Original Lecture #9: 29 October 1992
Topics: B-Splines
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1 Spline Curves

The beginning of this lecture was material covered in Handout # 23. Some of the material is presented again below, with clarifying figures.

1.1 The de Boor points

Suppose that we have chosen a degree n and a knot sequence (in which no knot has multiplicity greater than $n+1$). We complete our design of the spline curve F by choosing a sequence of control points, which are called *de Boor points* or *B-spline control points*. Each de Boor point is labeled by a block of n adjacent knots from the knot sequence, and successive de Boor points are labeled by blocks that are shifted by one step with respect to each other, that is, that overlap in all but one knot. That—and the de Boor generalization of the de Casteljau Algorithm—turns out to be all that you need to know to draw spline curves.

To see how this works, let's consider the affine, quadratic, and cubic cases in turn.

1.1.1 The affine case

Suppose that $n = 1$, so that we are designing an affine spline—that is, a polyline. And suppose that we have chosen the knot sequence

$$(\ldots, 1, 2, 4, 5, 5, 6, 8, \ldots).$$

Since $n = 1$, each de Boor point will be labeled by single knot, with adjacent de Boor points labeled by adjacent knots. So our spline F has, among its control points, the points P_1, P_2, P_4, P_5, P_5, P_6, and P_8. Note that there are two de Boor points labeled P_5, since 5 is a double knot.

The resulting affine spline F is quite straightforward. Over the time interval $[1\,..\,2]$, the spline F moves from P_1 to P_2 at a constant rate of speed. Over the time interval $[2\,..\,4]$, it moves from P_2 to P_4. Over $[4\,..\,5]$, it moves from P_4 to the first of the two points labeled P_5. Over $[5\,..\,6]$, it moves from the second point labeled P_5 to the point labeled P_6. And so on. Note that the joints corresponding to simple knots have C^0 continuity, as they should, while the joint corresponding to the double knot at time 5 has only C^{-1} continuity.

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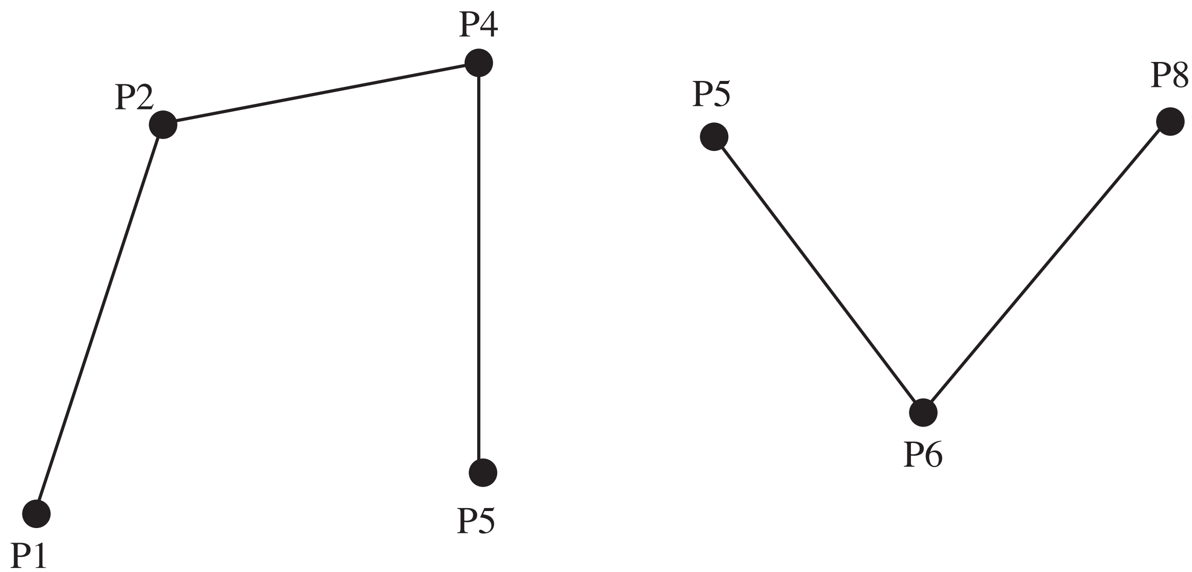

