Original Lecture #1: 1 October 1992
Topics: Affine vs. Projective Geometries
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1 Overview

In High School geometry classes, our consideration for geometric transformations was limited to *translations*, *rotations*, and *uniform scalings*. For instance, similar triangles can be defined as those triangles which can be transformed into each other using only the above transformations. These transformations are known as *Euclidean transformations*, and the geometry that is preserved by Euclidean transformations is known as *Euclidean geometry*.

However, for the purposes of Computer Graphics, we will be concerned with various geometries:

1. Euclidean Geometry
2. Affine Geometry
3. Projective Geometry

Affine transformations include all of the transformations of Euclidean geometry, but adds to them *non-uniform scaling* ("squashing" or "stretching") and *shearing*. Projective geometry adds *projections* to the transformations of affine geometry. It is important to note that the Euclidean transformations are a subset of the affine transformations, and the affine transformations are in turn a subset of the projective transformations.

The three geometric classifications can also be defined by the number of points which uniquely define a transformation in their respective geometries. A Euclidean transformation of the plane can transform any two points into any other two points (see Figure 1). We can transform the segment P_1P_2 to the segment Q_1Q_2 by concatenating up to three Euclidean transformations: First, *translate* the point P_1 to the point Q_1. Then, *rotate* the plane about the point Q_1 so that the image of P_2 lies on the line Q_1Q_2. Finally, *scale* the plane around Q_1 to make the image of P_2 coincide with Q_2.

Affine geometry allows us to transform any *three* non-collinear points into any other three (see Figure 2). First, using a translation and a rotation, transform P_1 to Q_1 and put P_2 onto the ray from Q_1 to Q_2. Then scale non-uniformly: In the direction parallel to Q_1Q_2, scale around Q_1 to take P_2 to Q_2. In the direction perpendicular to Q_1Q_2, scale around Q_1 so that P_3 ends up on the line through Q_3 parallel to Q_1Q_2. Finally, shear along lines parallel to Q_1Q_2 so that P_3 moves to the point Q_3.

Similarly, in projective geometry we can transform any four points, no three collinear, to any other four, although we will not demonstrate it here.

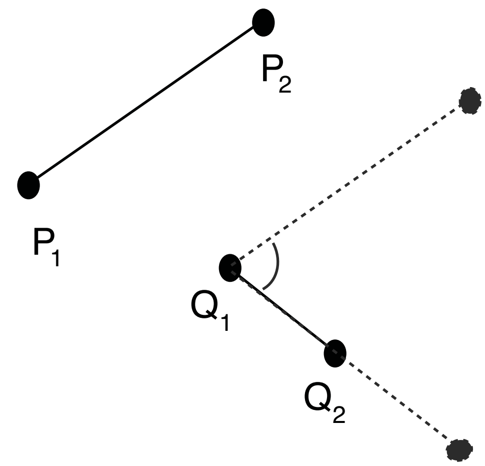

Figure 1: Euclidean Transformations

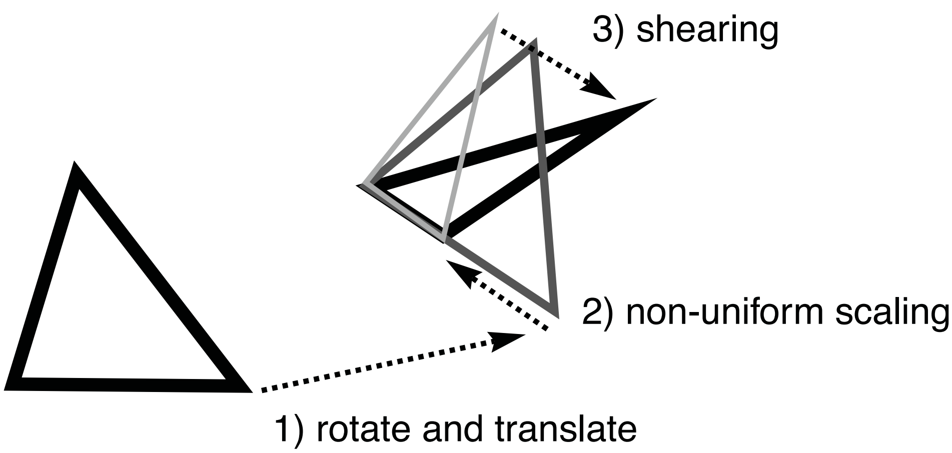

Figure 2: Affine Transformations

2 Points and Vectors

One way to describe points and vectors in 2-space is as ordered pairs. (The following discussion can be easily extended to consider 3-space, or n-space, for that matter). The point two units away from the y-axis and three units away from the x-axis is known as $(2,3)$. Similarly, the vector which can be decomposed into components two and three units long, parallel to the x and y axes respectively, is also known as $(2,3)$. This presents a small difficulty, since points and vectors are not completely interchangeable. For instance, vectors can be added to each other or multiplied by a scalar. Addition and scalar multiplication of points is illegal, but *averaging* is not.

Thus we choose an extended representation for both vectors and points which more fully encodes their nature. It may at first seem arbitrary, but as we will see, this notation is actually quite convenient mathematically as well. We will add another coordinate to our notation, which will be a 0 for vectors, or a 1 for points. This coordinate, called a *weight* coordinate, can either be made the first or the last coordinate. In this course, we will use the former notation. We will also further distinguish a weighted point or vector by using a semicolon as a separator for this weight coordinate. Thus, a weighted vector will take the form:

$$v = (0; v_x, v_y)$$

and a weighted point will take the form:

$$P = (1; P_x, P_y).$$

Whenever we refer to a point/vector in our new notation, but are not certain of its exact nature, we will call this point/vector a *site*.

