Original Lecture #12: 10 November 1992
Topics: Total-Degree Surfaces
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1 Surfaces in Parametric Form

Recall that a parametric description of a surface in 3-space has the form

$$P(U,V) = (X(U,V), Y(U,V), Z(U,V)).$$

The parametric form maps a point (U,V) in the 2-d parameter space to a point $P(U,V)$ in 3-space.

Two distinct interpretations of this parametric form are possible, depending on the method used to put a degree bound on the surface. Consider the X coordinate of a parametric surface introduced in the previous lecture

$$X(U,V) = 2UV + 3UV^2 + 7U^2.$$

In the tensor product case, the parameters U and V are considered separately, and the degree of the polynomial is computed for each while holding the other parameter constant. Thus the expression for $X(U,V)$ is viewed as two polynomials of degree (2;2), one with U held constant, the other with V constant. In the total degree case, however, we compute the total degree in U and V considered together. Thus for X above, the total degree is 3. Note that the distinction between tensor product and total degree interpretations is not a minor one, and leads to very different results after homogenization and polarization.

2 Homogenization of Total-degree Surfaces

Recall that in the tensor product case we made the separate substitutions $U = u/a$ and $V = v/b$ in order to homogenize X. In the total degree case, however, we introduce just one new homogenizing variable, say c, in order to map the coordinates $[1;U,V]$ to the homogeneous coordinates $(c;u,v)$ via the substitutions $U = u/c$ and $V = v/c$. Thus X becomes

$$X(U,V) = X\left(\frac{u}{c},\frac{v}{c}\right) = 2\left(\frac{u}{c}\right)\left(\frac{v}{c}\right) + 3\left(\frac{u}{c}\right)\left(\frac{v^2}{c^2}\right) + 7\left(\frac{u^2}{c^2}\right).$$

The rescaling that clears the denominators in this case will multiply all of the homogeneous coordinates w, x, y, and z by c^3, getting the final form

$$x(c,u,v) = 2uvc + 3uv^2 + 7u^2c.$$

*Heavily based on the 1991 notes of Steven Woo, Alice Yu, and Dan Yang.

3 Polarization of Total-degree Surfaces

Since, in the total degree case, $X(U,V)$ is the X-coordinate of a surface F of total degree 3, the polar form or blossom f of F takes three points in the (U,V) plane as arguments, and the X-coordinate x of the polar form f thus has the form $x((U_1,V_1),(U_2,V_2),(U_3,V_3))$.

To find the polar form of UV, we substitute all products of U_i and V_j such that $i \neq j$, and divide by the number of such products. Thus the polar form of UV becomes $\frac{1}{6}(U_1V_2 + U_1V_3 + U_2V_1 + U_2V_3 + U_3V_1 + U_3V_2)$.

To polarize the term UV^2, we substitute the products of U_i, V_j, and V_k where i, j, and k are distinct, and divide by the number of these products. So the polar form of UV^2 is $\frac{1}{3}(U_1V_2V_3 + U_2V_1V_3 + U_3V_1V_2)$.

Similarly, the polar form of U^2 is $\frac{1}{3}(U_1U_2 + U_1U_3 + U_2U_3)$.

The polar form x of the polynomial X, viewed as a polynomial of total degree 3 in U and V jointly, is the appropriate linear combination of the above polar forms:

$$\begin{aligned}
&x((U_1,V_1),(U_2,V_2),(U_3,V_3)) = \\
&\quad \frac{1}{3}(U_1V_2 + U_1V_3 + U_2V_1 + U_2V_3 + U_3V_1 + U_3V_2) + \\
&\qquad (U_1V_2V_3 + U_2V_1V_3 + U_3V_1V_2) + \frac{7}{3}(U_1U_2 + U_2U_3 + U_1U_3).
\end{aligned}$$

The real entities here are the *pairs*—we want something affine in terms of each *point* (U_i,V_i). Note that the blossoming degree is the maximum of all the coordinates of F, thus, had $Y(U,V)$ and $Z(U,V)$ been of degree less than three, the point (U,V) would still blossom into three points for the polar forms of Y and Z.

4 Developing a Bézier Theory for Surface Patches

Recall that for the tensor product case, the relevant region of the parameter plane to use as the domain of a patch was a rectangle. Since U and V were considered separately, intervals defined the ranges of U and V, and only one-dimensional interpolation was necessary.