Figure 1: Affine spline curve

1.1.2 The quadratic case

To take a less trivial case, suppose that $n = 2$, so that we are assembling parabolic segments into a quadratic spline, and suppose that we have chosen the knot sequence

$$(\ldots, 0, 1, 2, 3, 5, 6, 7, 7, 8, 9, 9, 9, \ldots).$$

Since $n = 2$, each de Boor point is now labeled by a pair of adjacent knots. So reasonable names for the de Boor points are

$$P_{01},\ P_{12},\ P_{23},\ P_{35},\ P_{56},\ P_{67},\ P_{77},\ P_{78},\ P_{89},\ P_{99},\ \text{and}\ P_{99}.$$

In order to figure out what the segments of the spline curve F are, it is helpful first to compute some auxiliary points, based on the de Boor points. For example, P_{01} and P_{12} share a common subscript. Thinking of P_{uv} rather like a 2-polar value $f(u, v)$, we will define the point P_{1t} for all t in $[0\,..\,2]$ by interpolating between P_{01} and P_{12}. In particular, we define

$$P_{11} := \frac{P_{01} + P_{12}}{2}.$$

In a similar way, we define

$$P_{22} := \frac{P_{12} + P_{23}}{2}.$$

But P_{33} is slightly different, since the knots in this part of the knot sequence are not uniformly spaced—in particular, 3 is only one-third of the way from 2 to 5:

$$P_{33} := \frac{2P_{23} + P_{35}}{3}.$$

And we define P_{55}, P_{66}, and P_{88} analogously. (See Figure 2.)

Over the time interval $[1\,..\,2]$, our spline F will follow a parabolic segment $F([1\,..\,2])$, which is part of some overall parabola—call it G. Of course, the parabola G has a polar form g. Note that F itself is a spline curve, so it doesn't have a polar form, strictly speaking—even though the de Boor points P_{uv} almost behave like 'polar values' $P_{uv} = f(u,v)$ of the spline F. We choose to determine the parabola G by giving the Bézier points of the segment $F([1\,..\,2]) = G([1\,..\,2])$, as follows: $g(1,1) := P_{11}$, $g(1,2) := P_{12}$, and $g(2,2) := P_{22}$.

In a similar way, over each interval $[s\,..\,t]$ between two adjacent, distinct knots, the spline F will follow a parabolic segment $F([s\,..\,t])$ that is part of some parabola G, and that parabola G is determined by the conditions $g(s,s) := P_{ss}$, $g(s,t) := P_{st}$, and $g(t,t) := P_{tt}$. The way in which we computed the auxiliary points, such as P_{11} and P_{22}, from the original de Boor points guarantees that the resulting spline curve F will have the required level of continuity at each joint. For example, the condition that $P_{33} = (2P_{23} + P_{35})/3$ guarantees that the ending tangent vector of the segment $F([2\,..\,3])$ is the same as the starting tangent vector of the segment $F([3\,..\,5])$; so the joint at time 3 is a C^1, as is appropriate, since 3 is a simple knot.