We can begin to see some of the utility of such a notation when we use the traditional rules for operations on points and vectors using sites instead. Adding and multiplying vectors is as before, since the weight coordinate is always 0:

$$rv = (0; rv_x, rv_y)$$

$$v + w = (0; v_x + w_x, v_y + w_y)$$

But if we should try to add or multiply points, we would quickly run aground:

$$2P = (2; 2P_x, 2P_y)$$

$$P + Q = (2; P_x + Q_x, P_y + Q_y)$$

The fact that the weight coordinate of the results are 2, an illegal quantity, signals to us that such operations are inappropriate for points. Averaging points, however, is still legal:

$$\frac{P+Q}{2} = \left(1; \frac{P_x + Q_x}{2}, \frac{P_y + Q_y}{2}\right)$$

Furthermore, you may see that the sum of a vector and a point is another point, and the difference of two points is a vector, both of which are expected results.

This site notation has a number of other desirable "side effects," some of which we will discuss in later sections. For instance, the signed area of an arbitrary triangle can be computed as the determinant of a 3×3 matrix as follows:

$$A = \frac{1}{2} \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

If two points coincide, the determinant is zero. If the three points are collinear, the rows are linearly dependent, and the determinant is also zero. Note that each row is simply a point in site notation.

3 Lines

We now add to our notation a representation for lines. Although one expression for a line is $Y = mX + r$, this representation has the limitation that there is no expression for vertical lines. One representation without such a bias is to write the equation for a line as $a + bX + cY = 0$. We will abbreviate these coefficients in our new notation as $[a; b, c]$. We retain the semicolon to remind us which member is the constant term. In this notation, the representation for the line $Y = mX + r$ is simply $[r; m, -1]$ or $[-r; -m, 1]$. Note that these coefficients are *homogeneous*: Multiplying the coefficients by any non-zero scalar does not change the line.

What can we deduce about a line from its coefficients?

1. If $a = 0$, the line passes through the origin.
2. If $b = 0$, the line is horizontal.
3. If $c = 0$, the line is vertical.
4. If $a = b = 0$, the line is the x-axis.
5. If $a = c = 0$, the line is the y-axis.
6. The case $b = c = 0$ is not a legal line in affine geometry (only in projective geometry).
7. The case $a = b = c = 0$ *never* represents a legal line.
8. The slope of the line $[a; b, c]$ is $-b/c$.
9. The x-intercept of the line $[a; b, c]$ is $-a/b$.
10. The y-intercept of the line $[a; b, c]$ is $-a/c$.

All of these follow directly from the definition.

4 Operations Using Site Notation

4.1 Coincidence of Sites and Lines

Site notation simplifies many common geometric operations. For instance, to find out if a point lies on a line, we can substitute its coordinates into the equation of a line and check for equality.

$$a + bp_x + cp_y = 0$$

However, note that this is exactly equivalent to taking the "dot product" of the point and the line!

$$(1; p_x, p_y) \cdot [a; b, c] = 0$$

Similarly, a vector lies on a line if

$$(0; v_x, v_y) \cdot [a; b, c] = 0,$$

since the slopes match ($\frac{v_y}{v_x} = -\frac{b}{c}$). Thus we have a mechanism to test whether a site lies on a line without knowing whether that site is a point or a vector:

$$(w; x, y) \cdot [a; b, c] = 0$$

4.2 The Line Joining Two Points

To determine the equation of the line joining any two points P and Q in the plane, we wish to find the set of all points R with the property that the slope of the line PR is the same as the slope of the line PQ. Thus, we need only solve:

$$\frac{Y - p_y}{X - p_x} = \frac{q_y - p_y}{q_x - p_x}$$

Here, $P = (1; p_x, p_y)$, $Q = (1; q_x, q_y)$, $R = (1; X, Y)$. The solution, in our canonical form, is:

$$(p_x q_y - p_y q_x) + (p_y - q_y)X + (q_x - p_x)Y = 0,$$

so the line PQ has the homogeneous coefficients

$$PQ = [p_x q_y - p_y q_x; p_y - q_y, q_x - p_x]$$

These coefficients are easily remembered as coefficients in the following determinant:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ 1 & p_x & p_y \\ 1 & q_x & q_y \end{vmatrix}$$

The coefficients a,b, and c of the line PQ are simply the coefficients in the determinant of α, β, and γ respectively. If the points are the same, all of the coefficients are zero, and we know immediately that there is no single line passing through both points. This matrix is easy to remember because the bottom two rows are simply the two points in site notation.

4.3 Intersection of Two Lines

Finally, the solution of the intersection of two lines has a similar mnemonic. To find the intersection of the lines $[a; b, c]$ and $[d; e, f]$, we want to solve the linear system of equations:

$$a + bp_x + cp_y = 0$$

$$d + ep_x + fp_y = 0$$

for p_x and p_y. The solution, in site notation, is:

$$\left(1; \frac{dc-af}{bf-ce}, \frac{ae-bd}{bf-ce}\right).$$

This is equivalent to the *homogeneous* point $[w; x, y]$, where w, x, and y are the coefficients of ϕ, ψ, and χ respectively, in the determinant:

$$\begin{vmatrix} \phi & \psi & \chi \\ a & b & c \\ d & e & f \end{vmatrix}$$