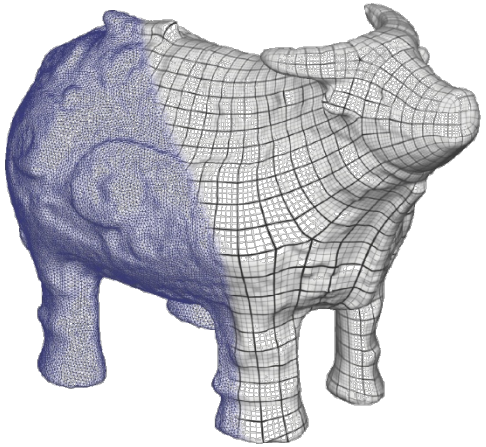
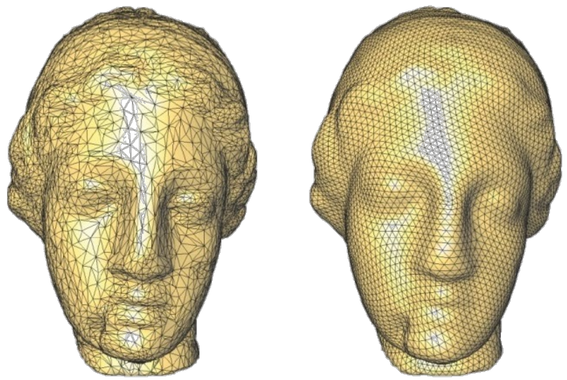
CS348a: Computer Graphics -- Geometric Modeling and Processing





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Computer Science Department
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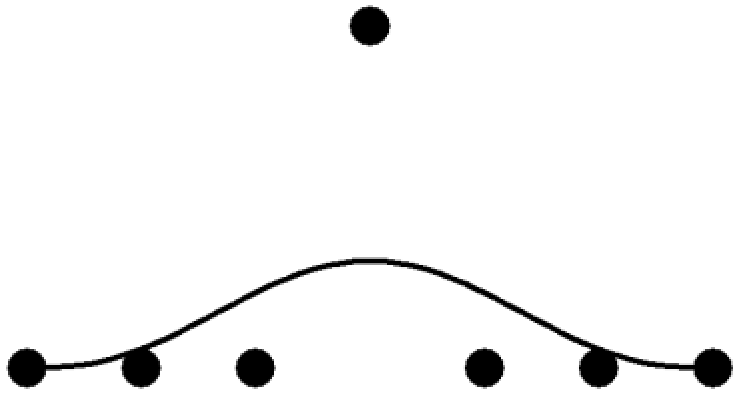
Leonidas Guibas Laboratory

Geometric Computing

Slides adapted from Scott Schaefer (TAMU)

Problems with Bézier Curves

- More control points means higher degree
- Moving one control point affects the entire curve



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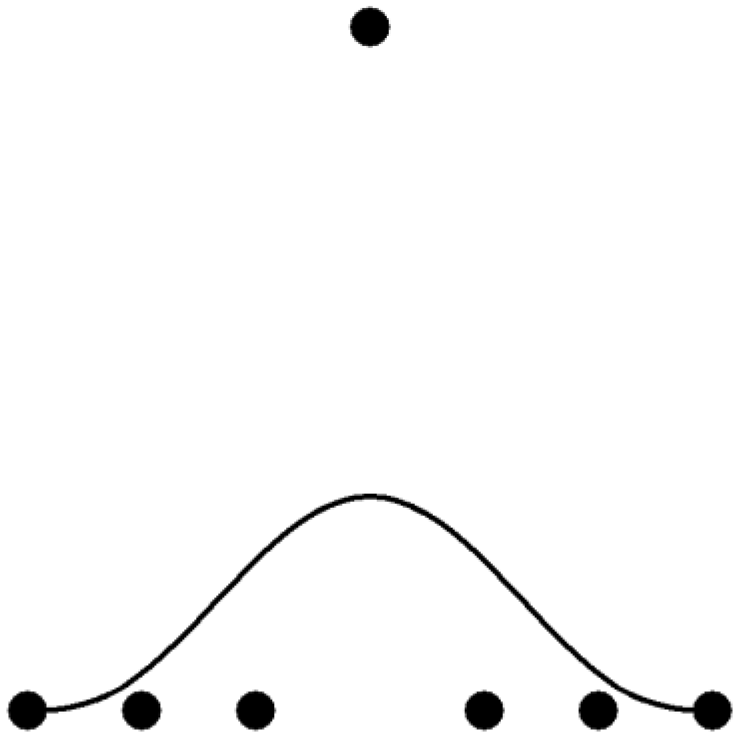

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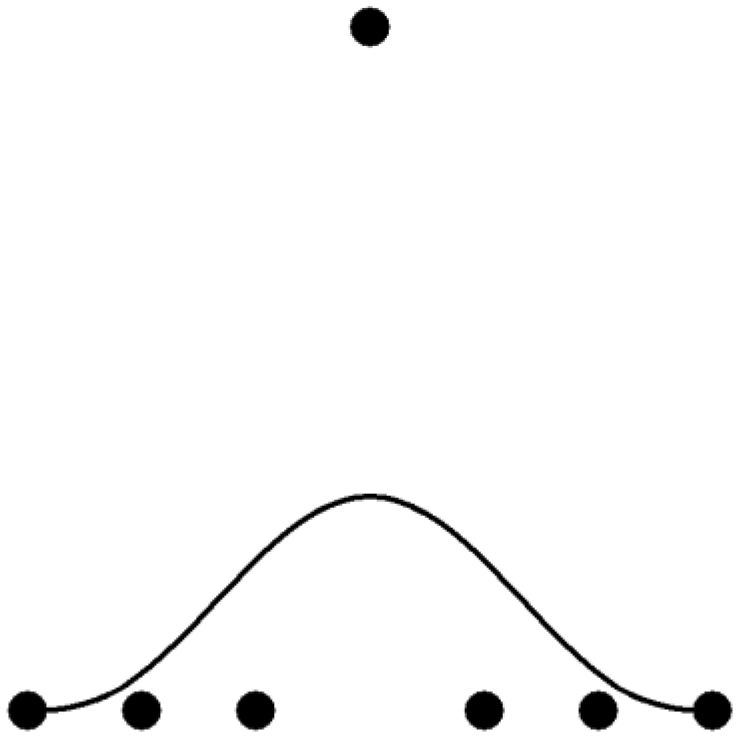

Solution: Use lots of Bezier curves and maintain C^k continuity

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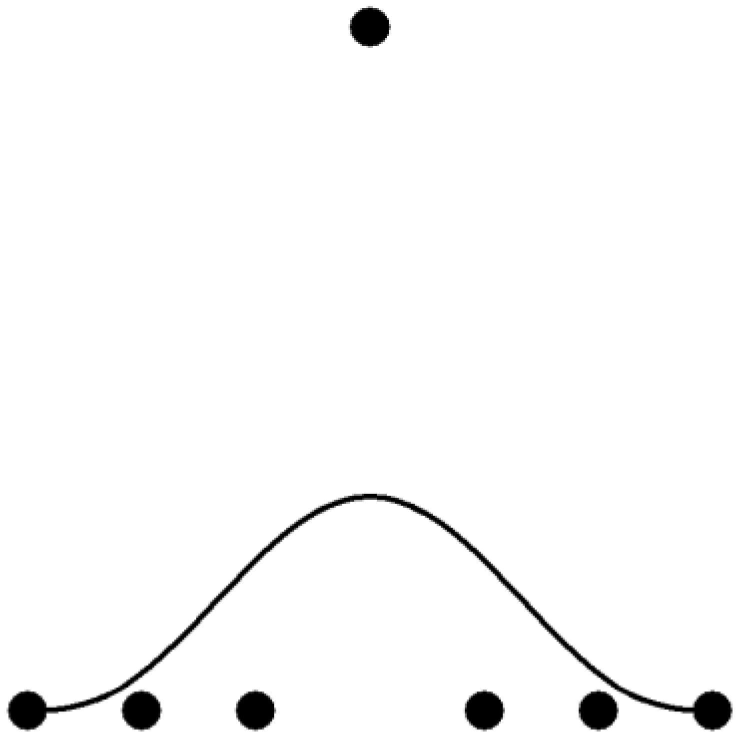
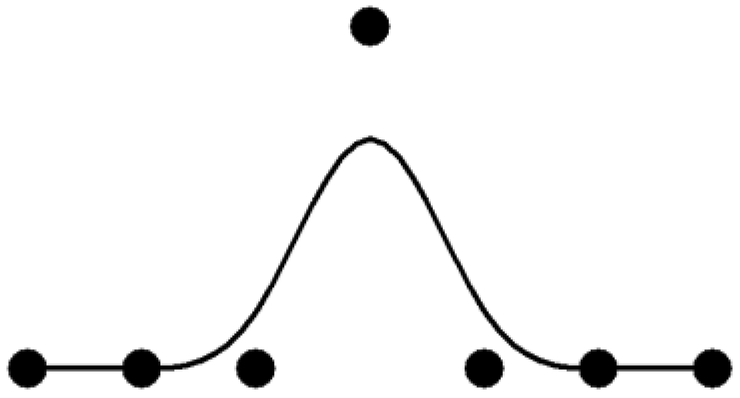


Solution: Use lots of Bezier curves and maintain C^k continuity

Difficult to keep track of all the constraints. ☹

B-spline Curves

- Not a single polynomial, but many of polynomials that meet together smoothly at their junctions
- Maintain local control



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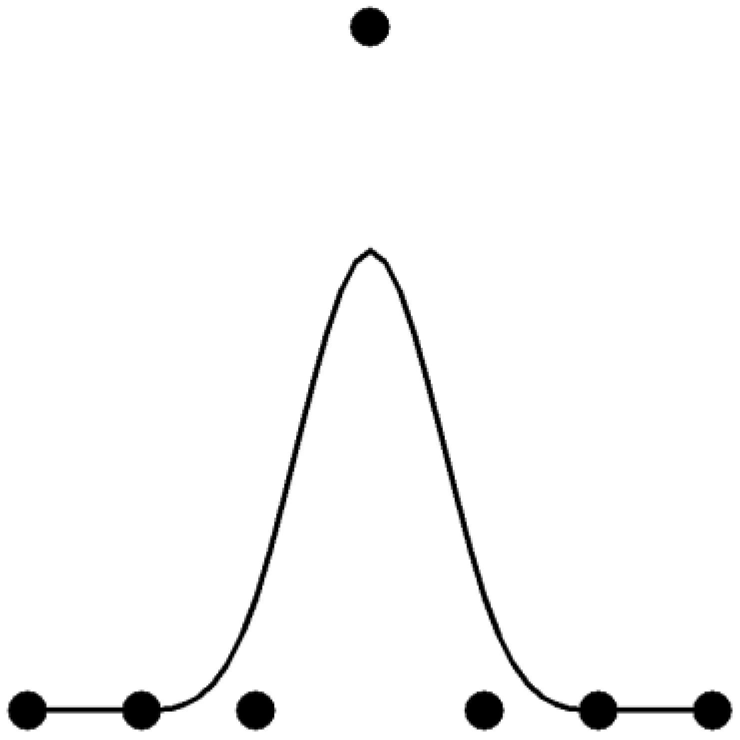


History of B-splines

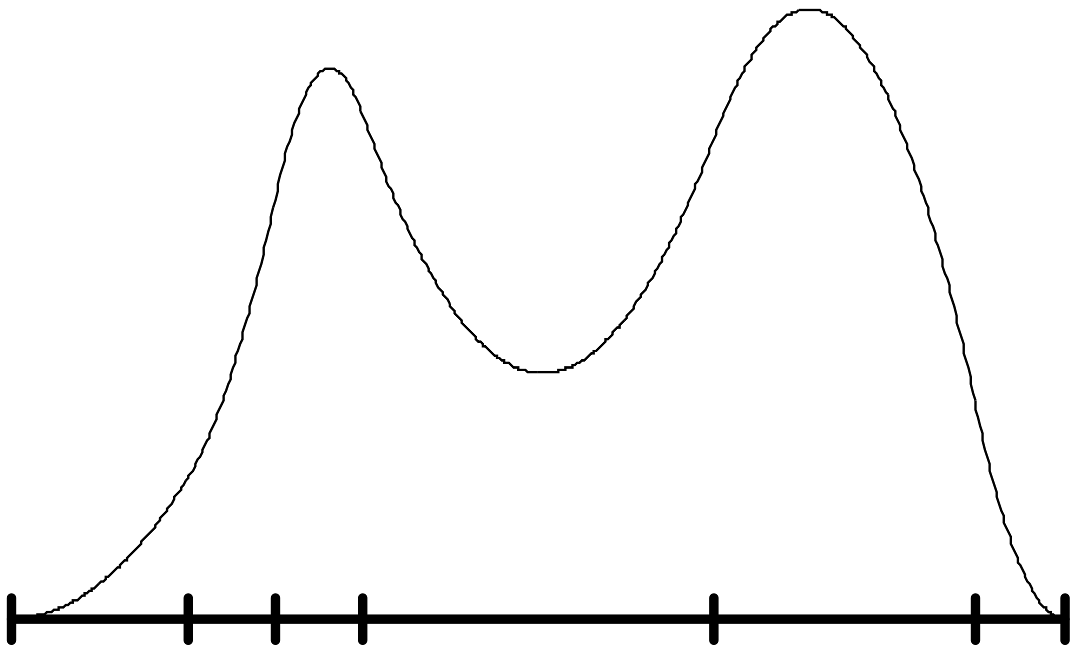
- Designed to create smooth curves
- Similar to physical process of bending wood
- Early Work
  - de Casteljau at Citroen
  - Bézier at Renault
  - de Boor at General Motors

