IV.13 Preserving Topology

The surface simplification algorithm of the last section works by contracting edges. We preserve the topological type by rejecting the contraction of edges that would change it. This section describes local conditions that characterize type preserving edge contractions. We first study manifolds, then manifolds with boundary, and finally general 2-complexes.

Manifolds. Suppose K is a 2-complex that triangulates a 2-manifold. Then every point $x \in \| K \|$ has a neighborhood homeomorphic to an open disk. To avoid lengthy sentences we just say the neighborhood *is* an open disk. This implies that in particular the star of every vertex u is an open disk. Strictly speaking this statement makes sense only if we replace the star by its underlying space, which we define as the union of simplex interiors, which is the set difference between the underlying spaces of two complexes,

$$\begin{aligned} | \mathrm{St}\, u | &= \bigcup_{\tau \in \mathrm{St}\, u} \mathrm{int}\, \tau \\ &= \| \mathrm{Cl}\, \mathrm{St}\, u \| - | \mathrm{Cl}\, \mathrm{St}\, u - \mathrm{St}\, u |. \end{aligned}$$

The condition on vertex stars is also sufficient. In other words, $| K |$ is a 2-manifold iff $\| \mathrm{St}\, K \| \approx \mathbb{R}^2$ for every vertex $u \in K$.

Now consider the contraction of an edge ab of K. Whether or not the contraction preserves the topological type depends on how the links of a and b meet. On a 2-manifold, the link of each vertex is a circle. In Figure IV.1 to the left, the two circles intersect in two points and the contraction preserves the topological type. To the right, the circles intersect in a point and an edge, and in this case the contraction pinches the manifold along a newly formed edge which forms the base of a fin similar to the one in Figure IV.5.

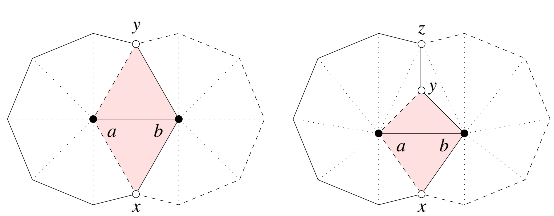

Figure IV.1: The edges of the link of a are solid and those of the link of b are dashed.

Link condition. The condition that distinguishes topology preserving edge contractions from others is that the vertex links intersect in the link of the edge.

Link Condition Lemma A. Let K be the triangulation of a 2-manifold. The contraction of $ab \in K$ preserves the topological type iff $\mathrm{Lk}\, a \cap \mathrm{Lk}\, b = \mathrm{Lk}\, ab$.

Proof. We prove only the more difficult direction, which is from the link condition to $| K | \approx | L |$. Since $| K |$ is a 2-manifold, the link of ab consists of exactly two vertices x and y, as shown in the left picture of Figure IV.1. The links of a and of b are two circles which meet at x and y. The outer pieces of the two circles glued at x and y form another circle, which is the link of $\overline{ab}$ in K and also the link of c in L. We construct isomorphic subdivisions of K and L by mapping the common link to the boundary of a regular n-gon in the plane, as shown in Figure IV.2. The stars

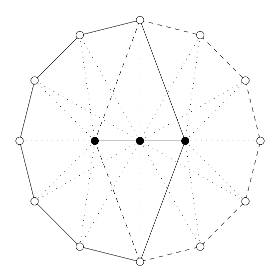

Figure IV.2: The superposition of the images of the stars of $\overline{ab}$ in K and of c in L.

of $\overline{ab}$ and of c are mapped to two triangulations of the n-gon. We superimpose the triangulations and get a decomposition into convex polygonal regions, which we further refine to another triangulation. This triangulation is mapped back to form subdivisions of the stars of $\overline{ab}$ and of c. The link has not been changed, so we can combine the subdivided star of $\overline{ab}$ with the unsubdivided rest of K, and similar for the star of c and L. The resulting subdivisions of K and L are isomorphic. We can now map corresponding triangles to each other and thus obtain a homeomorphism between $\| K |$ and $| L |$. ⊔⊓

A more formal description of how to create the homeomorphism from the isomorphic subdivisions requires simplicial maps, which will be introduced in Section **??**.

