CS348a: Computer Graphics Handout #21
Geometric Modeling Original Handout #17
Stanford University Thursday, 10 December 1992

Original Lecture #7: 22 October 1992
Topics: Polar Forms of Cubic Polynomial Curves
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Much of what was covered during this lecture is described in Handout 19 in the course reader. Also, a significant portion of this handout was taken from last years notes by Amala Mahadevan.

1 Introduction to Cubics

We have seen so far that we can interpolate between two points to get all the points along the line segment between the two points using the de Casteljau Algorithm. Figure 1 illustrates this.

For the quadratic case ($n = 2$), we have also seen how the de Casteljau Algorithm can be used to interpolate the points of a quadratic curve given the three Bézier control points of a parabolic segment. Figure 2 shows three control points of a parabolic segment. Figure 3 illustrates how the de Casteljau Algorithm can be used to interpolate to get a point at $F(t)$. Notice that we can determine the intermediary point $f(r,s)$ given $F(r)$ and $F(s)$ since it is just the intersection of the tangent lines at $F(r)$ and $F(s)$.

As we go to the cubic case, there is no simple rule to tell where along the tangent line to F at $F(r)$ the polar value $f(r,r,s)$ will lie, or where along the tangent line to F at $F(s)$ the polar value $f(r,s,s)$ will lie. However, we can still continue to use the de Casteljau Algorithm to find all the points of a cubic curve given the four Bézier control points.

2 Interpolation of Cubics

We start by reminding ourselves of the three properties of the polar form f of the curve $F(t)$. For the case when $F(t)$ is a cubic :

- $f(u,u,u) = F(u)$

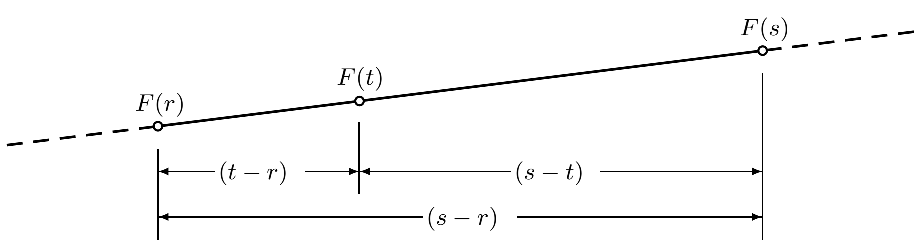

Figure 1: The de Casteljau Algorithm in the case $n = 1$

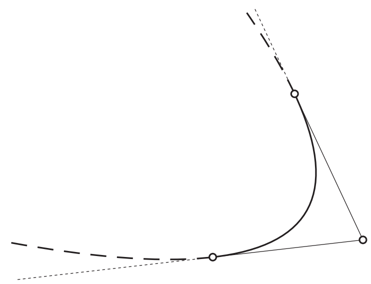

Figure 2: The three Bézier points of a parabolic segment

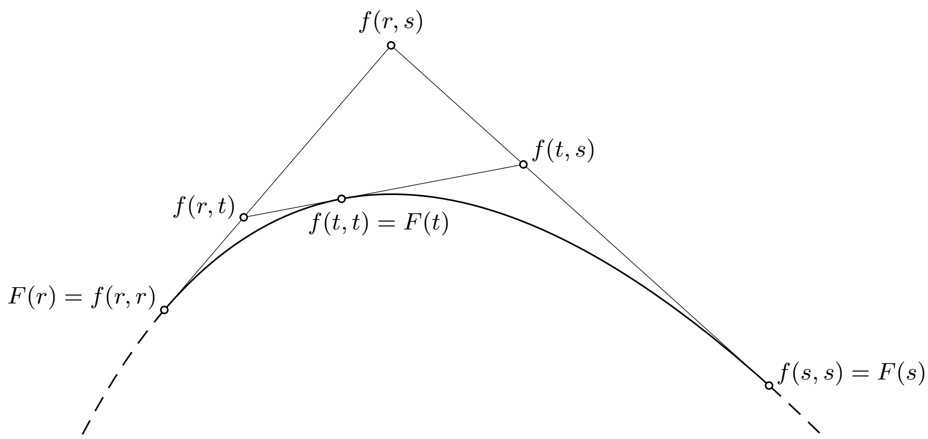

Figure 3: The de Casteljau Algorithm in the case $n = 2$

- f is symmetric, i.e.,

$$f(u,v,w) = f(v,u,w) = f(w,u,v) \text{ and so forth}$$

- f is triaffine.