In the total degree case, the "coupled" U and V interpretation requires three points to define the range, and necessitates two-dimensional interpolation. The relevant region of the parameter plane in this case is a triangle. The image of a domain triangle, say $\triangle PQR$, is a triangular surface patch $F(\triangle PQR)$. The Bézier points of this triangular patch are the polar values that result from using the points P, Q, and R, in all possible ways, as polar arguments to the polar form f of the surface F. For example, a cubic triangular surface patch F has ten Bézier points:

$$\begin{array}{ccccccc}
 & & & f(P,P,P) & & & \\
 & & f(P,P,Q) & & f(P,P,R) & & \\
 & f(P,Q,Q) & & f(P,Q,R) & & f(P,R,R) & \\
f(Q,Q,Q) & & f(Q,Q,R) & & f(Q,R,R) & & f(R,R,R)
\end{array}$$

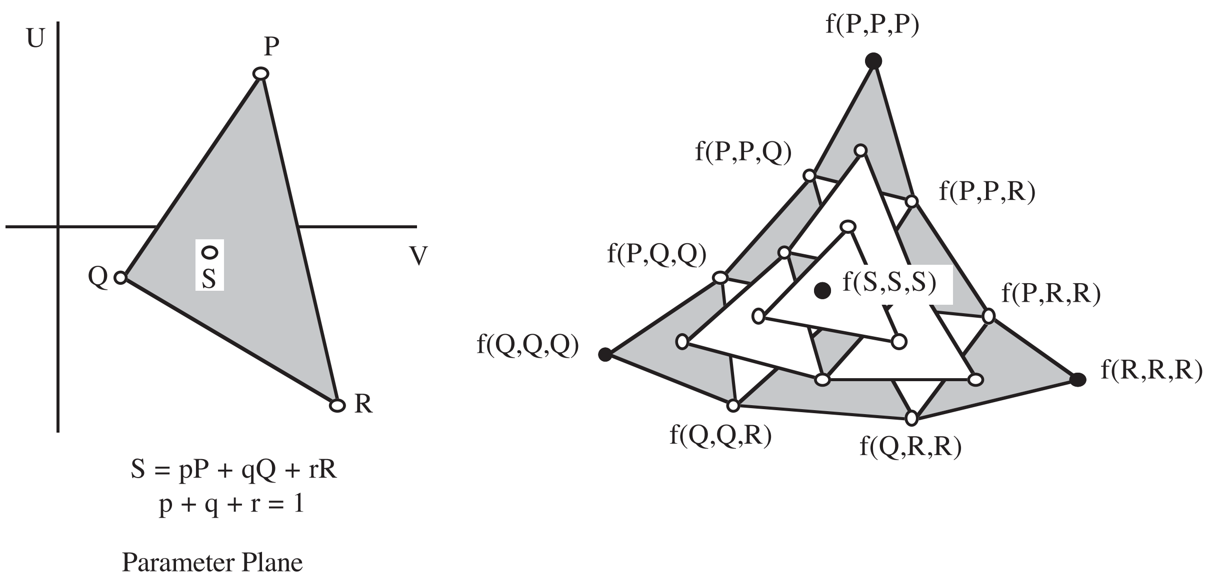

Figure 1: The de Casteljau Algorithm for a cubic surface

In the total-degree case, the de Casteljau algorithm does two-dimensional affine interpolations, as shown in Figure 1. For example, suppose that point S lies in the domain triangle $\triangle PQR$, given in barycentric coordinates by $S = pP + qQ + rR$, where $p + q + r = 1$. The point $f(P,P,S)$ can be found by interpolating in the triangle whose vertices are $f(P,P,P)$, $f(P,P,Q)$, and $f(P,P,R)$:

$$f(P,P,S) = pf(P,P,P) + qf(P,P,Q) + rf(P,P,R).$$

Similarly, other points with one S in their labels can be found by interpolating in other triangles with three Bézier points as their vertices. Then points with more than one S in their labels can be found be interpolating in triangles of points found earlier. The three corner Bézier points in the triangular array above, $F(P) = f(P,P,P)$, $F(Q) = f(Q,Q,Q)$, and $F(R) = f(R,R,R)$ are the three corners of the triangular surface patch $F(\triangle PQR)$. Also, the edge PQ of the domain triangle maps, under F, to the cubic curve whose four Bézier points are $f(P,P,P)$, $f(P,P,Q)$, $f(P,Q,Q)$, and $f(Q,Q,Q)$.

More generally, the image $F(AB)$ of an arbitrary line segment AB in the parameter plane is simply the cubic curve on the surface whose Bezier points are $f(A,A,A)$, $f(A,A,B)$, $f(A,B,B)$, and $f(B,B,B)$, as shown in Figure 2. Note that this differs from the tensor product case, where only line segments parallel to the U or V axis map to a quadratic curve (for the biquadratic surface example); a diagonal line may map to a curve of higher degree than the maximum degree of U or V separately.