In fact, we don't have to bother computing the auxiliary points P_{11}, P_{22}, and the like; we can work directly from the de Boor points. Consider the time interval $[2\,..\,3]$, for example. Above, we determined the parabola G that our spline F follows over $[2\,..\,3]$ by giving the Bézier points of the segment $G([2\,..\,3])$, which are P_{22}, P_{23}, and P_{33}. The first and last of these are auxiliary points, while the middle one is a de Boor point. Instead, we can determine the parabola G directly from three adjacent de Boor points by imposing the following three polar interpolation constraints on G: $g(1,2) = P_{12}$, $g(2,3) = P_{23}$, and $g(3,5) = P_{35}$. To prove that these three constraints are a legal way to specify a parabola G, we appeal to Theorem 4 from Handout 14, setting $r_1 := 2$, $r_2 := 1$, $s_1 := 3$, and $s_2 := 5$. From that theorem, we deduce that there exists a unique parabola G that satisfies $g(r_1,r_2) = g(1,2) = P_{12}$, $g(r_1,s_1) = g(2,3) = P_{23}$, and $g(s_1,s_2) = g(3,5) = P_{35}$. the double knot at time 5 has only C^{-1} continuity.

1.1.3 The cubic case

Suppose now that $n = 3$. It's worth going through several examples of knot sequences, in the cubic case. The first knot sequence has uniformly spaced simple knots for a while, then has a single longer parameter interval at $[5\,..\,7]$, and ends with a triple knot at 9:

$$(\ldots,0,1,2,3,4,5,7,8,9,9,9,\ldots).$$

The second starts with two triple knots in a row, later including a double knot in the middle of a string of simple knots:

$$(\ldots,0,0,0,1,1,1,2,3,4,4,5,6,7,8,\ldots).$$

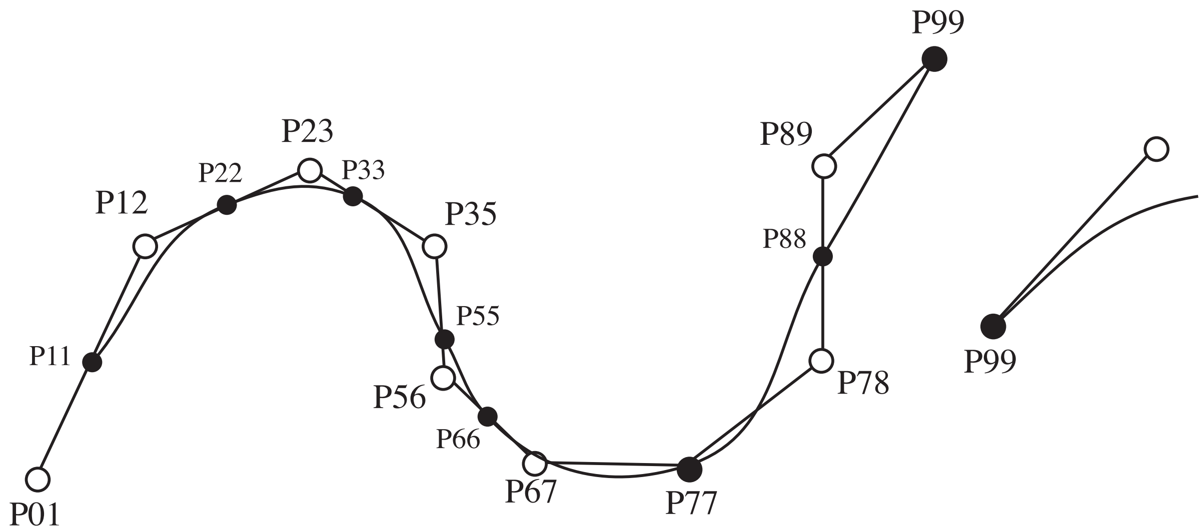

Figure 2: Quadratic spline curve

[Choose nine arbitrary de Boor points and draw a spline with the first knot sequence. Choose twelve arbitrary de Boor points and draw a spline with the second knot sequence.]

Since $n = 3$, each de Boor point is labeled by a triple of adjacent knots from the knot sequence. So, in the first example, the nine de Boor points are P_{012}, P_{123}, P_{234}, P_{345}, P_{457}, P_{578}, P_{789}, P_{899}, and P_{999}.