2.1 The de Casteljau Algorithm

We know that, given the Bezier points $f(0,0,0)$, $f(0,0,1)$, $f(0,1,1)$, and $f(1,1,1)$ of a cubic F, we can find the point $F(t)$ for any t in $[0\,..\,1]$ by repeated interpolation. This is shown in Figure 4. In particular, Figure 5 demonstrates how to find $F(0.4)$.

As is true in the case where $n = 1$ and $n = 2$, we can also find $f(u,v,w)$ when one or more of u, v, and w lie outside the range $[0\,..\,1]$ by extrapolating instead of interpolating. Hence we may find $F(t) = f(t,t,t)$ for any t. Figure 6 illustrates finding $F(2)$ for a cubic, starting from the Bezier points $f(0,0,0)$, $f(0,0,1)$, $f(0,1,1)$, and $f(1,1,1)$ of the segment $F([0\,..\,1])$.

If we carry out the algebra for the interpolations above to find $F(t)$, we get the following result:

$$F(t) = f(t,t,t) = (1-t)^3P + 3t(1-t)^2Q + 3t^2(1-t)R + t^3S \tag{1}$$

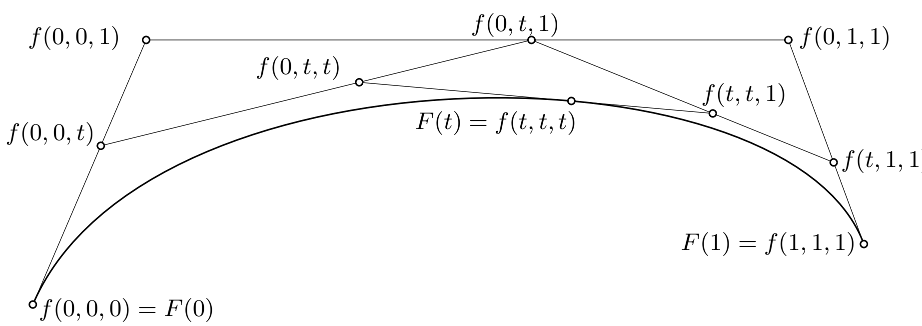

Figure 4: The de Casteljau Algorithm manipulating polar values

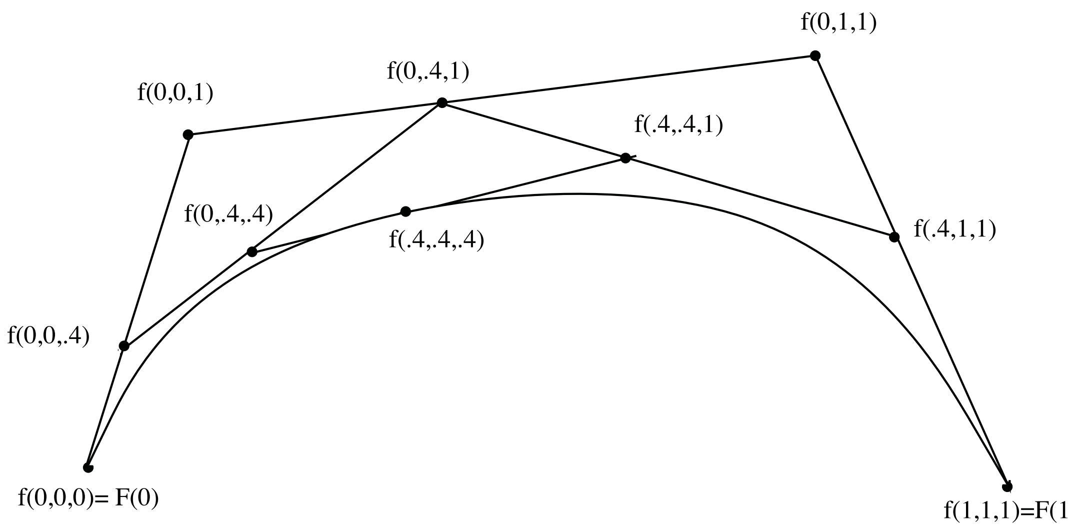

Figure 5: Interpolating using Polar Forms

where $P = f(0,0,0)$, $Q = f(0,0,1)$, $R = f(0,1,1)$, and $S = f(1,1,1)$. The four coefficient polynomials here are known as the Berstein polynomials for the interval $[0\,..\,1]$.