B-spline Curves

- Curve defined over a set of parameters $t_0,\ldots,t_k$ ($t_i \leq t_{i+1}$) with a polynomial of degree n in each interval $[t_i, t_{i+1}]$ that meet with C^{n-1} continuity
- t_i do not have to be evenly spaced
- Commonly called NURBS
  - **N**on-**U**niform **R**ational **B**-**S**plines

B-Spline Curves

$$p(t) = \sum_{i=1}^{m} p_i N_i^n(t)$$

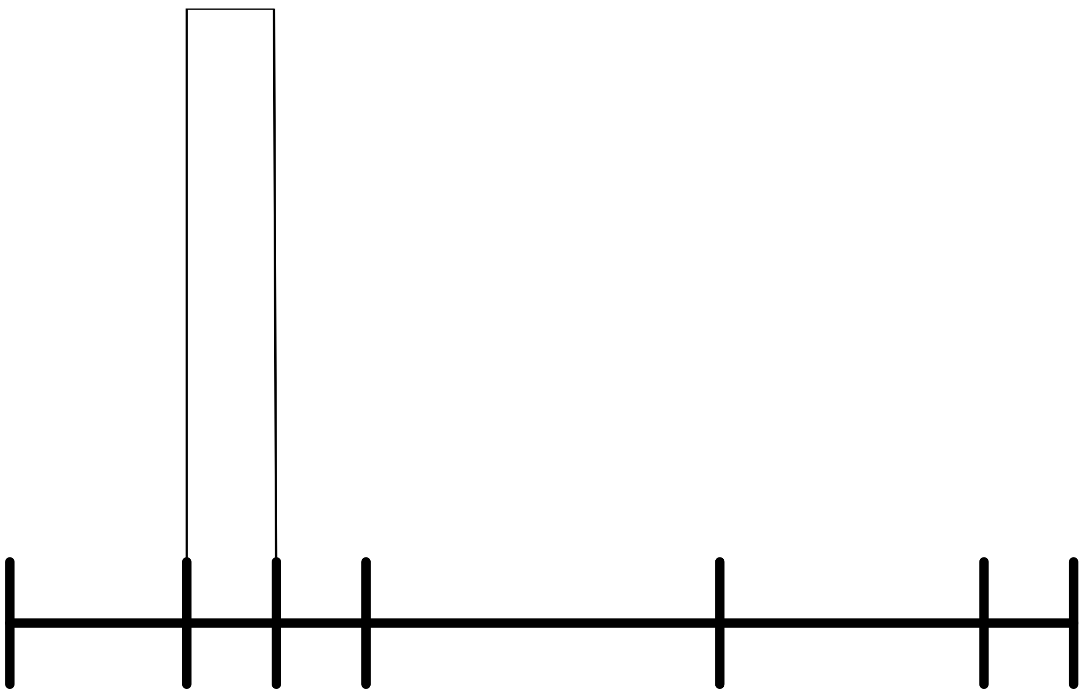


B-Spline Basis Functions

$$N_i^0(t) = \begin{cases} 1 & t_{i-1} \le t < t_i \\ 0 & o.w. \end{cases}$$

$$N_i^n(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_i^{n-1}(t) + \frac{t_{n+i} - t}{t_{i+n} - t_i} N_{i+1}^{n-1}(t)$$

$$N_3^0(t) =$$



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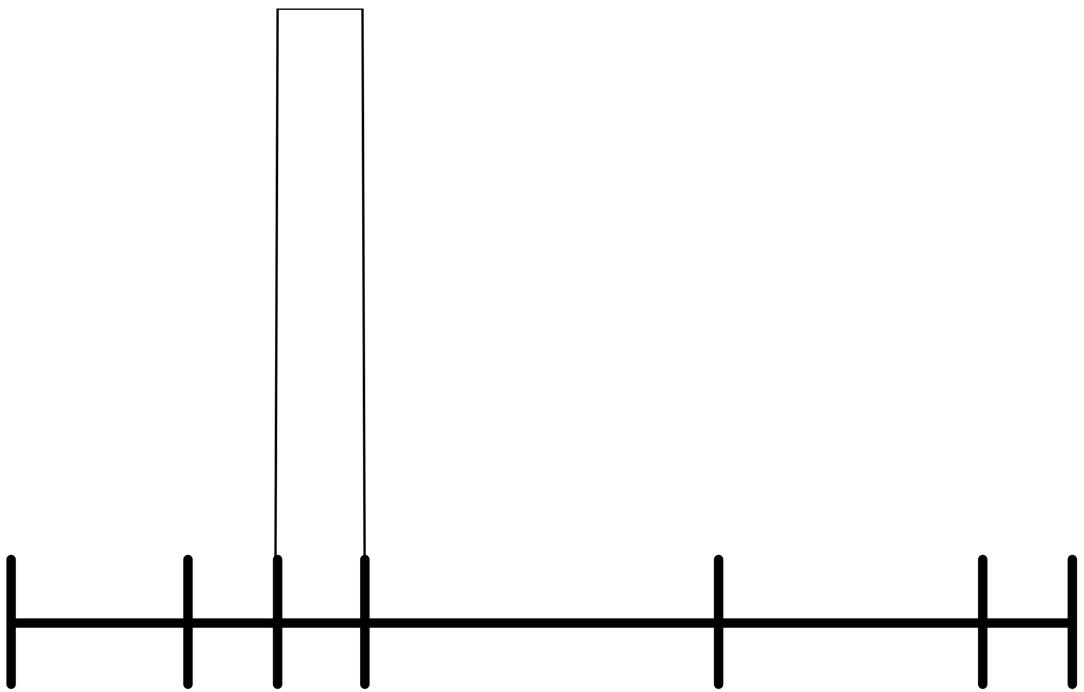

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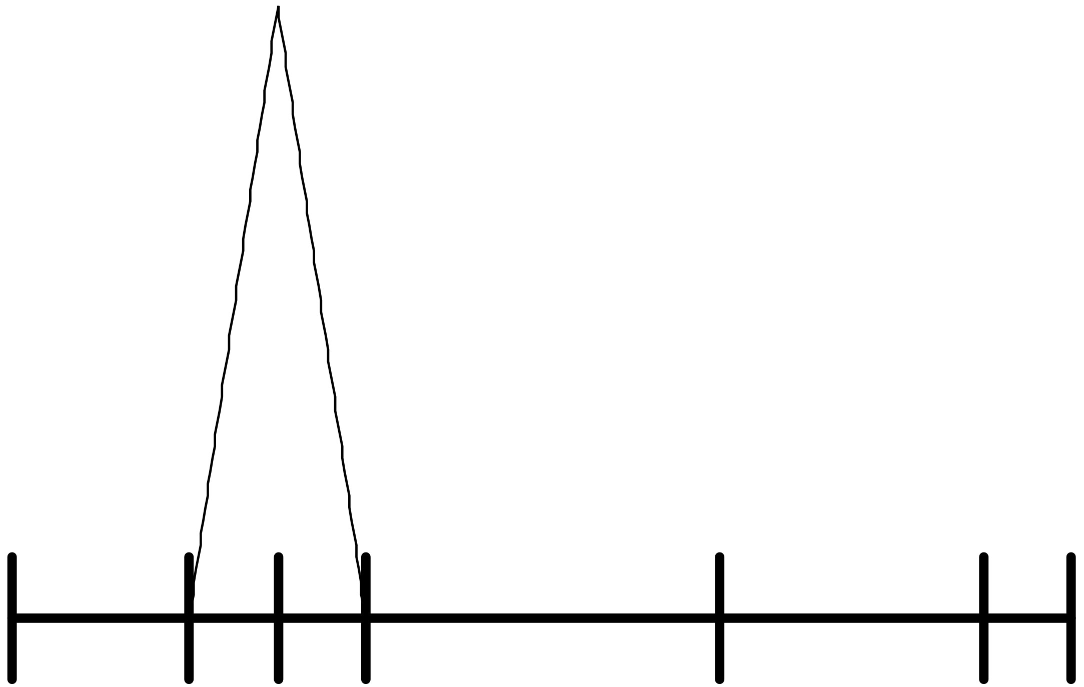

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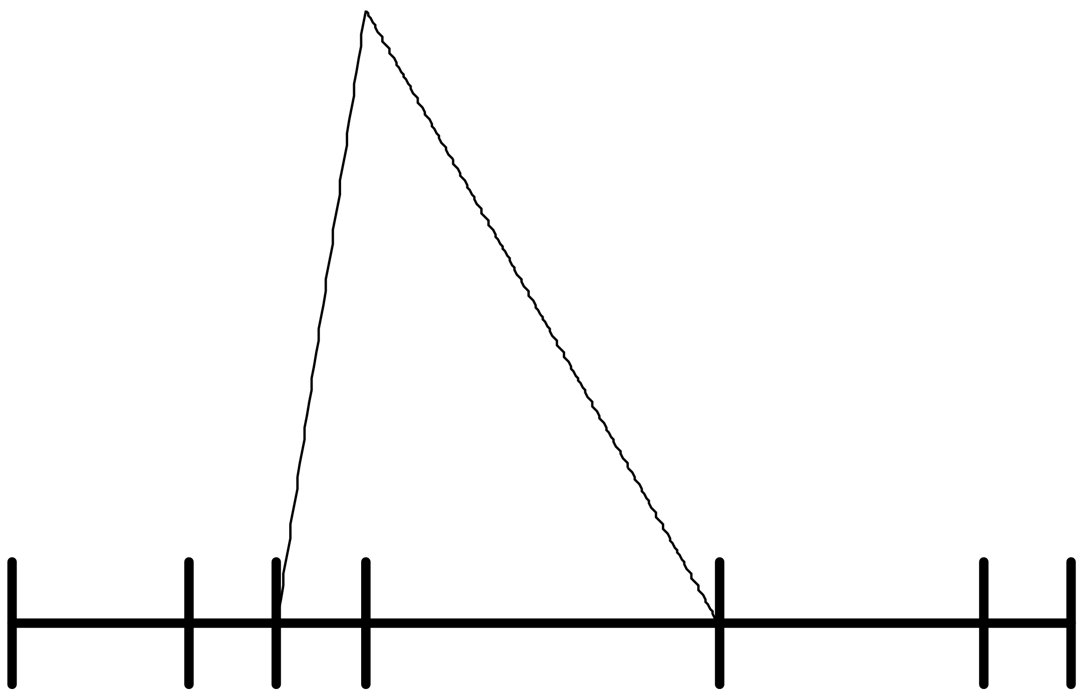

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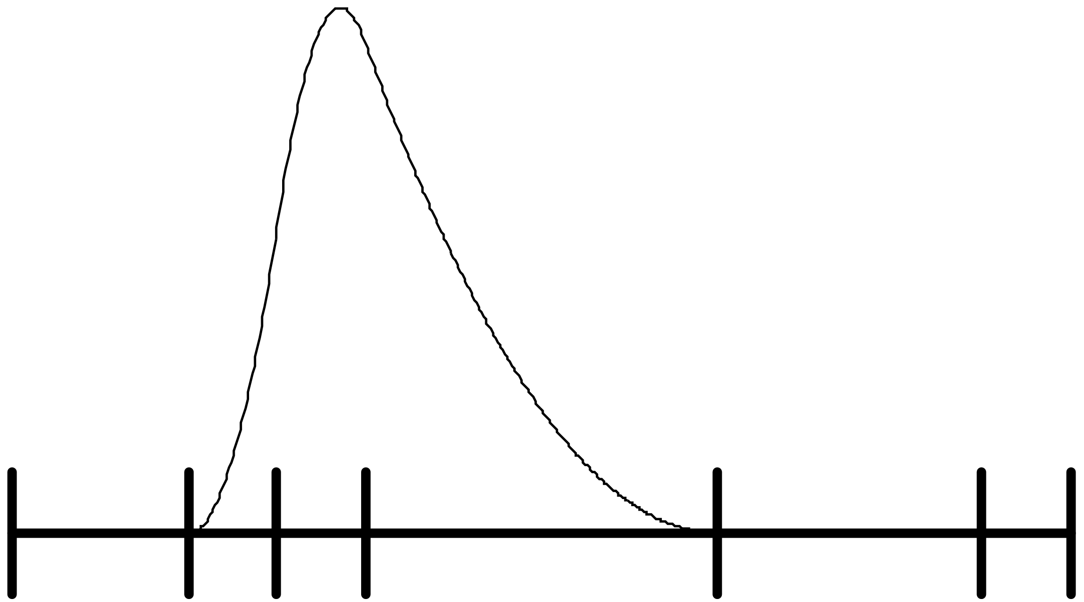