Going from polynomial surfaces to rational surfaces is analogous to going from two dimensional polynomial curves to rational curves. Points are allowed different weight values, and our point S in barycentric coordinates would be defined as $S = (pw_pP + qw_qQ + rw_rR)/(pw_p + qw_q + rw_r)$.

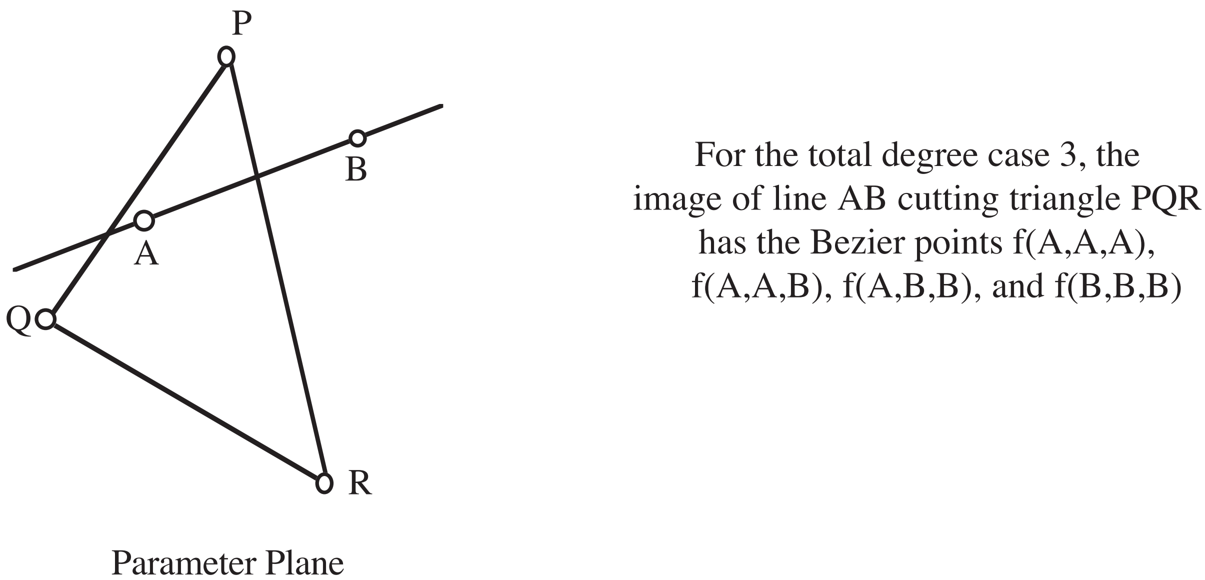

Figure 2: The image of a segment under a total-degree surface

5 Continuity Constraints For Triangular Surface Patches

We now examine the continuity constraints of triangular patches. Consider again the cubic total degree case. Suppose we have two patches A and B with common boundary along the edge st (Figure 3).

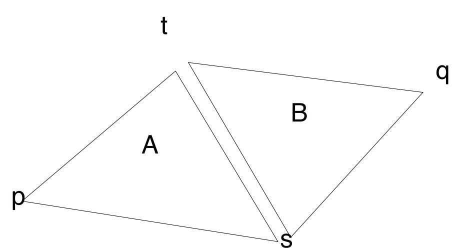

Figure 3: triangular patches share common boundary

5.1 C^0-Continuity

For these two patches to be continuous along the boundary, we have to have (Figure 4)