But we can generalize the situation to have the four control points correspond to any interval $[r\,..\,s]$, instead of using just the interval $[0\,..\,1]$. So given the four Bézier points as is shown in Figure 7, we can interpolate using the de Casteljau Algorithm repeatedly as is shown in Figure 8 to get a point $F(t)$.

The following triangular array records the progress of the de Casteljau Algorithm symbolically:

$$\begin{array}{ccccccc} f(r,r,r) & & f(r,r,s) & & f(r,s,s) & & f(s,s,s) \\ & f(r,r,t) & & f(r,t,s) & & f(t,s,s) & \\ & & f(r,t,t) & & f(t,t,s) & & \\ & & & f(t,t,t) & & & \end{array} \tag{2}$$

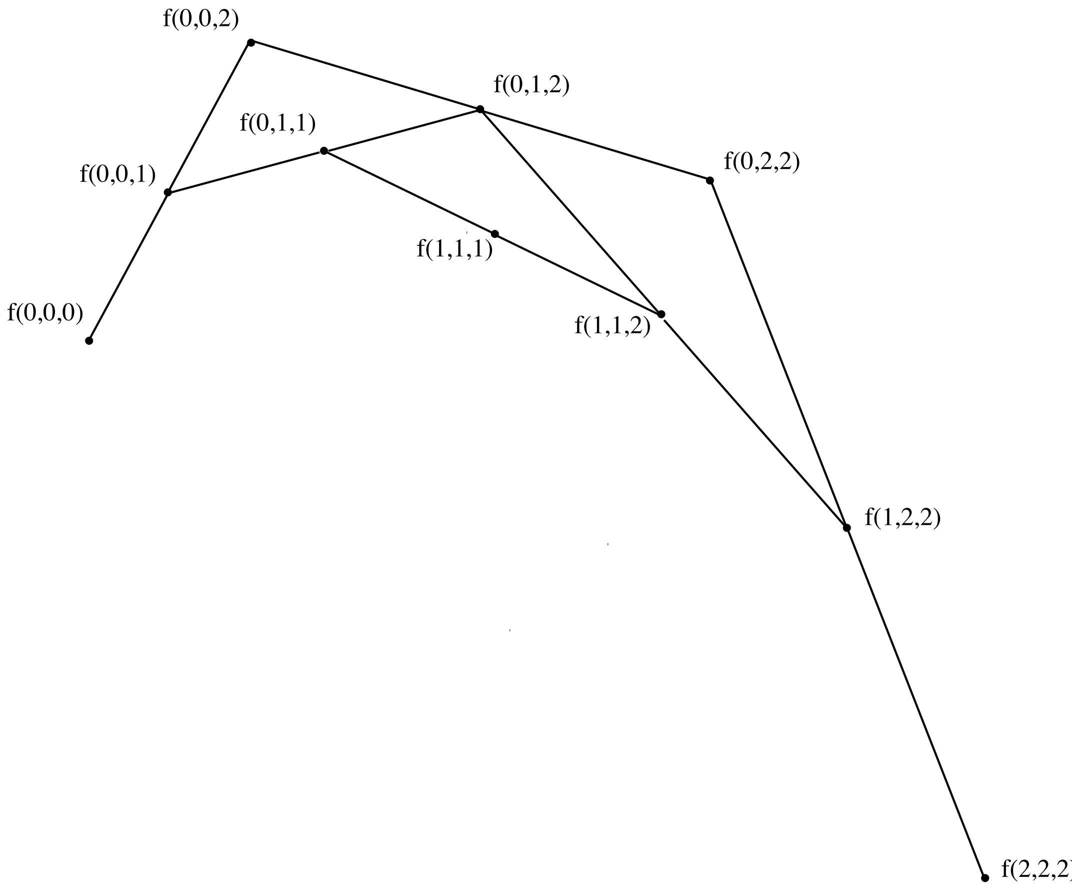

Figure 6: Extrapolating using Polar Forms

If we carry out these interpolations algebraically, we get the following result for $F(t)$:

$$\begin{aligned} F(t) = \quad & \left(\tfrac{s-t}{s-r}\right)^3 f(r,r,r) + 3\left(\tfrac{s-t}{s-r}\right)^2 \left(\tfrac{t-r}{s-r}\right) f(r,r,s) \\ & +3\left(\tfrac{s-t}{s-r}\right)\left(\tfrac{t-r}{s-r}\right)^2 f(r,s,s) + \left(\tfrac{t-r}{s-r}\right)^3 f(s,s,s) \end{aligned}$$

The coefficients in this expansion are called the Berstein polynomials for the interval $[r \,..\, s]$.

The de Casteljau Algorithm can be used to compute polar values as well as diagonal values. For example, Figure 9 shows one of the ways of finding $f(2,3,4)$, starting from the Bezier points $f(0,0,0)$, $f(0,0,6)$, $f(0,6,6)$, and $f(6,6,6)$ of the segment $F([0 \,..\, 6])$. Note that the polar value $f(u,v,w)$ lies on the curve, in general, only when $u = v = w$; that is, diagonal values lie on the curve, but polar values in general don't. In Figure 9, we controlled the first stage of linear interpolations with the polar argument 2, the second stage with 3, and the last stage with 4. Using the arguments in some other order would generate different intermediate points and lines but the same final value $f(2,3,4)$.