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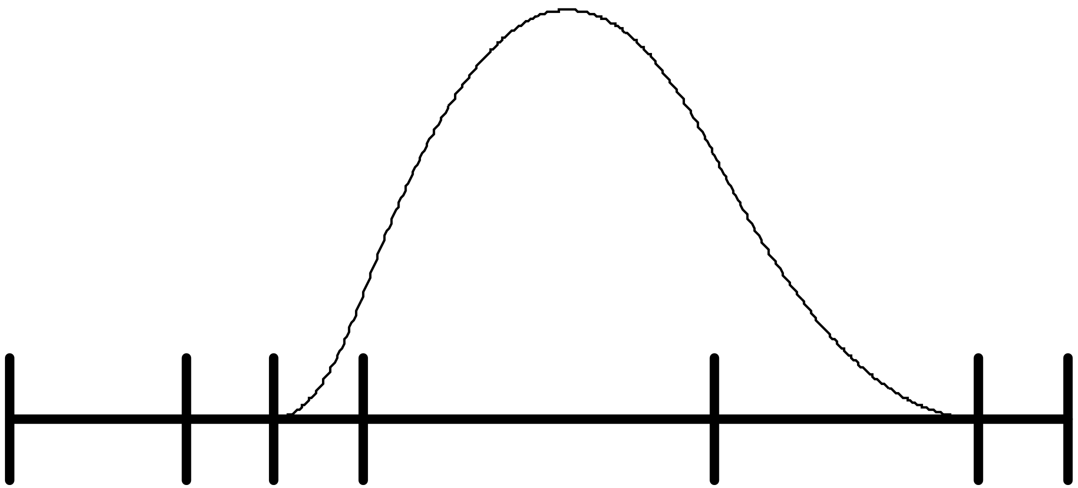

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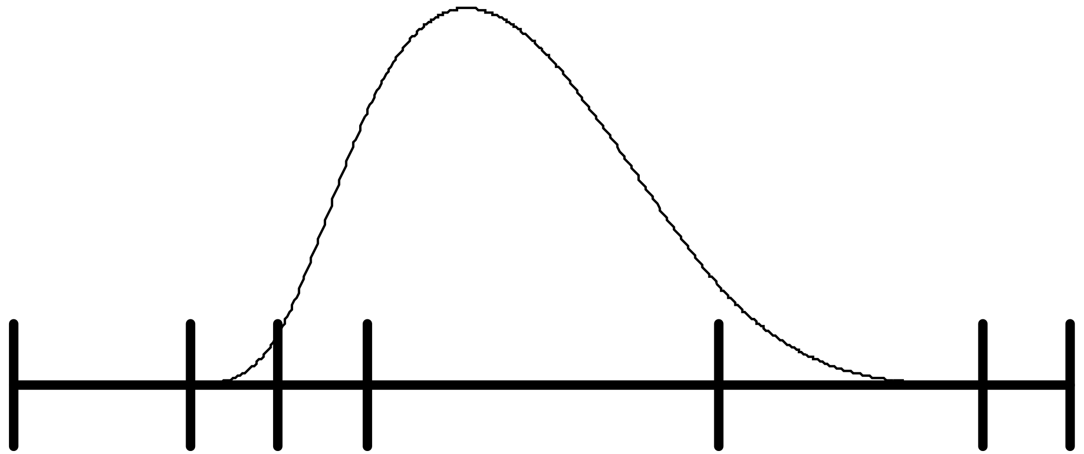

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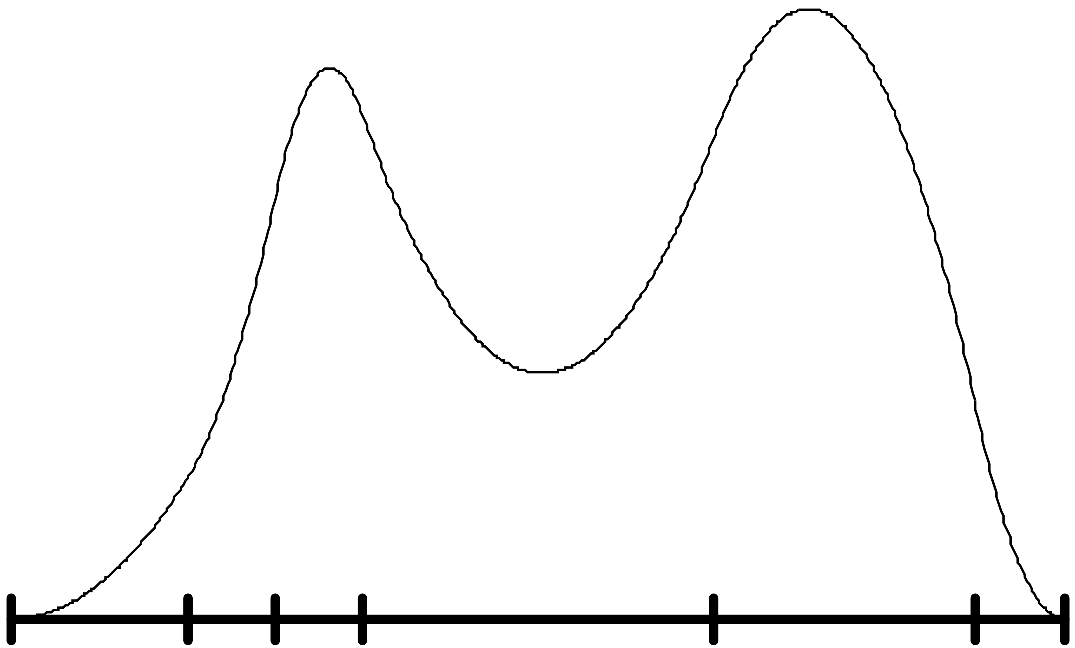
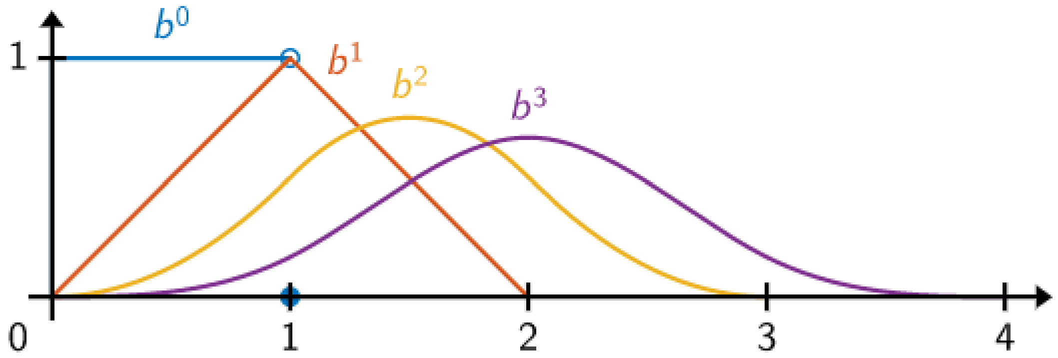
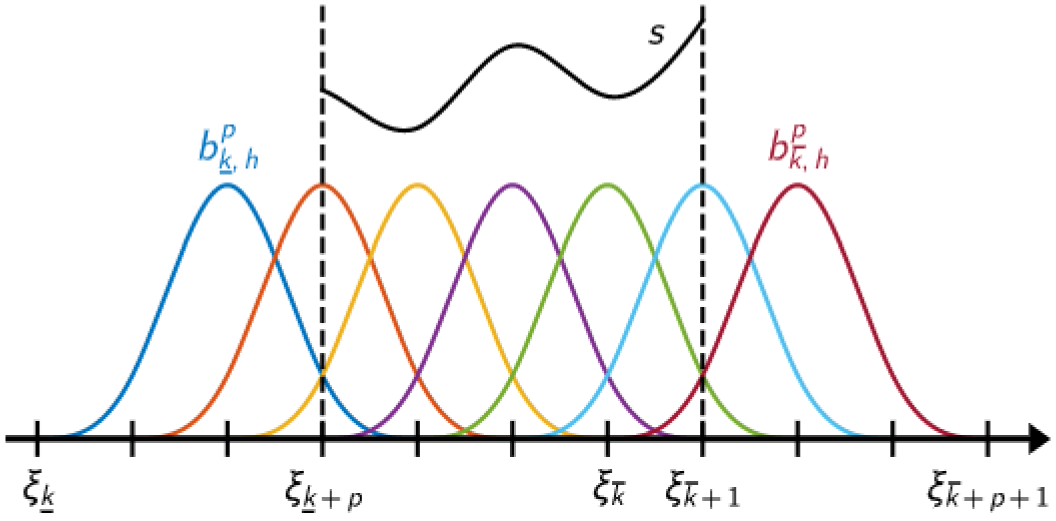
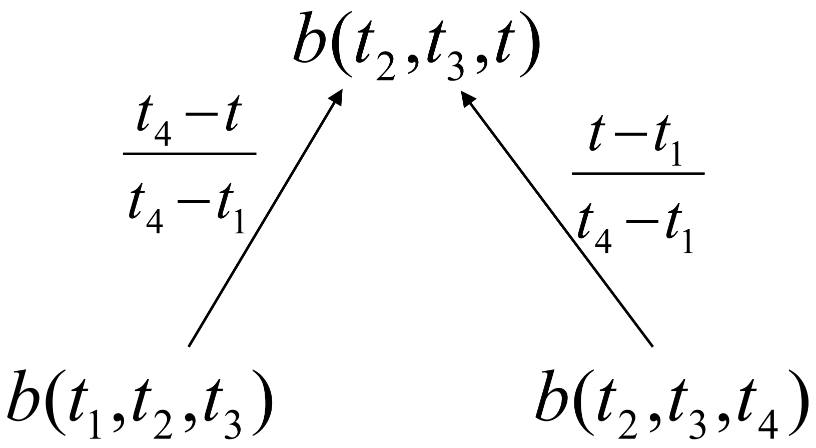


Uniform B-splines





B-Splines via Blossoming



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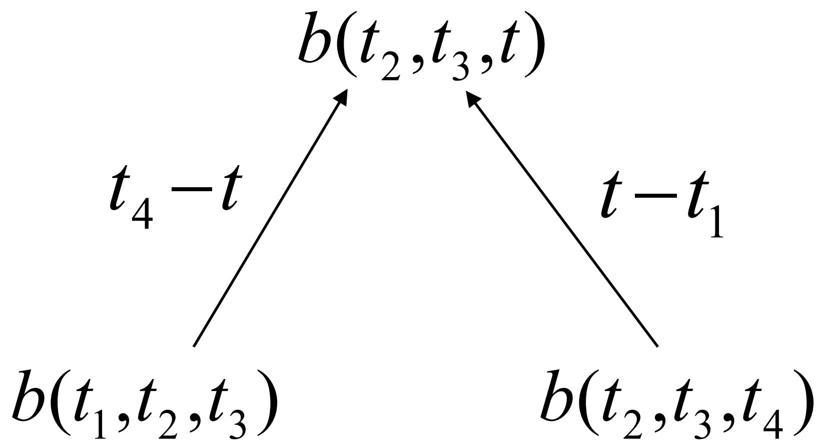