Manifolds with boundary. A triangulation K of a manifold with non-empty boundary also has vertices whose stars are open half-disks, $|\operatorname{St} u| \approx \mathbb{H}^2$. To keep the number of cases small, we add a dummy vertex ω and the cone from ω to each boundary circle. This idea is illustrated in Figure IV.3. The boundary of

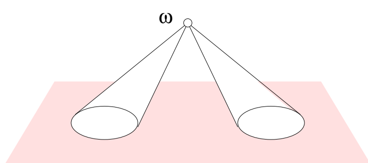

Figure IV.3: The two holes in the manifold are filled by adding the cone from ω to the circles bounding the holes.

$|K|$ consists of $\ell \geq 1$ circles triangulated by cycles $C_i \subseteq K$. We fill the holes by adding the cone from ω to every cycle,

$$K^\omega = K \cup (\omega \cdot \bigcup_{i=1}^{\ell} C_i).$$

In K^ω, every vertex star is an open disk except possibly the star of ω. We denote the link of a vertex u in K^ω as $\operatorname{Lk}^\omega u$. The condition that distinguishes topology preserving edge contractions from others is now the same as for manifolds.

Link Condition Lemma B. Let K be the triangulation of a 2-manifold with boundary. The contraction of $ab \in K$ preserves the topological type iff $\operatorname{Lk}^\omega a \cap \operatorname{Lk}^\omega b = \operatorname{Lk}^\omega ab$.

The proof is only mildly more complicated than that of the weaker Lemma A.

Open books. To attack the problem for general 2-complexes, we need a better understanding of the different types of neighborhoods that are possible. We classify stars using a new type of space. The *open book with p pages* is the topological space $\mathbb{K}_p^2$ homeomorphic to the union of p copies of $\mathbb{H}^2$ glued along the common boundary line. For example, the open book with one page is the open half-disk and the open book with two pages is the open disk. The *order* of a simplex $\tau \in K$ is

$$\operatorname{ord} \tau = \begin{cases} 0 & \text{if } |\operatorname{St} \tau| \approx \mathbb{R}^2, \\ 1 & \text{if } |\operatorname{St} \tau| \approx \mathbb{K}_p^2,\ p \neq 2, \\ 2 & \text{otherwise.} \end{cases}$$

Figure IV.4 illustrates the definition with sketches of four different types of vertex stars. The order of an edge in a 2-complex can only be 0 or 1, and the order of a triangle is always 0.

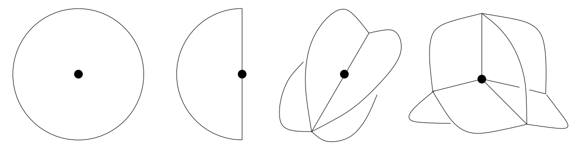

Figure IV.4: The underlying space of the vertex star in (a) is an open disk, in (b) is an open half-disk, in (c) is an open book with 4 pages, and in (d) is not an open book. The corresponding order of the vertex is 0 in (a), 1 in (b), 1 in (c), and 2 in (d).

Boundary. We generalize the notion of boundary in such a way that only triangulations of 2-manifolds have no boundary. At the same time we use the order information to distinguish between different types of boundaries. Specifically, the *j-th boundary* of a 2-complex K is the collection of all simplices with order j or higher,

$$\operatorname{Bd}_j K = \{\sigma \in K \mid \operatorname{ord} \sigma \geq j\}.$$

As an example consider the shark fin complex shown in Figure IV.5. It is constructed by gluing two closed disks along a simple path. This path is a contiguous piece of the boundary of one disk (the fin) and it lies in the interior of the other disk. Note that $|K|$ is a 2-

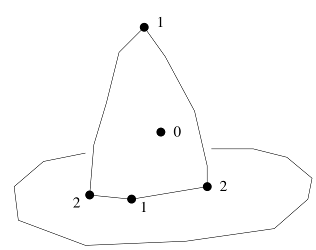

Figure IV.5: The shark fin 2-complex. A few of the vertices are high-lighted and marked with their order.

manifold iff $\operatorname{Bd}_1 K = \operatorname{Bd}_2 K = \emptyset$. The 2-nd boundary of a 2-manifold with boundary is empty, but there are other spaces with this property. For example, the sphere together with its equator disk has empty 2-nd boundary. Its 1-st boundary is a circle of edges and vertices (the equator) whose stars are open books of 3 pages each.