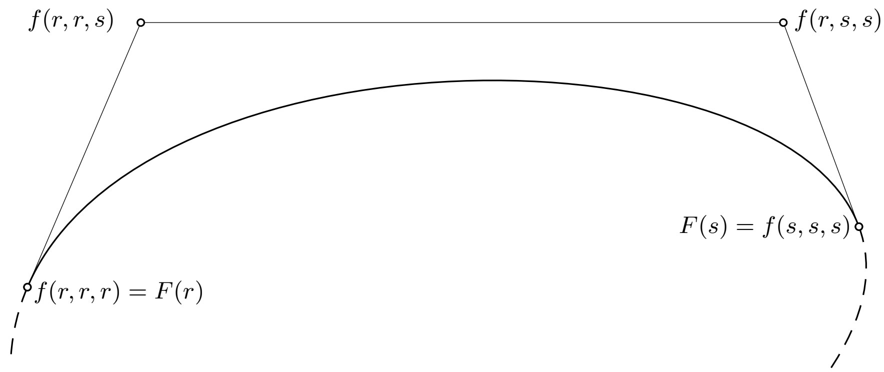

Figure 7: The four Bézier points of a cubic segment

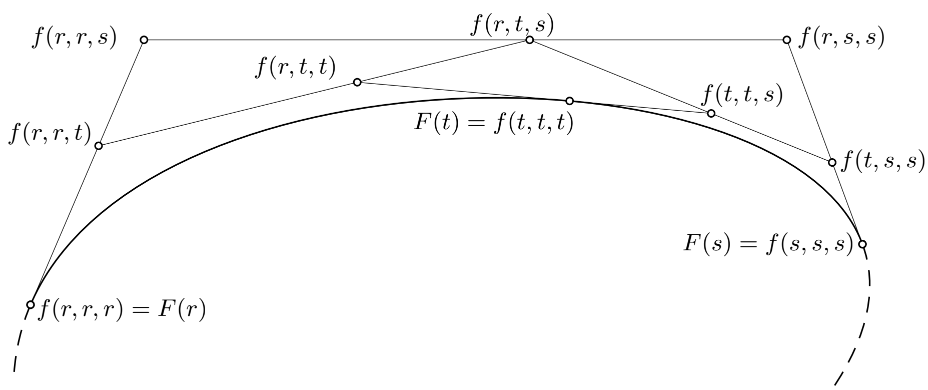

Figure 8: The de Casteljau Algorithm in the case $n = 3$

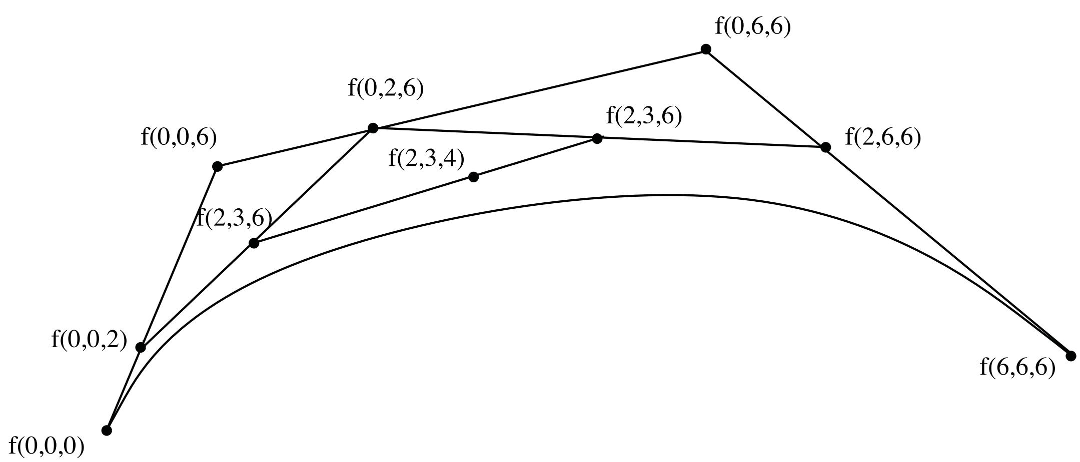

Figure 9: To find $f(2,3,4)$

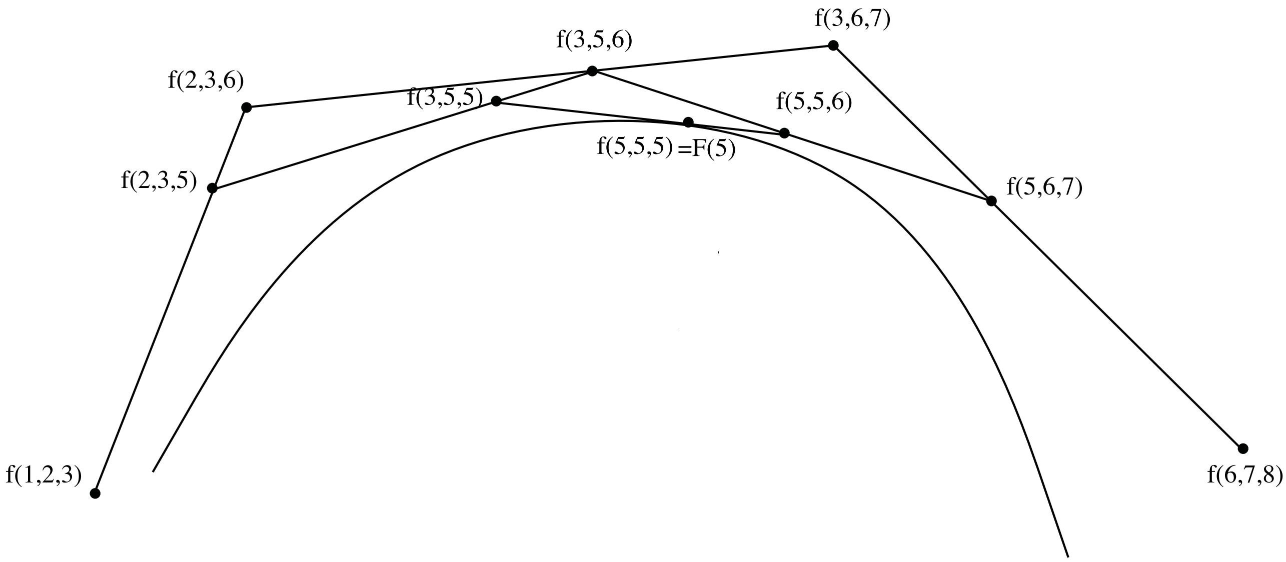

Figure 10: The de Boor Algorithm

2.2 The de Boor Algorithm

The de Boor algorithm is a more general form of the de Casteljau algorithm. Given the de Boor points

$$f(t_1,t_2,t_3), f(t_2,t_3,t_4), f(t_3,t_4,t_5), f(t_4,t_5,t_6),$$

we find $f(u,v,w)$ by interpolation as shown below.