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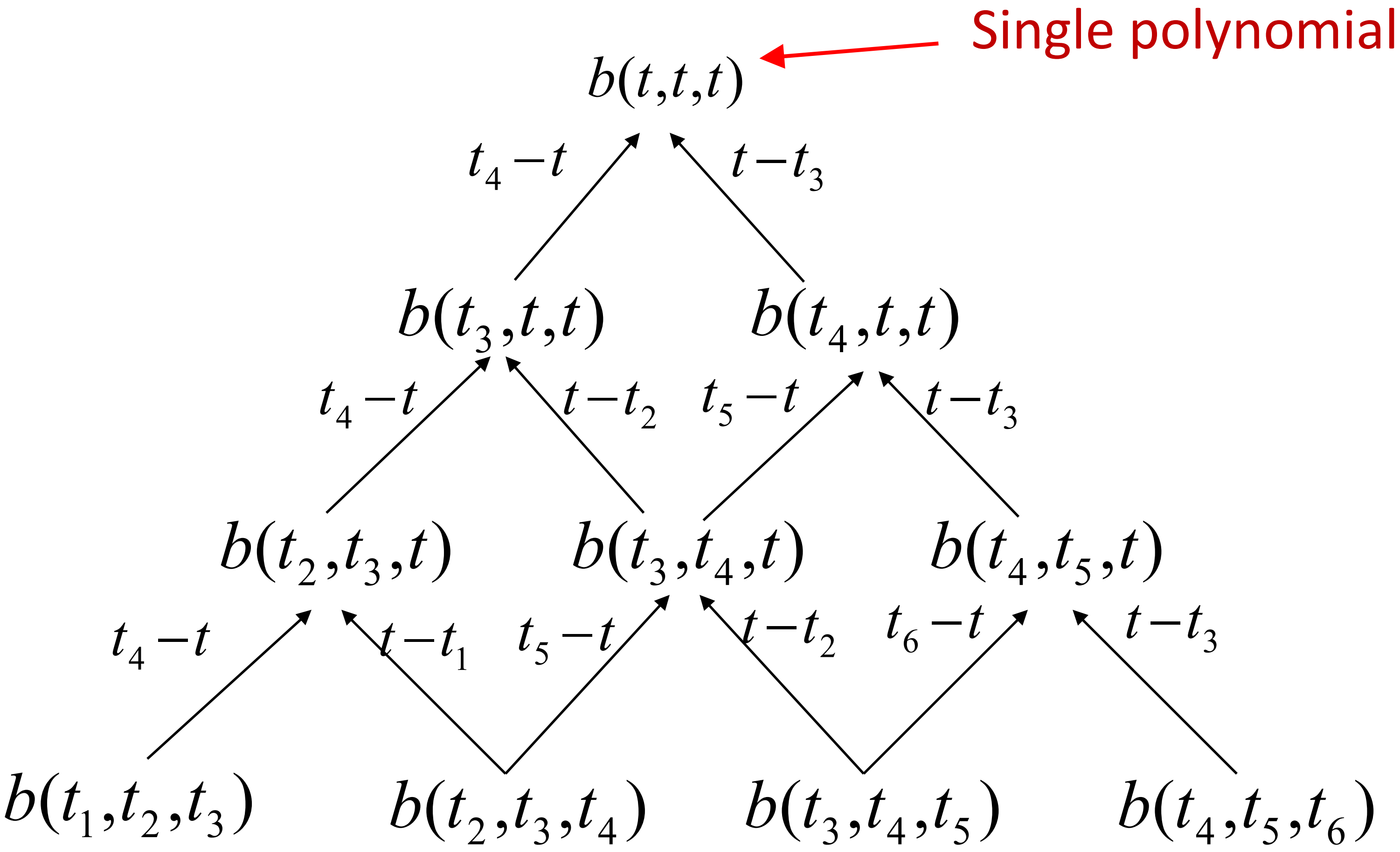

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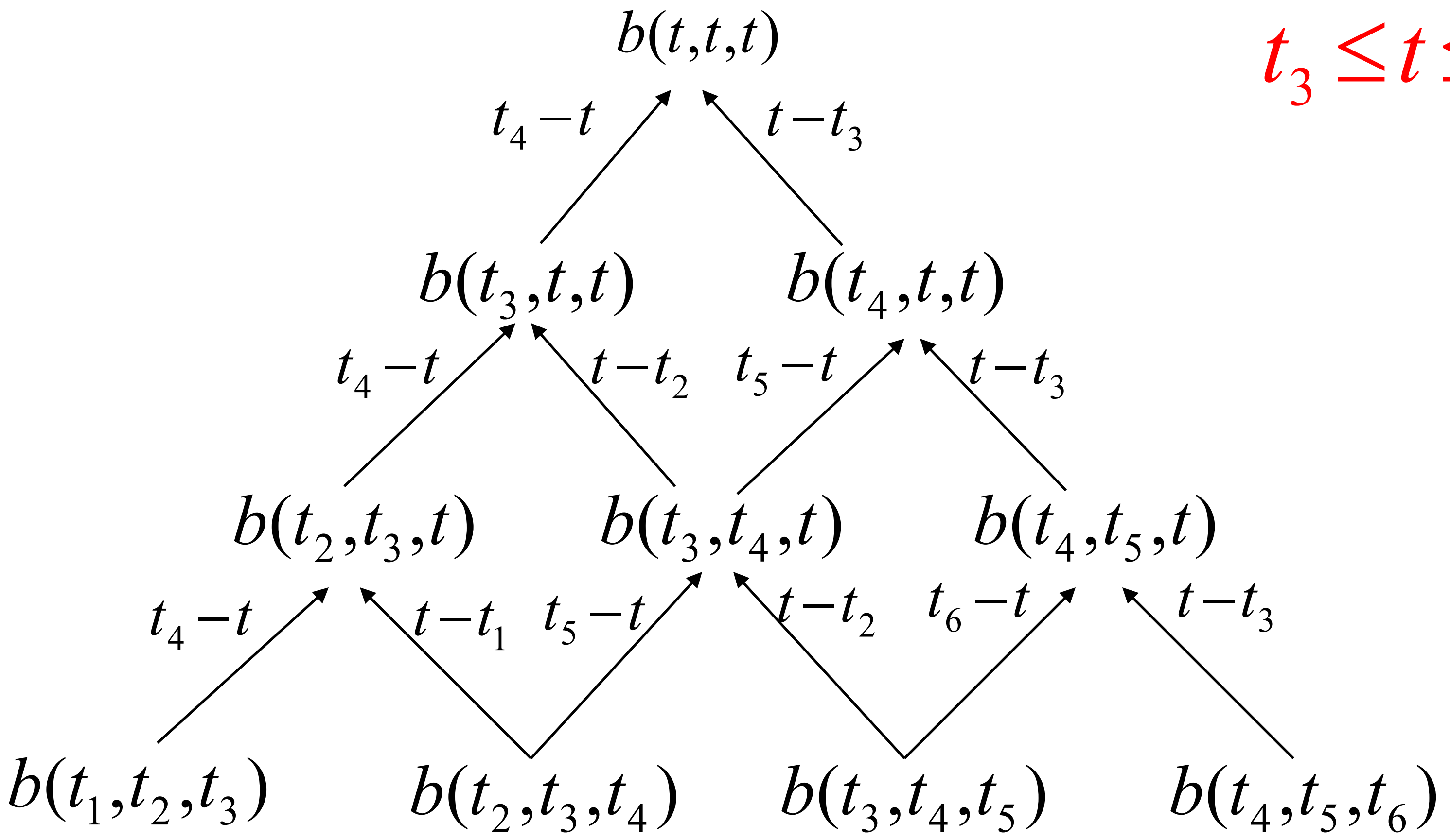
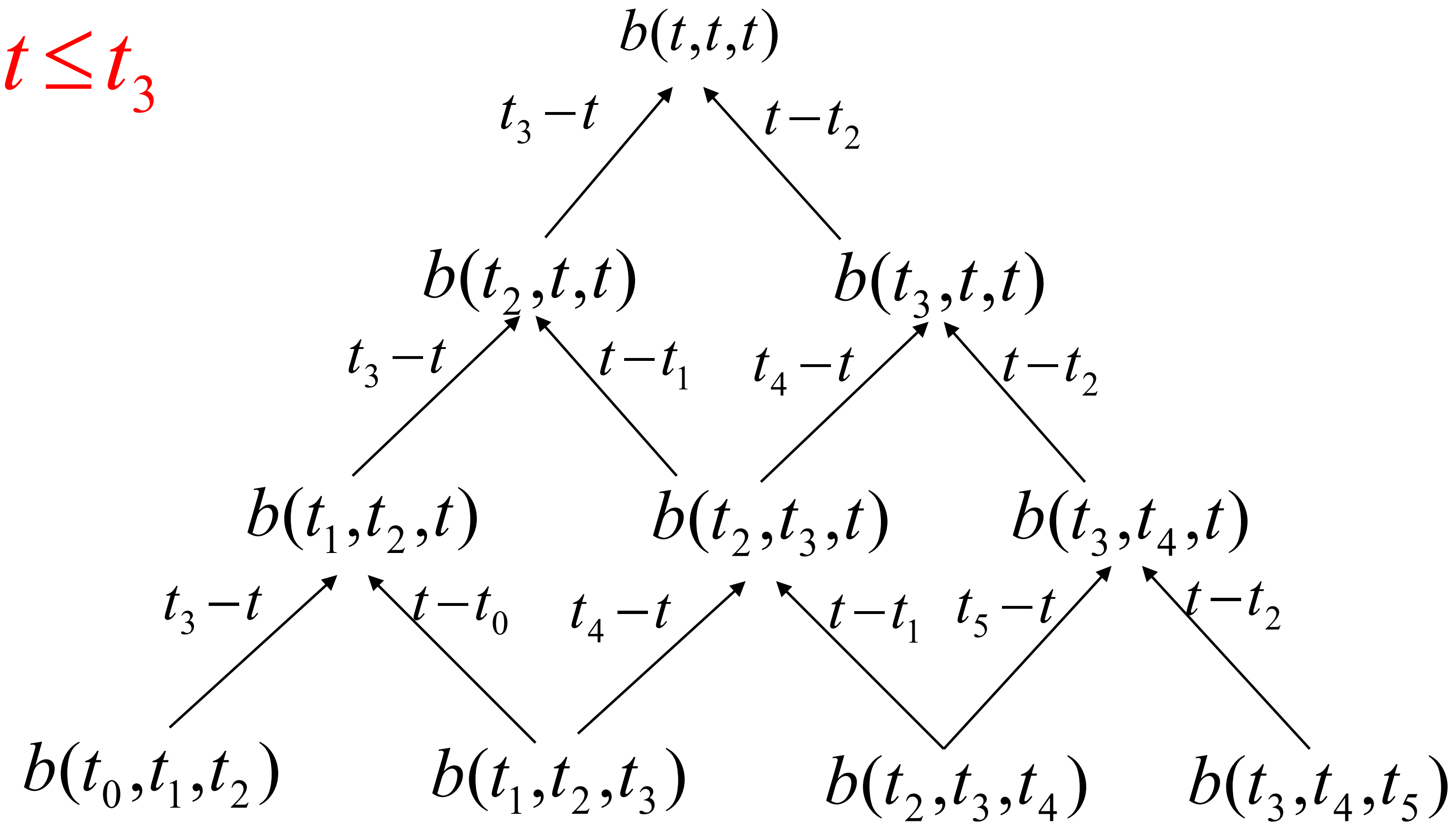


$$t_3 \le t \le t_4$$

B-Splines via Blossoming

$t_2 \le t \le t_3$



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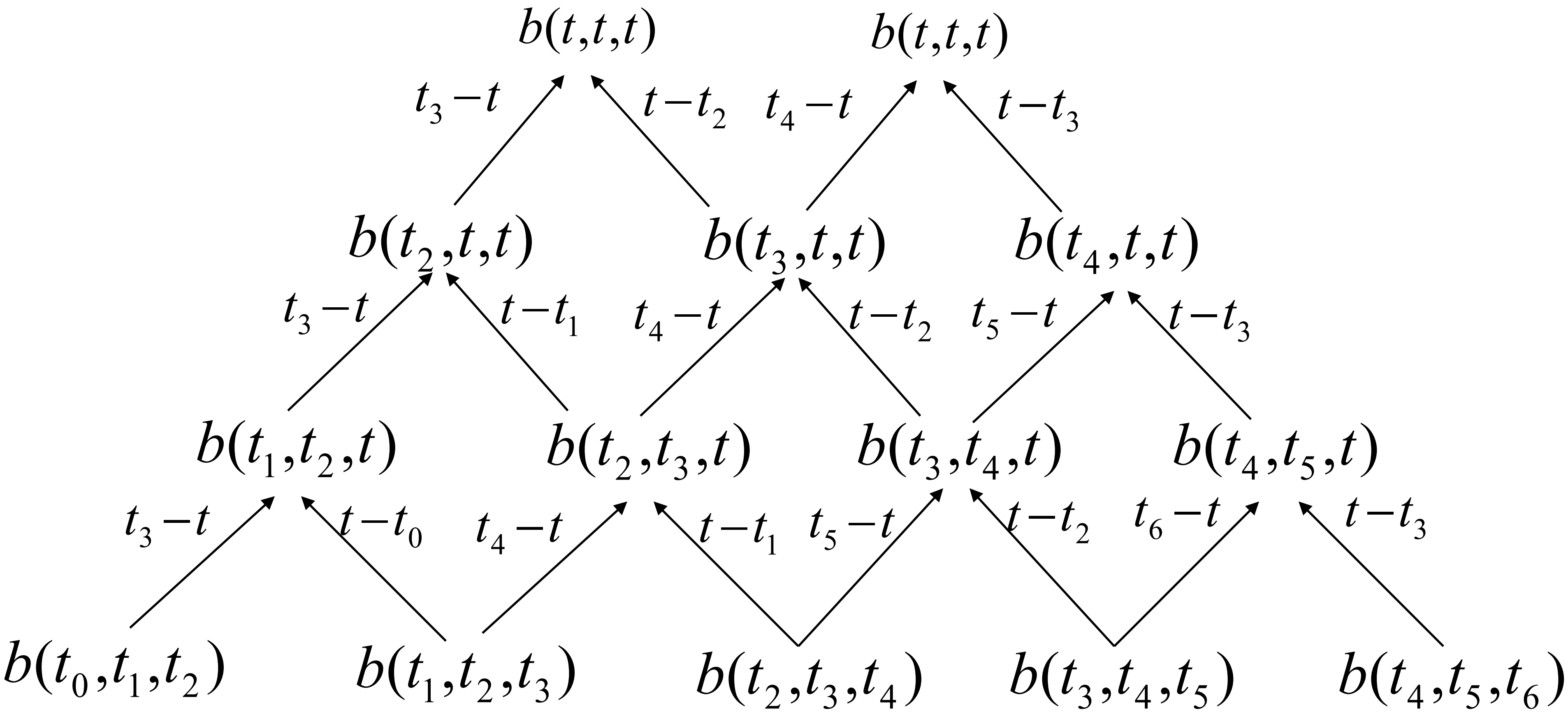