2-complexes. We are now ready to study conditions under which an edge contraction in a general 2-complex preserves the topological type of that complex. As it turns out, there does not exist a local condition that is sufficient and necessary, but there is a characterizing local condition for a more restrictive notion of type preservation. Let L be the 2-complex obtained from K by contracting an edge $ab \in K$. A *local unfolding* is a homeomorphism $f : |K| \to |L|$ that differs from the identity only outside the star of $\overline{ab}$, that is, $f(x) = x$ for all $x \in \| K - \mathrm{St}\, \overline{ab} |$. The condition refers to links $\mathrm{Lk}_0^\omega \tau$ in $K^\omega = K \cup (\omega \cdot \mathrm{Bd}_1 K)$ and to links $\mathrm{Lk}_1^\omega \tau$ in $\mathrm{Bd}_1^\omega K = \mathrm{Bd}_1 K \cup (\omega \cdot \mathrm{Bd}_2 K)$.

LINK CONDITION LEMMA C. Let K be a 2-complex, ab an edge of K, and L the complex obtained by contracting ab. There is a local unfolding $|K| \to |L|$ iff

(i) $\mathrm{Lk}_0^\omega a \cap \mathrm{Lk}_0^\omega b = \mathrm{Lk}_0^\omega ab$ and
(ii) $\mathrm{Lk}_1^\omega a \cap \mathrm{Lk}_0^\omega b = \emptyset$.

The proof that conditions (i) and (ii) suffice for the existence of a local unfolding is similar to proof of sufficiency in Lemma A, only more involved. The necessity of conditions (i) and (ii) requires a somewhat tedious case analysis.

Non-local homeomorphism. Instead of proving Lemma C, we show that there cannot be a similar condition that recognizes the existence of a general homeomorphism $\|K| \to \|L|$. The example we use is the folding chair complex displayed in Figure IV.6. Before the contraction of ab, it consists of five triangles in the star of x and four disks U, V, Y, Z glued to the link of x. Vertices a and b belong to the 1-st boundary, but ab does not. It follows that ω violates condition (i) of the Lemma C. Hence, there is no local unfolding from $|K|$ to $|L|$. After the contraction there is one less triangle in the star of x, U loses two triangles, and V, Y, Z are unchanged. The contraction of ab exchanges left and right in the asymmetry of the complex. We can find a homeomorphism $|K| \to |L|$ that acts like a mirror and maps U to V, V to U, Y to Z, Z to Y. The homeomorphism is necessarily global. To detect that homeomorphism, we can force any algorithm to look at every triangle of K.

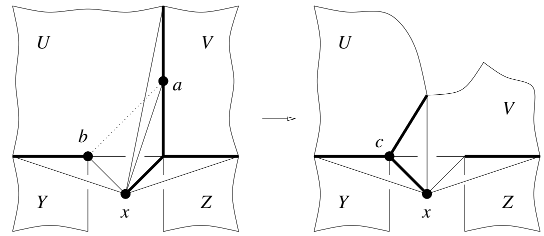

Figure IV.6: The folding chair complex. The bold edges belong to three triangles each.

Bibliographic notes. The material of this lecture is taken from a paper by Dey et al. [1], which studies edge contractions in general simplicial complexes and proves results for 2- and for 3-complexes. The order of a simplex has already been defined in 1960 by Whittlesey [4], although in different words and notation. He uses the concept to study the topological classification of 2-complexes. O'Dunlaing et al. [2] use his results to show that deciding whether or not two 2-complexes have the same topological type is as hard as deciding whether or not two graphs are isomorphic. No polynomial time algorithm is known, but it is also not known whether the graph isomorphism problem is NP-complete [3].

[1] T. K. DEY, H. EDELSBRUNNER, S. GUHA AND D. V. NEKHAYEV. Topology preserving edge contraction. *Publ. Inst. Math. (Beograd)* **66** (1999), to appear.

[2] C. O'DUNLAING, C. WATT AND D. WILKINS. Homeomorphism of 2-complexes is equivalent to graph isomorphism. Rept. TCDMATH 98-04, Math. Dept., Trinity College, Ireland, 1998.

[3] M. R. GAREY AND D. S. JOHNSON. *Computers and Intractibility. A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, California, 1979.

[4] E. F. WHITTLESEY. Finite surfaces: a study of finite 2-complexes. *Math. Mag.* **34** (1960), 11–22 and 67–80.