$$\begin{array}{c} f(t_1,t_2,t_3) \quad f(t_2,t_3,t_4) \quad f(t_3,t_4,t_5) \quad f(t_4,t_5,t_6) \\ f(t_2,t_3,u) \quad f(t_3,t_4,u) \quad f(t_4,t_5,u) \\ f(t_3,u,v) \quad f(t_4,u,v) \\ f(u,v,w) \end{array} \tag{3}$$

Figure 10 is an example in which we find $F(5)$ for the case when

$$t_1 = 1,\ \ t_2 = 2,\ \ t_3 = 3,\ \ t_4 = 6,\ \ t_5 = 7,\ \ t_6 = 8.$$

3 The Convex Hull Rule

The Convex Hull Rule states that the portion of the curve which lies between the first and the last control point must lie within the convex hull of the control points.

This allows us to intersect the convex hulls of two curves to easily determine if the two curves might intersect. If the two convex hulls do not intersect, then the curves will not intersect.

We can combine this fact with the de Boor Algorithm to get better and better estimates of a curve. We do this by creating smaller and smaller cubic control points, and thus we know that the curve for these smaller segments must lie within these smaller convex hulls.

For example, in Figure 8, we know that after we find $f(t,t,t)$ that the curve must lie within the two convex hulls defined by the four control points to the left, and the four control points to the right. Specifically, the segment $F(r \mathinner{..} t])$ must lie within the convex hull defined by $f(r,r,r)$, $f(r,r,t)$, $f(r,t,t)$, and $f(t,t,t)$, while the segment $F([t \mathinner{..} s])$ must lie within the convex hull defined by $f(t,t,t)$, $f(t,t,s)$, $f(t,s,s)$, and $f(s,s,s)$.

4 Degree Raising

An n-ic curve is a curve of degree *at most n*. Let $F(t)$ be a parabola (degree=2), and let $G(t)$ be $F(t)$ viewed as a degenerate cubic (degree=3).

Let $g(u,v,w)$ be the polar form of G and let $f(u,v)$ be the polar form of F. Then, we have

$$F(u) = f(u,u) = g(u,u,u) = G(u) \tag{4}$$

$$g(u,v,w) = \frac{1}{3}(f(u,v) + f(u,w) + f(v,w)) \tag{5}$$

To see why Equation (5) is correct, note that the function g so defined has the proper values on the diagonal, and hence satisfies Equation (4). Second, the function g so defined is symmetric. Finally, if we hold two arguments to g fixed and vary the third, the resulting value is the average of three values of f, each of which is either moving at a constant rate along a straight line or staying constant. Thus, the average will move at a constant speed along a straight line, so g is triaffine.

4.1 Is a cubic a quadratic in disguise?

Suppose we are given the four Bézier points $g(0,0,0)$, $g(0,0,1)$, $g(0,1,1)$, and $g(1,1,1)$ of a cubic segment $G([0 \mathinner{..} 1])$. We know that, if G is actually a quadratic F in disguise, then it can be specified by three Bézier points $f(0,0)$, $f(0,1)$, and $f(1,1)$. In that case,

$$f(0,0) = g(0,0,0) \text{ and } f(1,1) = g(1,1,1).$$

We try to find $f(0,1)$ using two different approaches. If the points found by the two approaches coincide, then G is indeed a quadratic in disguise.

First, we have

$$g(0,0,1) = \frac{1}{3}(f(0,0) + f(0,1) + f(0,1)) \tag{6}$$

so, solving for $f(0,1)$, we get

$$f(0,1) = \frac{3}{2}g(0,0,1) - \frac{1}{2}g(0,0,0) \tag{7}$$

Second, we have

$$g(0,1,1) = \frac{1}{3}(f(0,1) + f(0,1) + f(1,1)) \tag{8}$$

so we get

$$f(0,1) = \frac{3}{2}g(0,1,1) - \frac{1}{2}g(1,1,1) \tag{9}$$

If the $f(0,1)$ found from Equation (7) is the same as that found from Equation (9), then G is a degenerate cubic, a quadratic in disguise, and we have $G(t) = F(t)$ for all t, where the quadratic F is specified by the Bézier points $f(0,0)$, $f(0,1)$, and $f(1,1)$. The method of degree raising described here can be extended to curves of higher degree.