To determine the Bézier points of the various cubic segments in the first example, we treat each de Boor point P_{uvw} like a polar value $f(u,v,w)$ of the spline F. For example, consider the cubic segment $F([2\,..\,3])$. If G is the overall cubic curve that the spline F follows over $[2\,..\,3]$, then the segment $F([2\,..\,3]) = G([2\,..\,3])$ is determined by the positions of the four Bézier points $g(2,2,2)$, $g(2,2,3)$, $g(2,3,3)$, and $g(3,3,3)$. None of the points P_{222}, P_{223}, P_{233}, or P_{333} is a de Boor point. But P_{123} and P_{234} are de Boor points, and we can find both of the points P_{223} and P_{233} by interpolating between P_{123} and P_{234}:

$$P_{223} = \tfrac{2}{3}P_{123} + \tfrac{1}{3}P_{234}$$
$$P_{233} = \tfrac{1}{3}P_{123} + \tfrac{2}{3}P_{234}$$

In a similar way, we can find P_{112} and P_{122} by interpolating between P_{012} and P_{123}, and we can find P_{334} and P_{344} by interpolating between P_{234} and P_{345}. Finally, the point P_{222} must be halfway between P_{122} and P_{223}, while the point P_{333} must be halfway between P_{233} and P_{334}. [Draw all of these points on your picture.]

The other segments of the first example and all of the segments in the second example can be found, in terms of their Bézier points, in a similar way.

Finding the Bézier points of the various segments helps to reveal the geometric structure of our spline curves. If we prefer, however, we can also dispense with the Bézier points and work straight from the de Boor points. Figure 13 in Handout 14 shows an example of determining

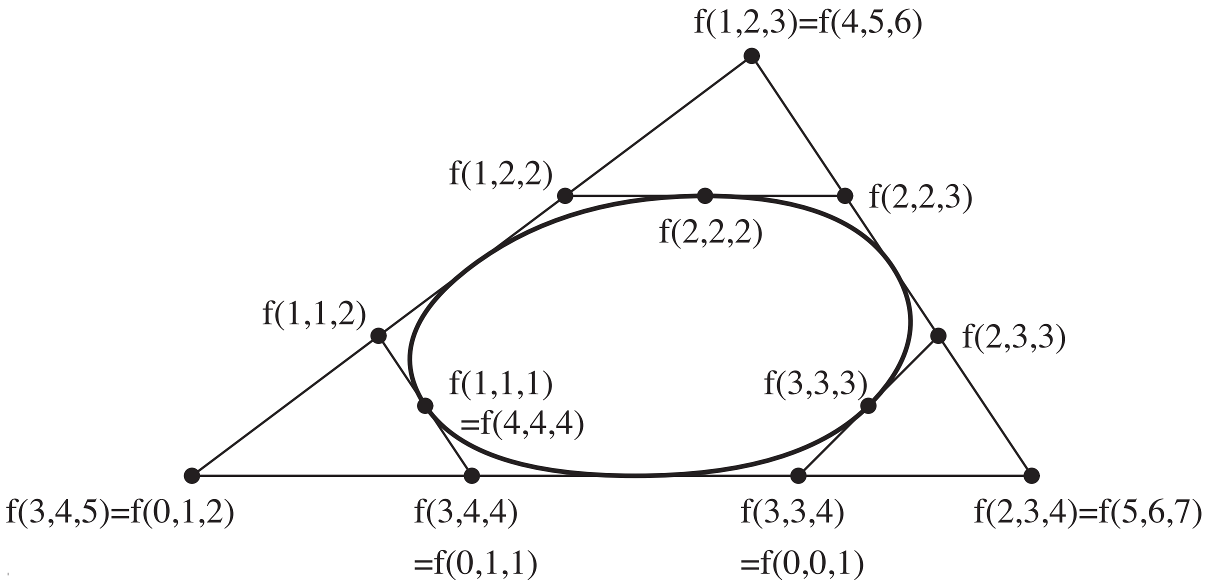

Figure 3: A closed curve with three joints

the segment $F([4\,..\,7]) = G([4\,..\,7])$ of a cubic spline curve F on the knot sequence