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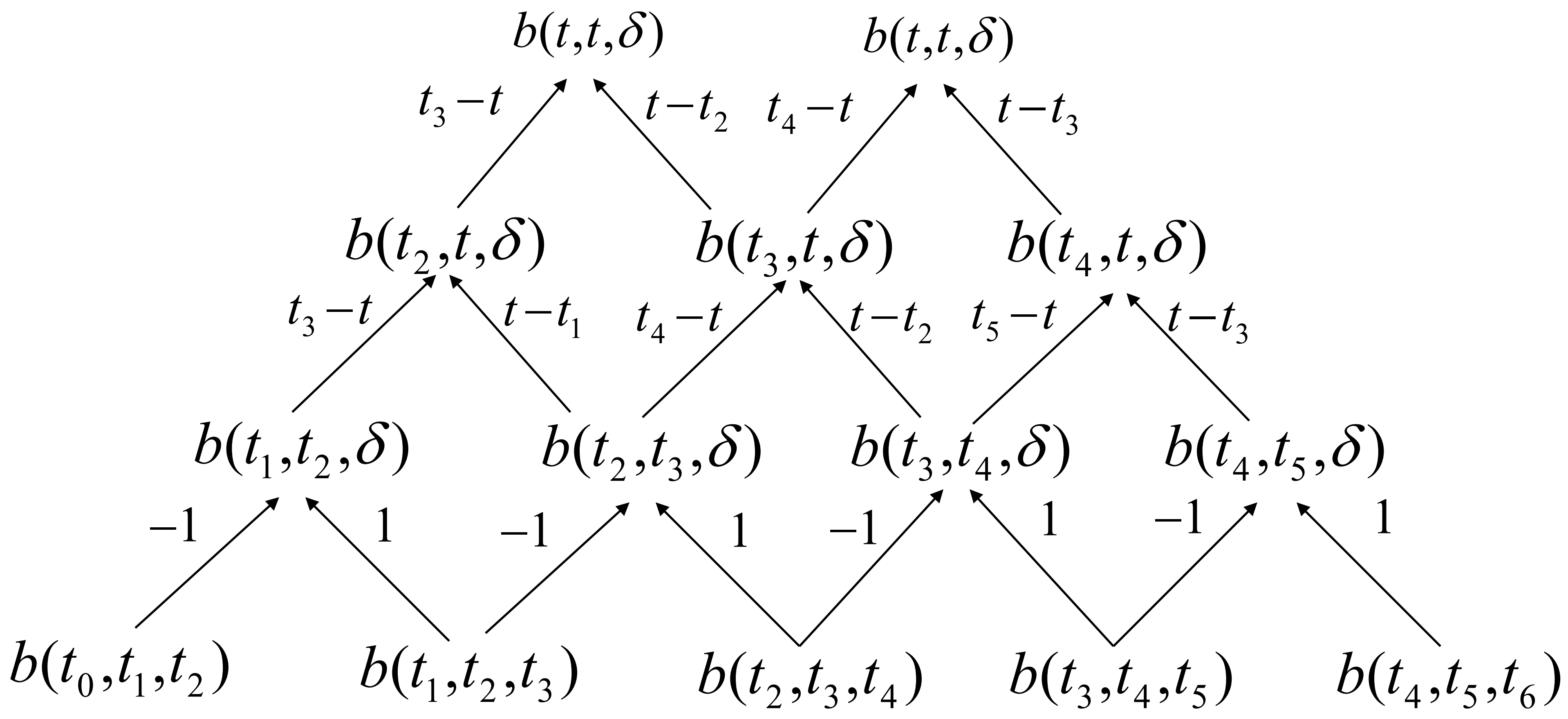

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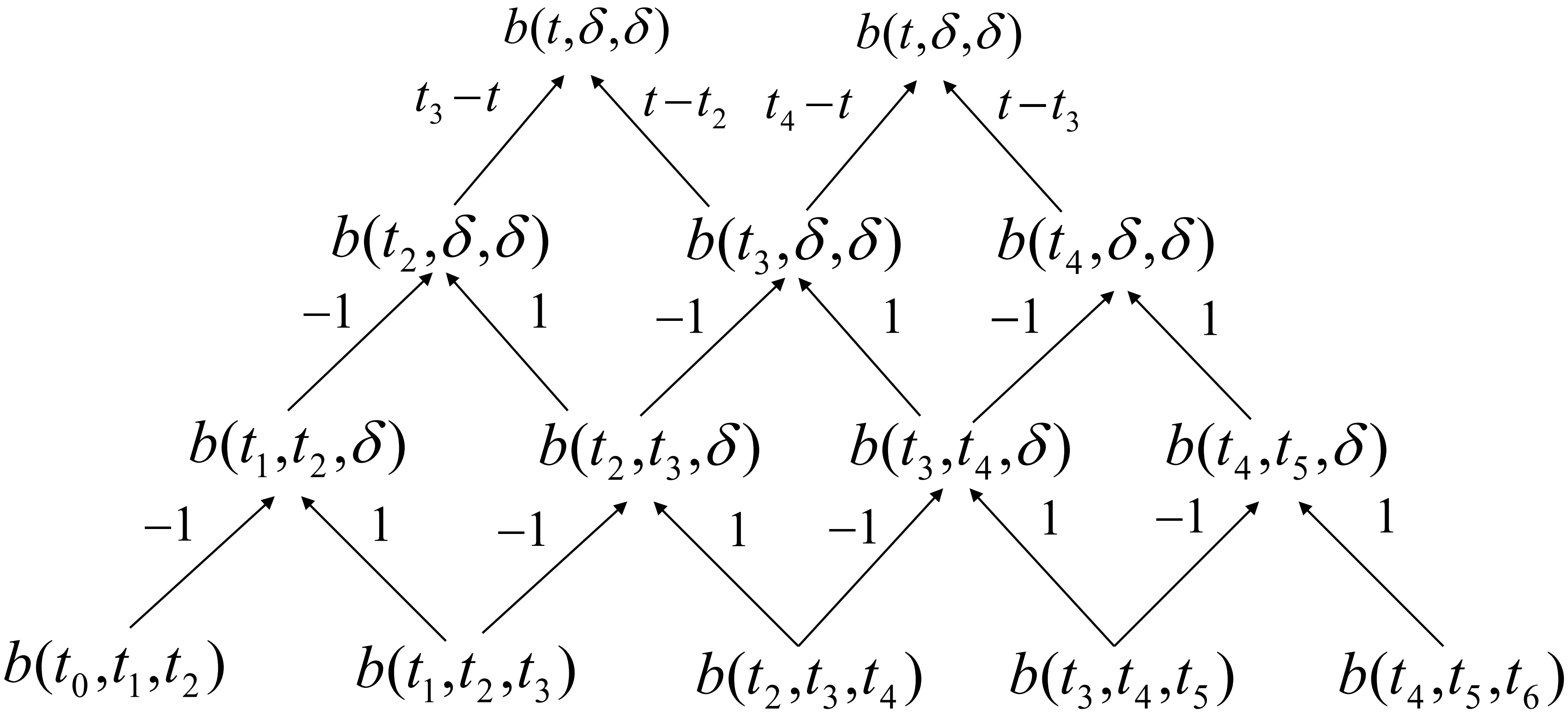
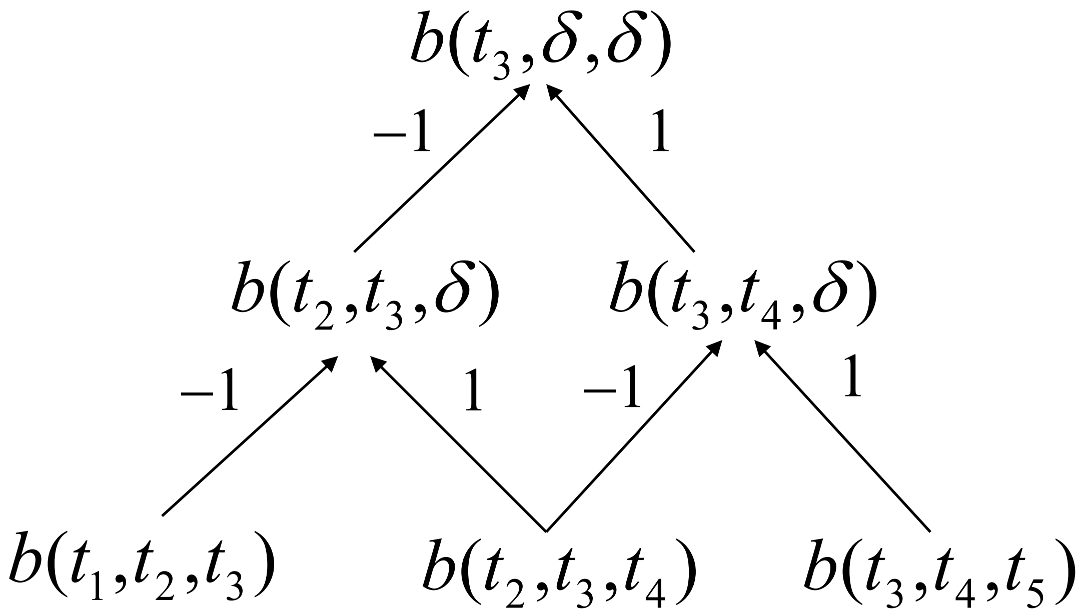


B-Splines via Blossoming

n-1 derivatives are equal yielding C^{n-1} continuity



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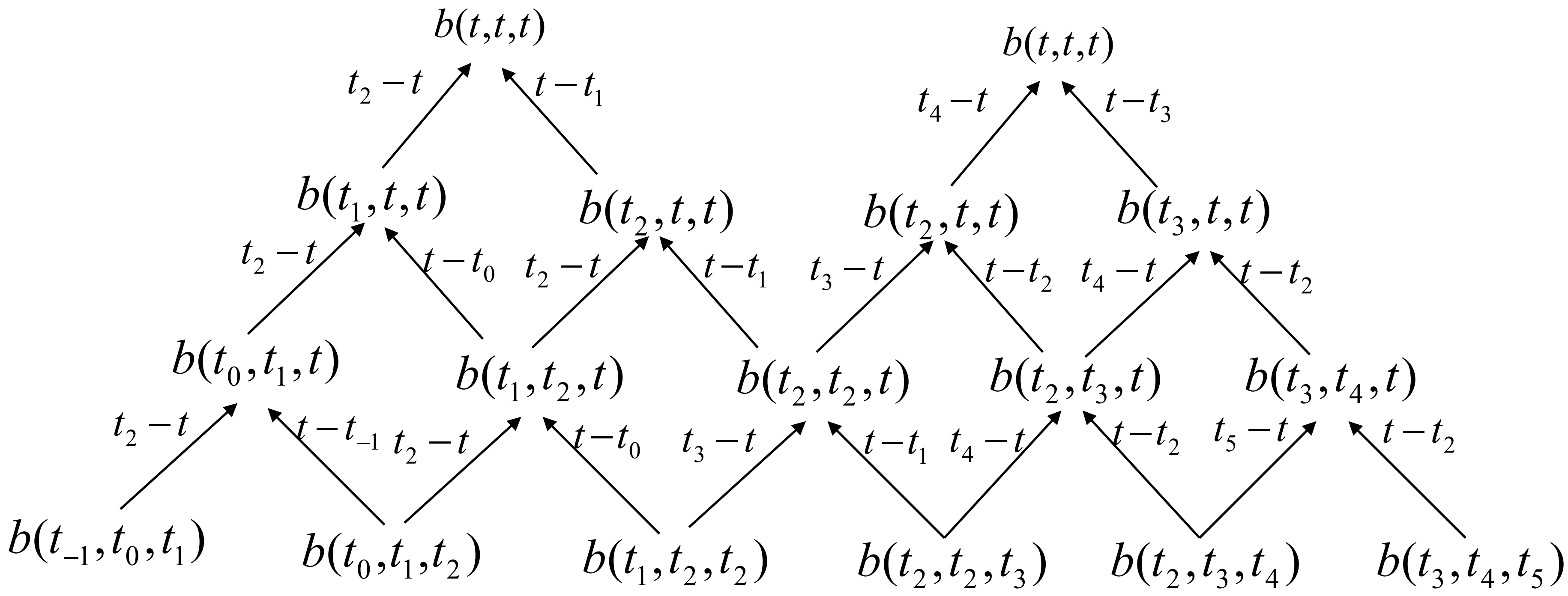
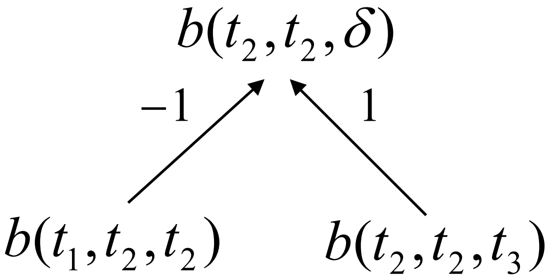