- $A(sss) = B(sss)$
- $A(sst) = B(sst)$
- $A(stt) = B(stt)$

- $A(ttt) = B(ttt)$

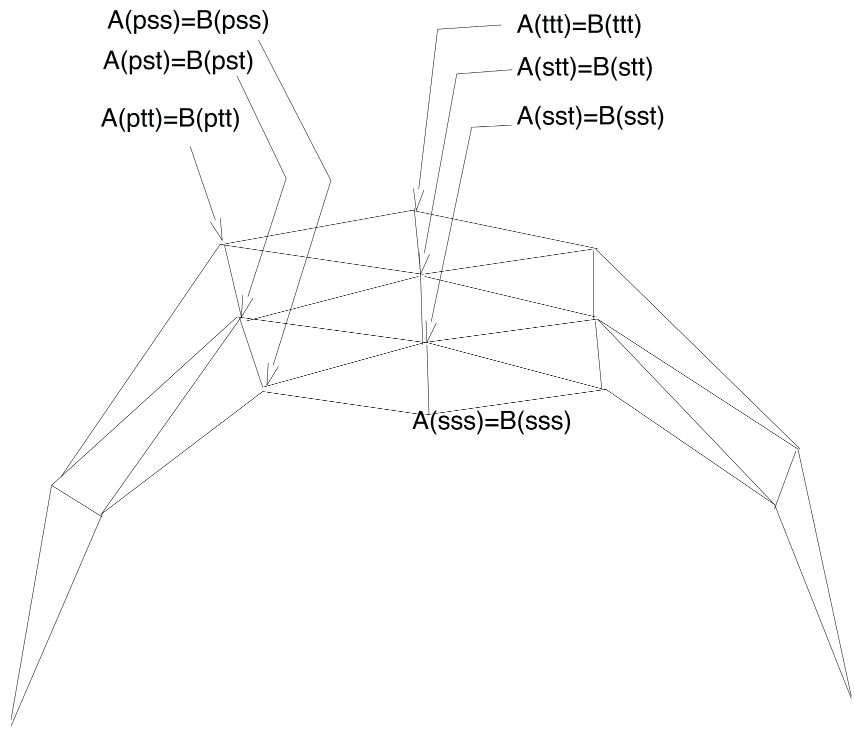

Figure 4: The Bezier points of two cubic surfaces that join with C^0 continuity

5.2 C^1-Continuity

Even though q is not inside the patch domain pst, we still can calculate $A(qss)$, $A(qst)$, $A(qtt)$ by extrapolation. Similarly, we can calculate $B(pss)$, $B(pst)$, $B(ptt)$. For these two patches to be C^1-continuous, in addition to the conditions for C^0, we also need (see Figure 5)

- $A(xss) = B(xss) \quad \text{for } x = p, q$
- $A(xst) = B(xst) \quad \text{for } x = p, q$
- $A(xtt) = B(xtt) \quad \text{for } x = p, q$

That is, groups of four Bezier points along the common boundary must be affine images of the quadrilateral $ptqs$ in the parameter planes.

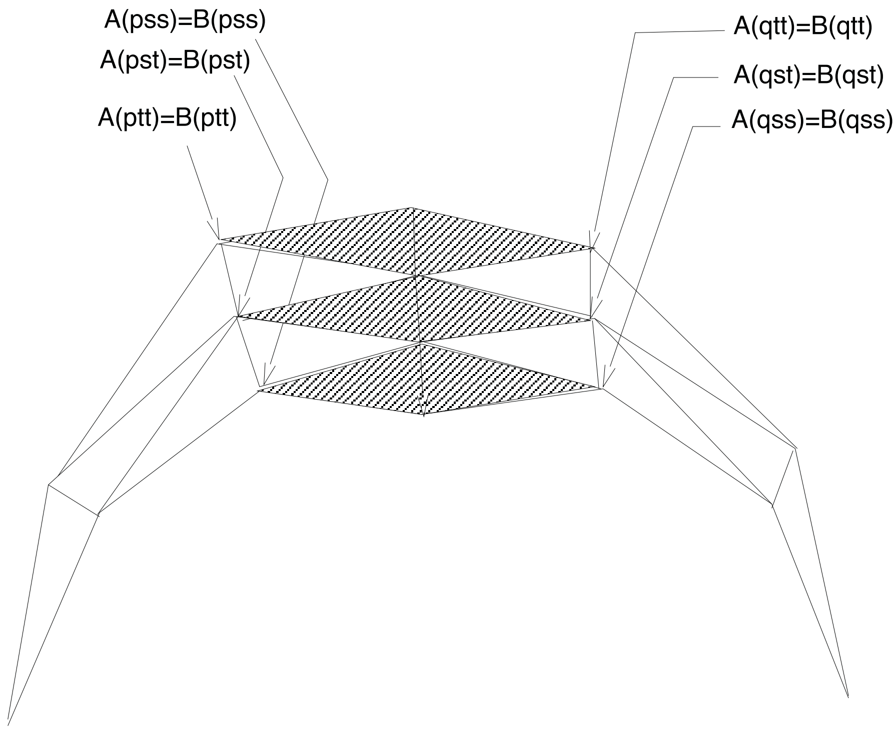

Figure 5: The Bezier points of two cubic surfaces that join with C^1 continuity

5.3 C^2-Continuity

For two patches to be C^2-continuous, in addition to the conditions for C^1, we also need (see Figure 6)

- $A(ppx) = B(ppx)$ for $x = s, t$
- $A(qqx) = B(qqx)$ for $x = s, t$

5.4 C^3-Continuity

For two patches to be C^3-continuous, in addition to the conditions for C^3, we also need (see Figure 7)

- $A(xxx) = B(xxx)$ for $x = p, q$

In other words, two patches agree on all 10 Bezier points.