$$(\ldots, 2, 3, 4, 7, 8, 9, \ldots)$$

from the four adjacent de Boor points $P_{234} = g(2,3,4)$, $P_{347} = g(3,4,7)$, $P_{478} = g(4,7,8)$, and $P_{789} = g(7,8,9)$. The instance of the de Boor Algorithm illustrated in Figure 13 is computing the point $F(5) = G(5) = g(5,5,5)$ on the resulting segment $F([4\,..\,7])$.

1.2 Closed curves

We have not yet dealt with closed curves. These can be handled by taking an infinite knot sequence with a periodicity condition.

For example, if there are three joints, we can take the knot sequence

$$(\ldots, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots) \tag{1}$$

with condition

$$F(t+3) = F(t) \tag{2}$$

holding for all t. Figure 3 shows an example.

1.3 Interpolating splines

So far, we have been constructing our spline curves by treating the de Boor points as the control points. Once the de Boor points are known, we can compute the Bézier points of the spline segments and the points of the spline via affine interpolations.

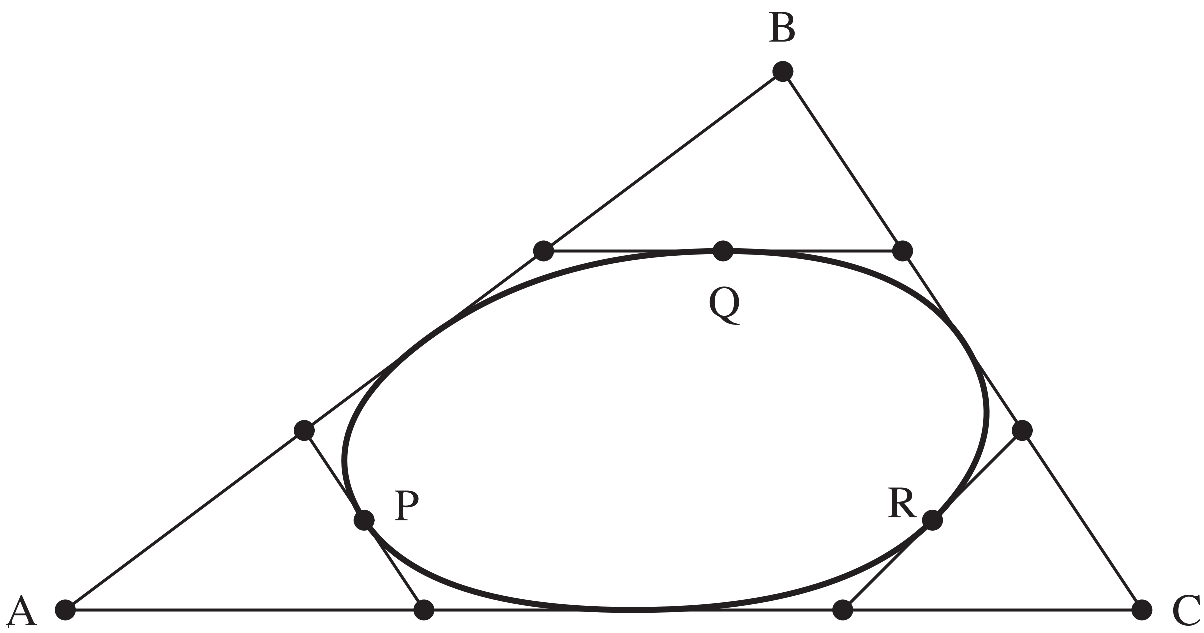

Figure 4: Bézier points A, B, and C; joints P, Q, and R

An alternative is to treat the joints of the spline as the control points. One advantage of this scheme is that the resulting curve actually interpolates the control points; hence, this technique is called the technique of *interpolatory splines*.