B-Splines via Blossoming

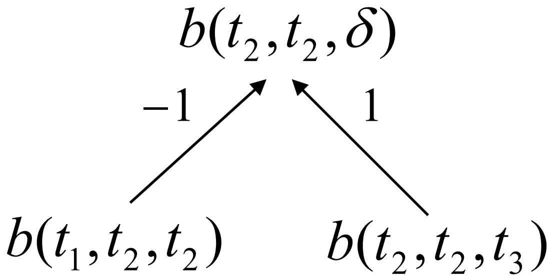
n-2 derivatives are equal yielding

$C^{n\text{-}2}$ continuity at doubled knot

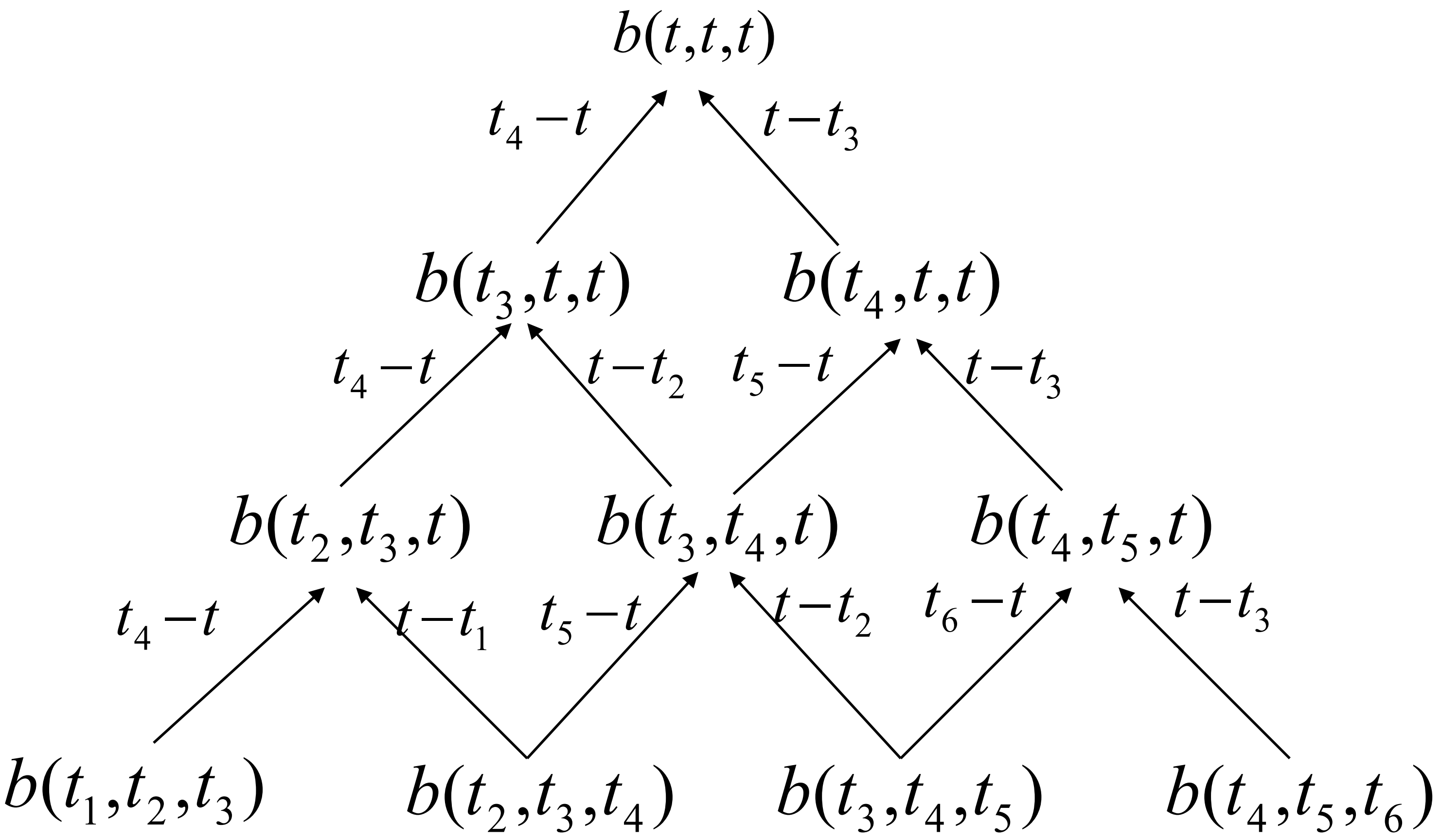


B-Splines Via Blossoming

In general, curves have C^{n-u} continuity at knot of multiplicity u



Conversion to Bézier Form



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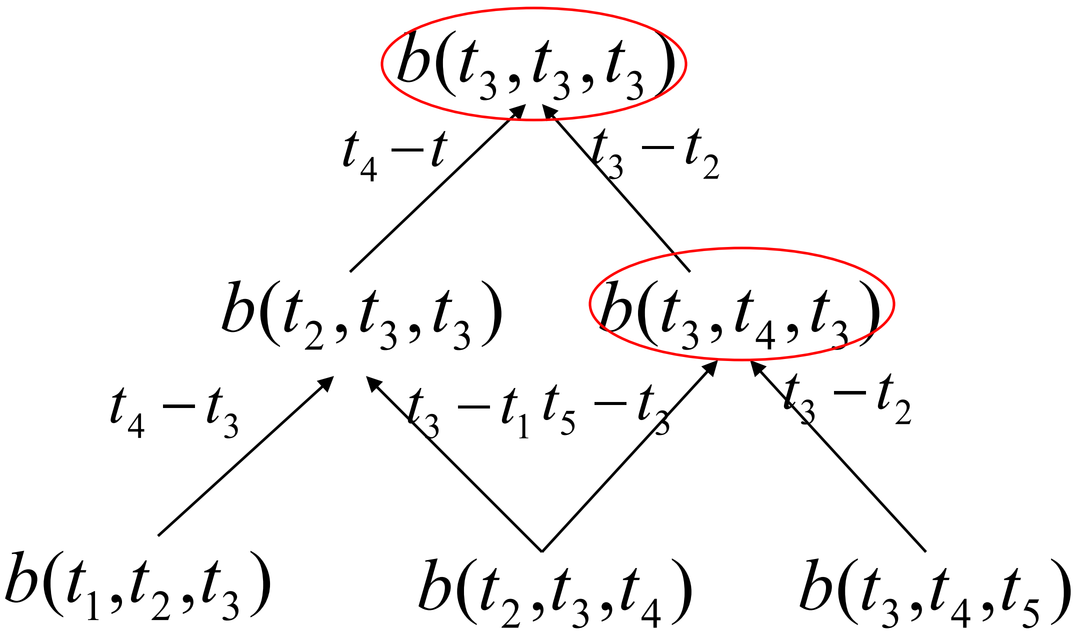

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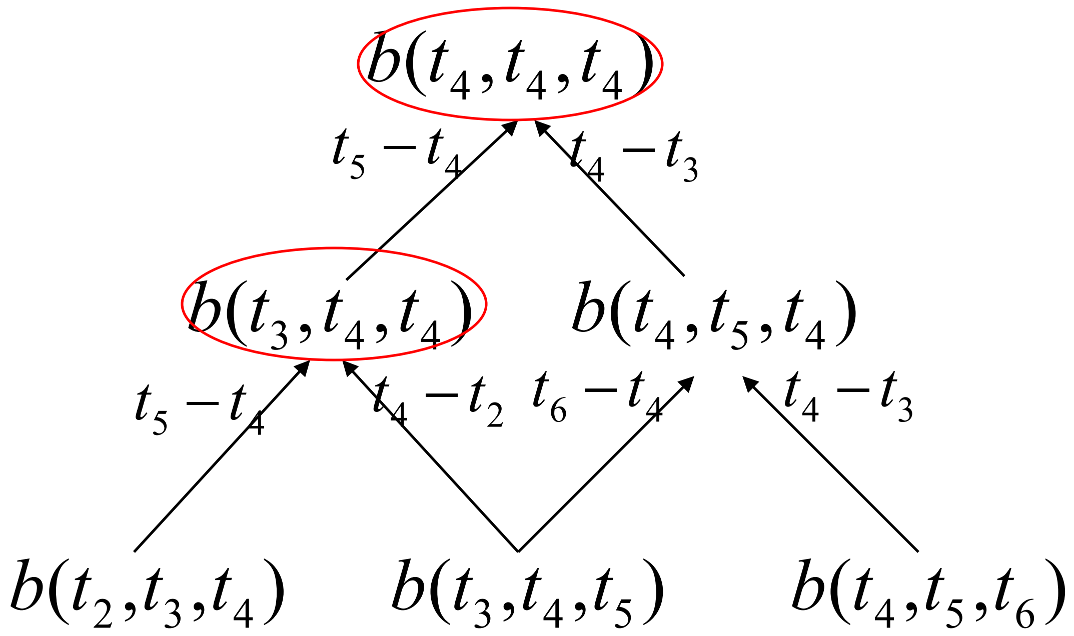

Polynomial Reproduction

- Given a polynomial $p(t)$ and a set of knots t_1, t_2, t_3, …, find control points for the b-spline curve that produces $p(t)$

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$$b(u_1, u_2, u_3) = 1 + 2\frac{u_1 + u_2 + u_3}{3} + 3\frac{u_1 u_2 + u_2 u_3 + u_1 u_3}{3} - 4u_1 u_2 u_3$$

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$$b(t_1, t_2, t_3),\ b(t_2, t_3, t_4),\ b(t_3, t_4, t_5),\ b(t_4, t_5, t_6), \ldots$$

Control points

Knot Insertion

- Given a B-spline curve with knot sequence …, t_{k-2}, t_{k-1}, t_k, t_{k+1}, t_{k+2}, t_{k+3}, … generate the control points for an identical B-spline curve over the knot sequence …, t_{k-2}, t_{k-1}, t_k, u, t_{k+1}, t_{k+2}, t_{k+3}, …

Boehm's Knot Insertion Algorithm

- Given curve with knots t_1, t_2, t_3, t_4, t_5, t_6, find curve with knots t_1, t_2, t_3, u, t_4, t_5, t_6

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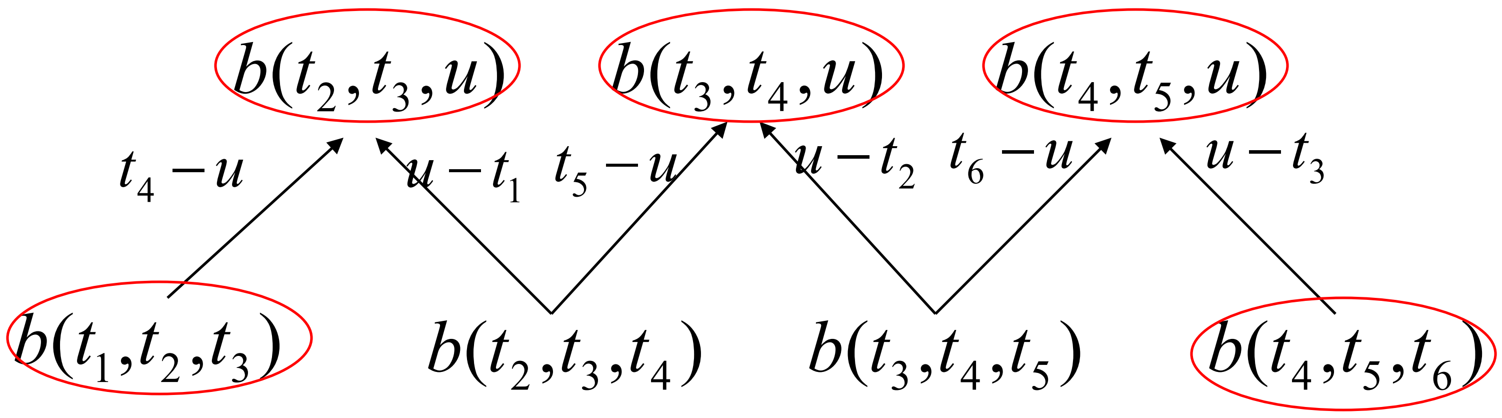