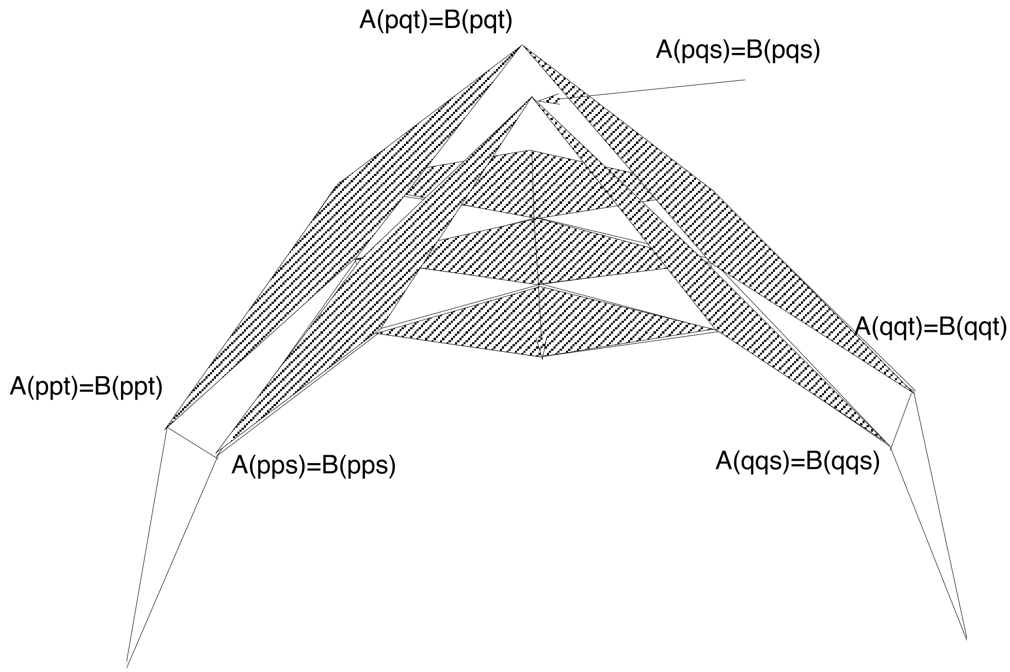

Figure 6: The Bezier points of two cubic surfaces that join with C^2 continuity

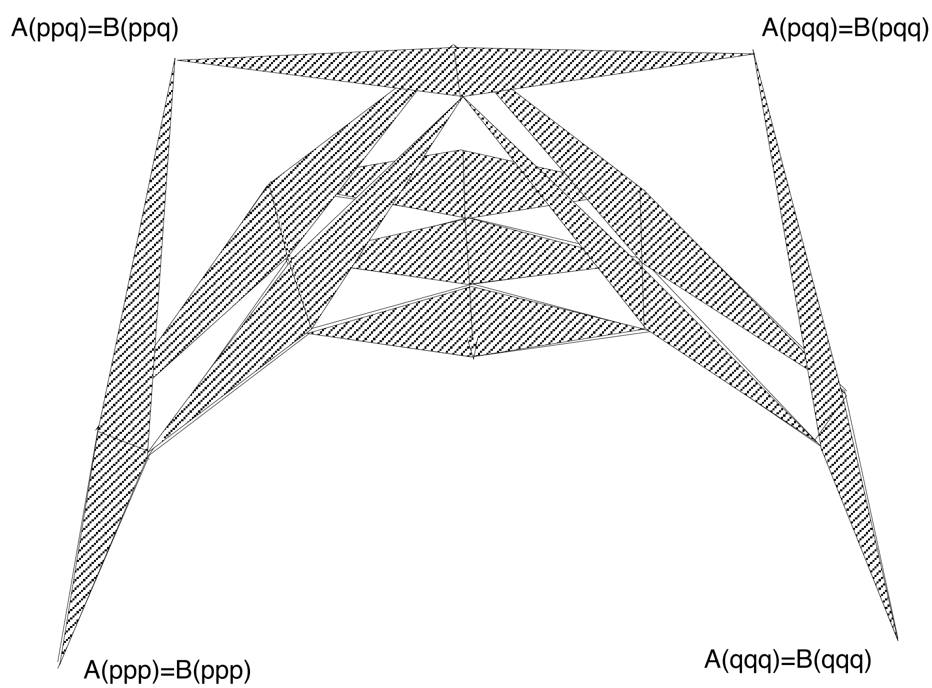

Figure 7: The Bezier points of two cubic surfaces that join with C^3 continuity