Consider, for example, a closed cubic spline with three segments, as shown in Figure 4. Observe that, if A, B and C are the de Boor points, the joints P, Q and R can be expressed in terms of A, B and C as follows:

$$\begin{aligned} P &= \tfrac{2}{3}A + \tfrac{1}{6}B + \tfrac{1}{6}C \\ Q &= \tfrac{1}{6}A + \tfrac{2}{3}B + \tfrac{1}{6}C \\ R &= \tfrac{1}{6}A + \tfrac{1}{6}B + \tfrac{2}{3}C \end{aligned} \tag{3}$$

So, if we are given the joints P, Q and R, we can solve a linear system of equations to obtain the de Boor points A, B, and C.

One disadvantage of this method is that there is no local control. Each time a joint is moved, a new linear system of equations (3) must be solved, and all of the de Boor points can change positions. If the designer moved only one joint, the de Boor points far from that joint probably don't move very far, but they do move.

1.4 Open curves

Now suppose we are given a sequence of joints in the plane, and we wish to find an *open* cubic spline that interpolates those joints. The condition that the spline be C^2 continuous at every

Figure 5: Uniform parametrization

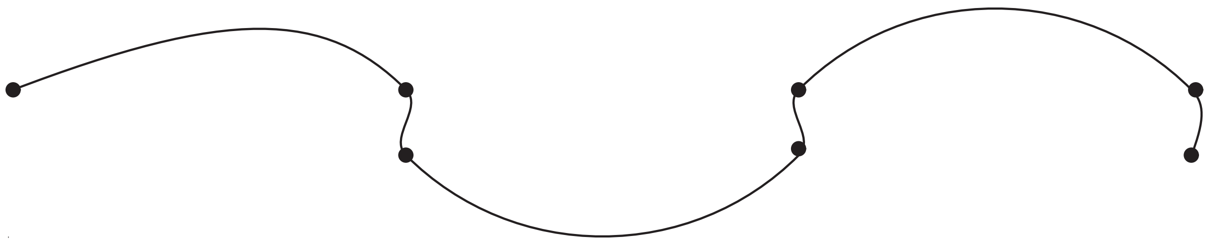

Figure 6: Chord-length parametrization

joint still leaves us with four degrees of freedom, two for each end of the open spline. This is the problem of **end conditions**. There are lots of strategies for using up these final degrees of freedom, including:

- "Natural" end conditions: set zero acceleration at the endpoints, i.e., $F''(\text{start}) = 0$, $F''(\text{end}) = 0$.
- Allow designer to specify $F'(\text{start})$ and $F'(\text{end})$.
- "Not-a-knot" end conditions: set C^3 continuity at the two joints closest to the endpoints.

1.5 Choosing the knot spacing

The most obvious way to form a knot sequence is to use all single knots, evenly spaced; this is called **uniform parameterization**. However, this sometimes leads to undesirable results. In the example of Figure 5, the knot sequence is uniform:

$$(0, 1, 2, 3, 4, 5, 6). \tag{4}$$

The resulting interpolating spline streaks straight across the long gaps, working hard to get there in time, and then does a long, wild S-turn during all the extra time that it has to get across the short gaps.

More sensible curves often arise if **chord-length parametrization** is used. That is, the difference between successive knots is made proportional to the length of the chord between the corresponding joints. Figure 6 shows the spline arising from the same joints as in Figure 5, but with the knot sequence

$$(0, 6, 7, 13, 14, 20, 21). \tag{5}$$

References

[1] Carl de Boor (1978), *A Practical Guide to Splines*, Springer-Verlag.

[2] Lyle Ramshaw (1989), Blossoms are polar forms, Computer Aided Geometric Design **6**, 323–358.

[3] Gerald Farin (1988), *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*, Academic Press.