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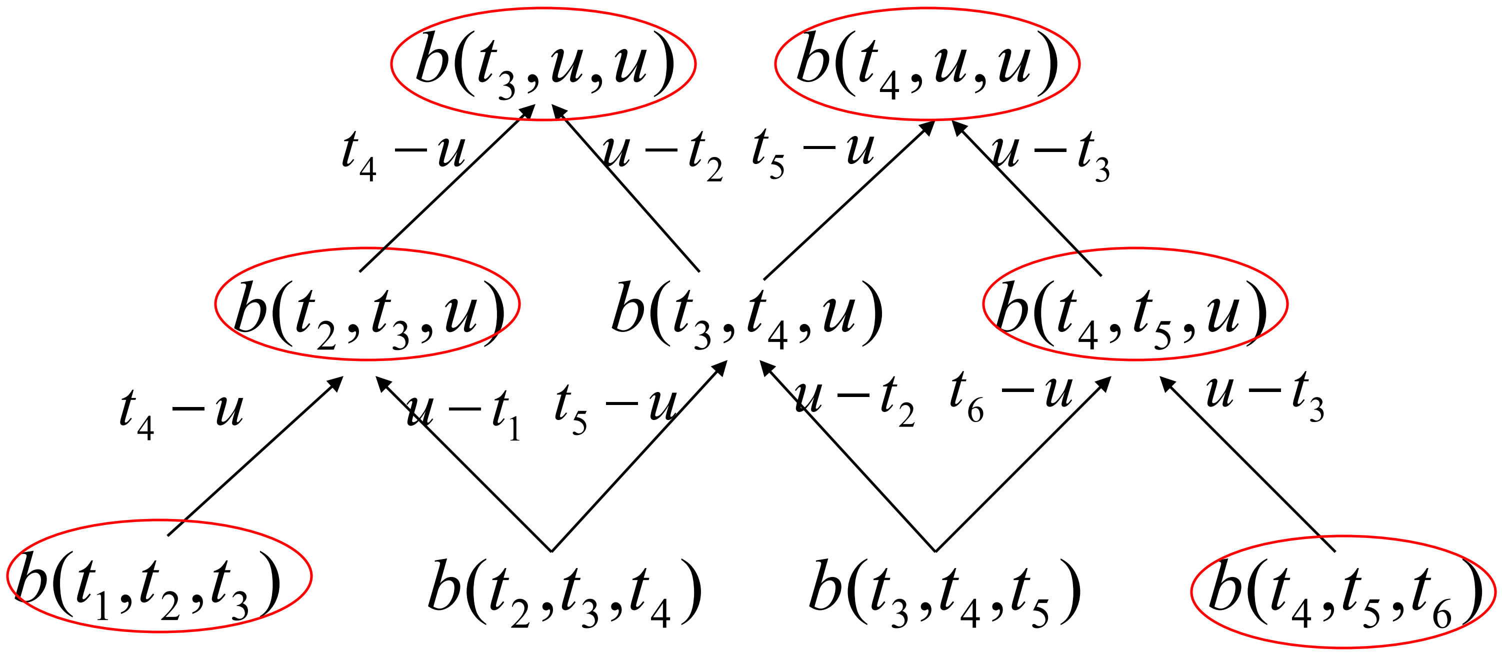

The Oslo Algorithm

- Given curve with knots t_1, t_2, t_3, t_4, t_5, t_6, find curve with knots t_1, t_2, t_3, u_1, u_2, u_3, u_4, t_4, t_5, t_6

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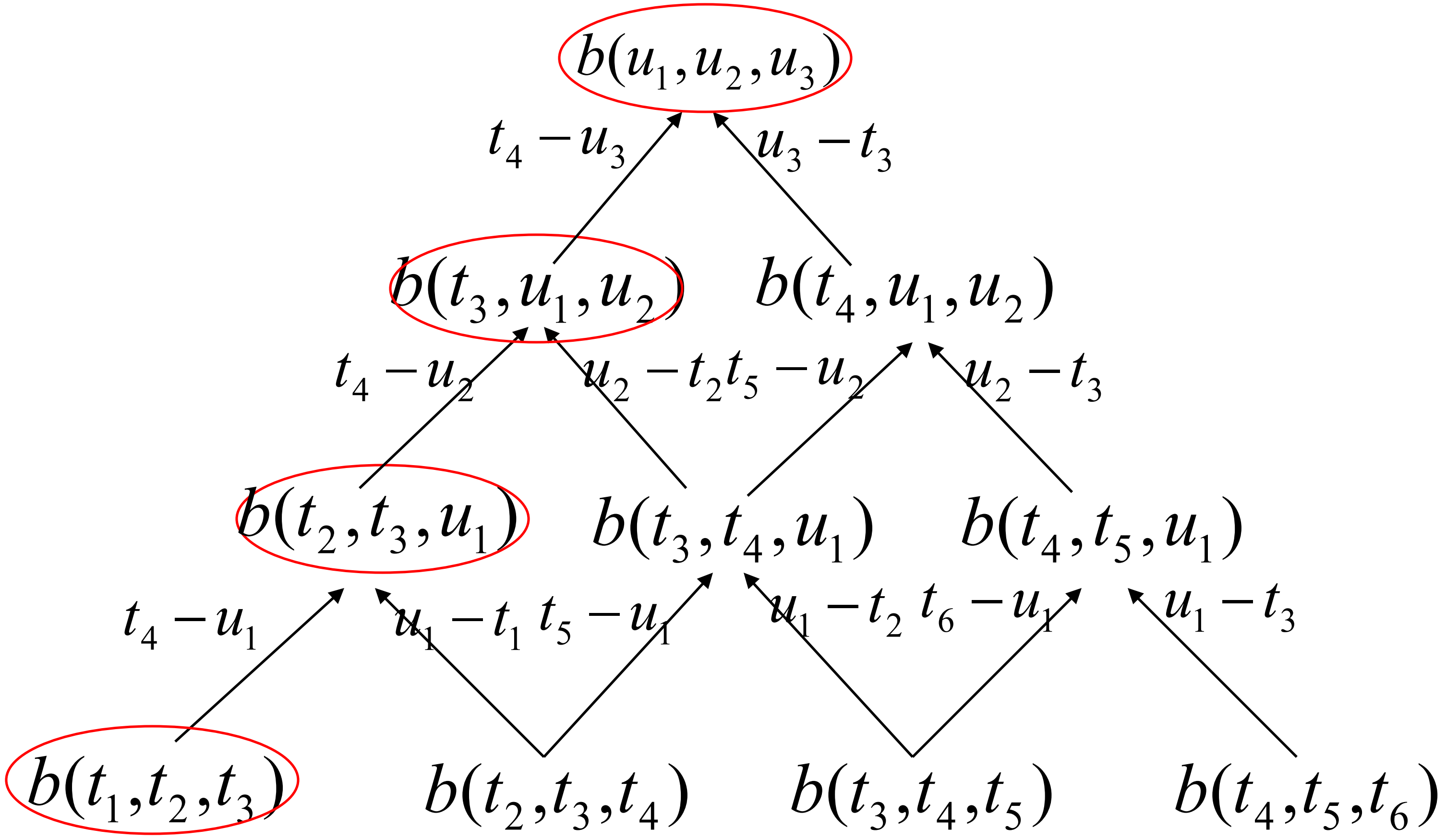

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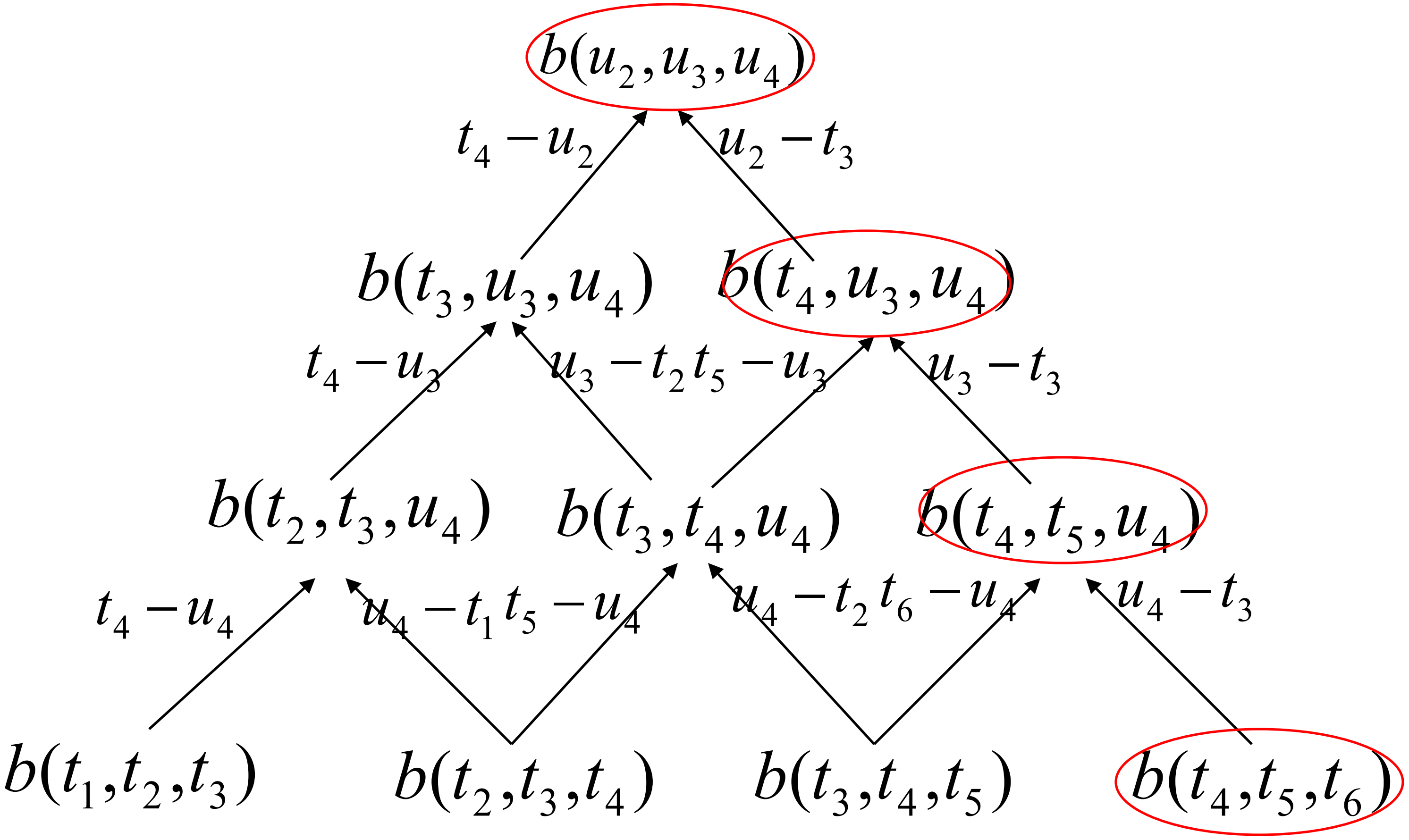
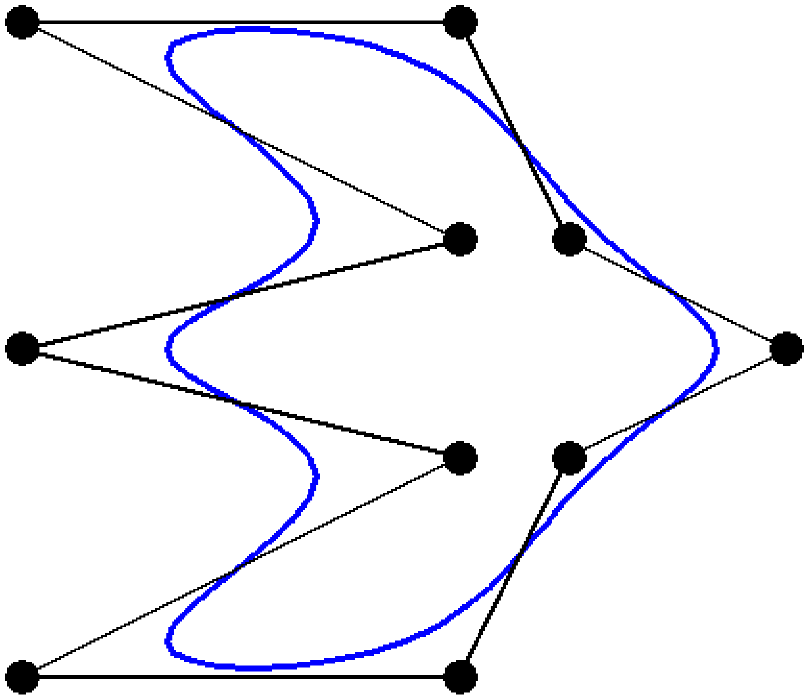


B-spline Properties

- Piecewise polynomial
- C^{n-u} continuity at knots of multiplicity u
- Compact support
- Non-negativity implies local convex hull property
- Variation diminishing

