaller sets are below larger sets, and inclusions are edges. Figure III.5 shows the Hasse diagrams of simplices of dimension 0 through 3. Implied inclusions are usually not drawn.

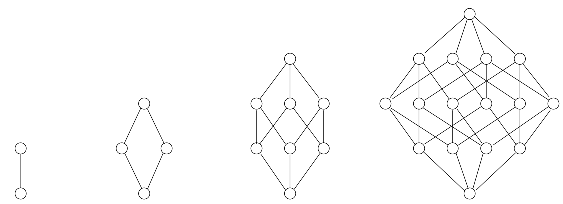

Figure III.5: From left to right, the poset of a vertex, an edge, a triangle, a tetrahedron.

Here is a recursive way to construct the Hasse diagram of a $k$-simplex $\alpha$. First draw the Hasse diagrams for two $(k-1)$-simplices. One is the diagram of a $(k-1)$-face $\beta$ of $\alpha$ and the other is the diagram for the star of the vertex $u \in \alpha - \beta$. Finally, connect every simplex $\gamma$ in the star of $u$ with the simplex $\gamma - \{u\}$ in the closure of $\beta$.

**Geometric realization.** We can think of an abstract simplicial complex as an abstract version of a geometric simplicial complex. To formalize this idea, we define a *geometric realization* of an abstract simplicial complex $A$ as a simplicial complex $K$ together with a bijection $\varphi : \operatorname{Vert} A \to \operatorname{Vert} K$ such that $\alpha \in A$ iff $\operatorname{conv} \varphi(\alpha) \in K$. $A$ is sometimes called an *abstraction* of $K$.

Given $A$, we can ask for the smallest number of dimensions that allow a geometric realization. For example, graphs are 1-dimensional abstract simplicial complexes and can always be realized in $\mathbb{R}^3$. Two dimensions are sometimes but not always sufficient. This result generalized to $k$-dimensional abstract simplicial complexes. They can always be realized in $\mathbb{R}^{2k+1}$ and sometimes $\mathbb{R}^{2k}$ does not suffice. To prove the sufficiency of the claim, we show that the $k$-skeleton of every $n$-simplex can be realized in $\mathbb{R}^{2k+1}$. Map the $n+1$ vertices to points in general position in $\mathbb{R}^{2k+1}$. Specifically, we require that any $2k+2$ of the points are affinely independent. Two simplices $\sigma$ and $\upsilon$ of the $k$-skeleton have a total of at most $2(k+1)$ vertices, which are therefore affinely independent. In other words, $\sigma$ and $\upsilon$ are faces of a common simplex of dimension at most $2k+1$. Hence, $\sigma \cap \upsilon$ is a common face of both.

**Nerves.** A convenient way to construct abstract simplicial complexes starts from an arbitrary finite set. The *nerve* of such a set $C$ is the system of subsets with non-empty intersection,

$$\mathrm{Nrv}\, C \quad = \quad \{\alpha \subseteq C \mid \bigcap \alpha \neq \emptyset\}.$$

If $\beta \subseteq \alpha$ then $\bigcap \alpha \subseteq \bigcap \beta$. Hence $\alpha \in \mathrm{Nrv}\, C$ implies $\beta \in \mathrm{Nrv}\, C$, which shows that the nerve is an abstract simplicial complex. Consider for example the case where $C$ covers some geometric space, such as the union of elliptic regions in Figure III.6. Every set in the covering corresponds to a vertex, and $k+1$ sets with non-empty intersection define a $k$-simplex.

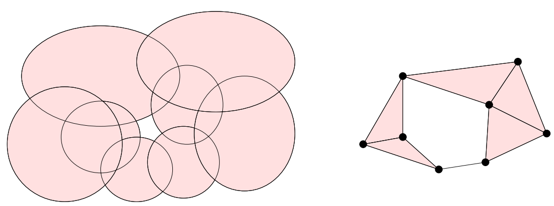

Figure III.6: A covering with eight sets to the left and a geometric realization of its nerve to the right. The sets meet in triplets but not in quadruplets, which implies that the nerve is 2-dimensional.

We have seen an example of such a construction earlier. The Voronoi regions of a finite set $S \subseteq \mathbb{R}^2$ define a covering $C = \{V_a \mid a \in S\}$ of the plane. Assuming general position, the Voronoi regions meet in pairs and in triplets, but not in quadruplets. The nerve contains abstract vertices, edges, triangles, but no abstract tetrahedra. Consider the function $\varphi : C \to \mathbb{R}^2$ that maps a Voronoi region to its generator, $\varphi(V_a) = a$. This function defines a geometric realization of $\mathrm{Nrv}\, C$, namely

$$D \quad = \quad \{\mathrm{conv}\, \varphi(\alpha) \mid \alpha \in \mathrm{Nrv}\, C\}.$$

This is the Delaunay triangulation of $S$. What happens if the points in $S$ are not in general position? If $k+1 \geq 4$ Voronoi regions have a non-empty common intersection then $\mathrm{Nrv}\, C$ contains the corresponding abstract $k$-simplex. So instead of making a choice among the possible triangulations of the $(k+1)$-gon, the nerve takes all possible triangulations together and interprets them as subcomplexes of a $k$-simplex. The disadvantage of this method is of course that a $k$-simplex for $k \geq 3$ cannot be realized in $\mathbb{R}^2$.

**Bibliographic notes.** During the first half of the twentieth century, combinatorial topology was a flourishing field within Mathematics. We refer to Paul Alexandrov [1] as a comprehensive text originally published as a series of three books. This text roughly coincides with a fundamental reorganization of the field triggered by a variety of technical results in topology. One of the successors of combinatorial topology is modern algebraic topology, where the emphasis shifts from combinatorial to algebraic structures. We refer to James Munkres [5] for an introductory text in that area.

We have seen that every $k$-complex can be geometrically realized in $\mathbb{R}^{2k+1}$. Examples of $k$-complexes that require $2k+1$ dimensions are provided by Flores [2] and independently by van Kampen [3]. One such example is the $k$-skeleton of the $(2k+2)$-simplex. For $k = 1$, this is the complete graph of five vertices, which is one of the two obstructions of graph planarity identified by Kuratowski [4].


[1] P. S. Alexandrov. *Combinatorial Topology.* Dover, New York 1956.

[2] A. Flores. Über $n$-dimensionale Komplexe die in $R_{2n+1}$ selbstverschlungen sind. *Ergeb. Math. Koll.* **6** (1933/34), 4-7.

[3] E. R. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Univ. Hamburg* **9** (1933), 72–78.

[4] K. Kuratowski. Sur le problème des courbes en topologie. *Fund. Math.* **15** (1930), 271–283.

[5] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.