B-spline Curve Example



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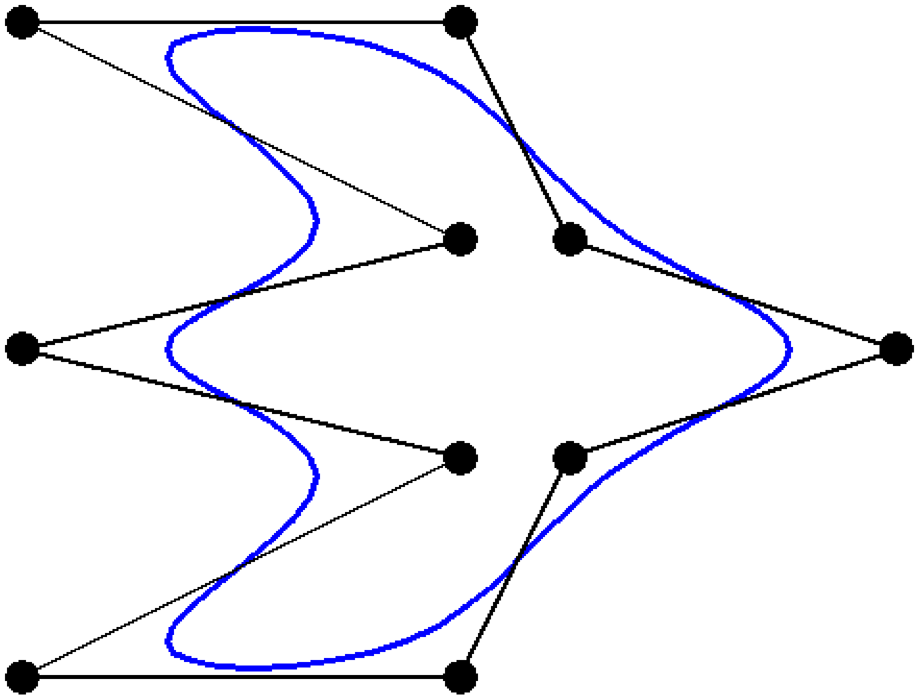

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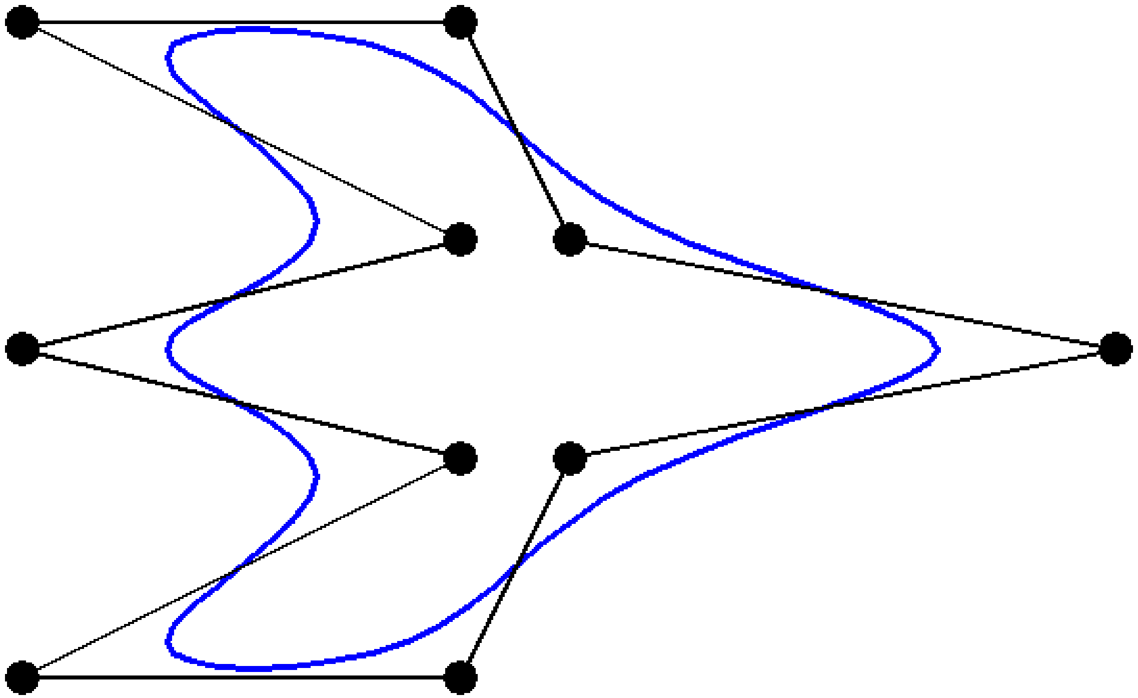

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- B-splines dependent on choice of knots t_i
- Can we choose t_i automatically?

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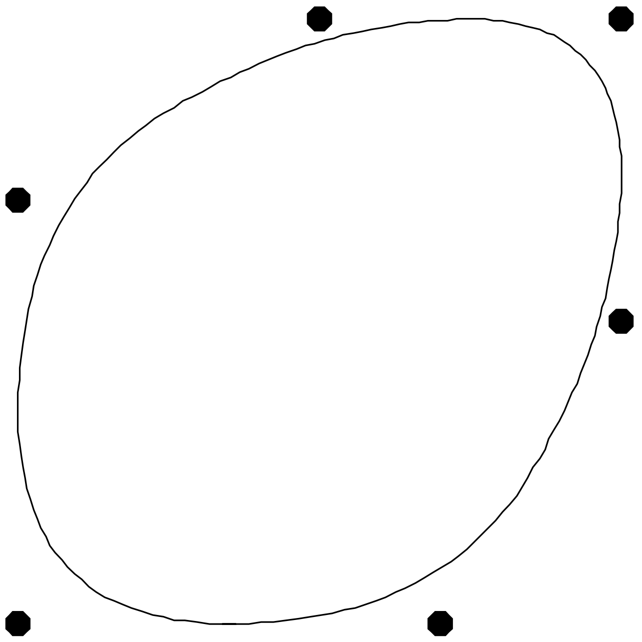

Uniform parameterization

$$\alpha = 0$$

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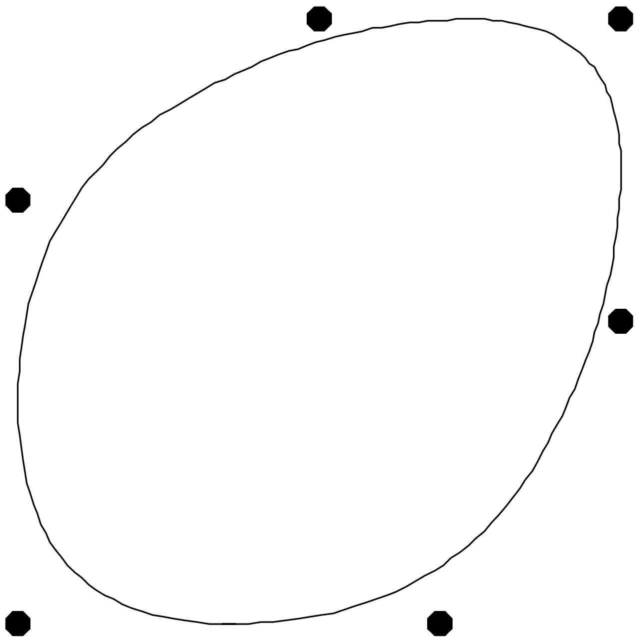

Centripetal parameterization

$\alpha = ½$

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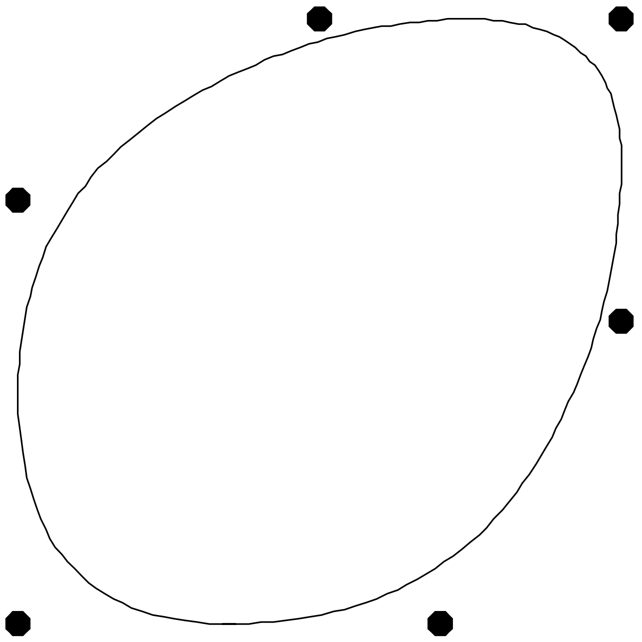

Chord length parameterization

$\alpha = 1$

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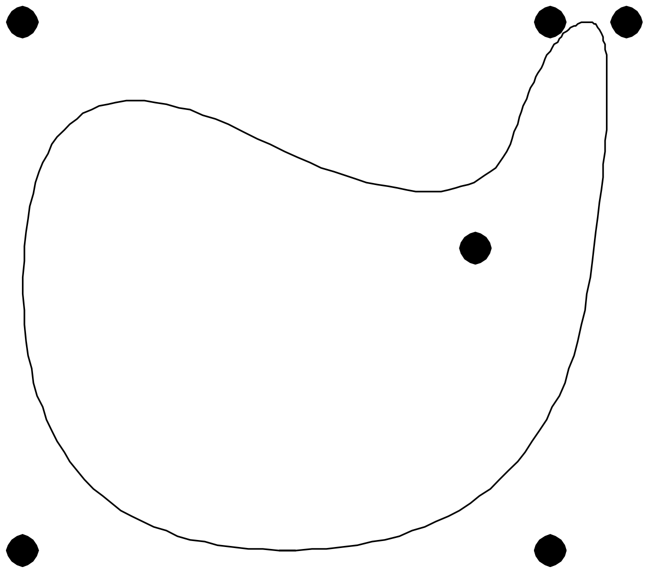

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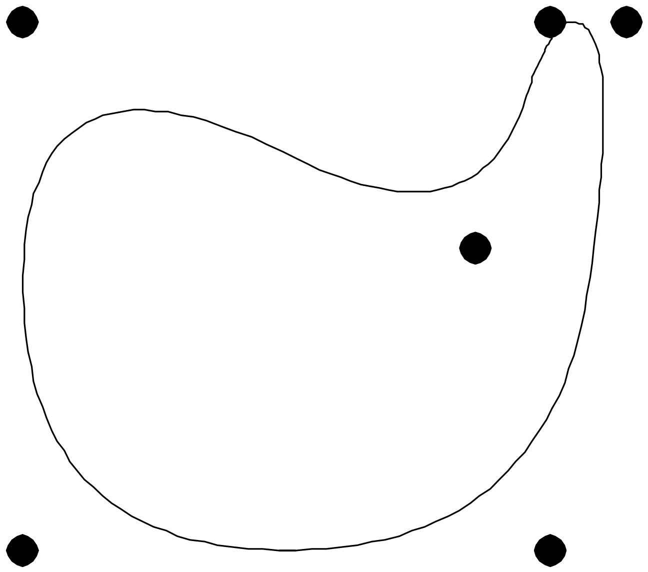
- Can we choose t_i automatically?

$$t_i = \left| p_{i+k} - p_{i+1+k} \right|^{\alpha}$$



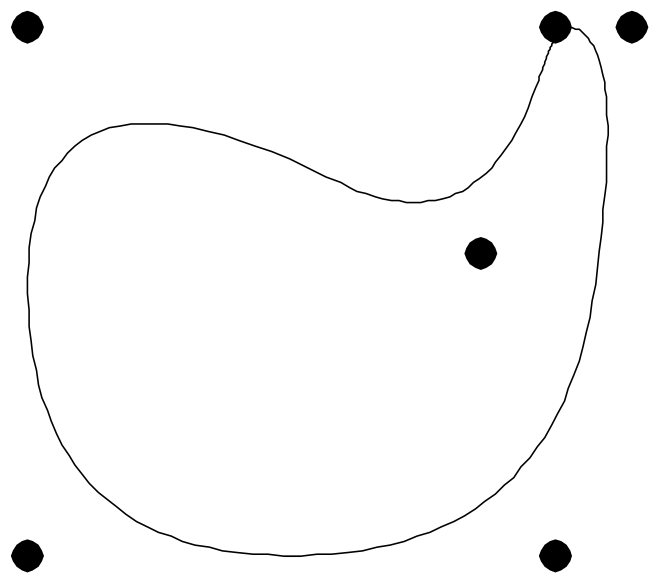
Centripetal parameterization

$\alpha = ½$

Choosing Knot Values

- B-splines dependent on choice of knots t_i
- Can we choose t_i automatically?

$$t_i = \left| p_{i+k} - p_{i+1+k} \right|^{\alpha}$$



Chord length
parameterization
$\alpha = 1$